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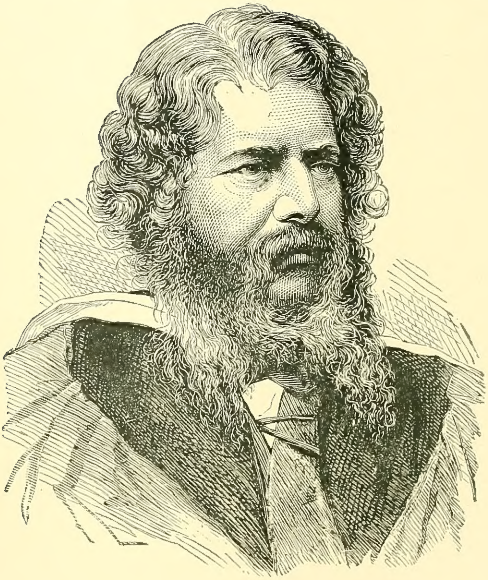

William John Macquorn Rankine, ll.d.

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ELEMENTARY

APPLIED MECHANICS

BY

homas

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TO THE MEMORY OF

DR. WILLIAM JOHN MACQUORN RANKINE

LATE PROFESSOR OF CIVIL ENGINEERING AND MECHANICS
IN THE UNIVERSITY OF GLASGOW

This Work is Dedicated by his Pupils

THOMAS ALEXANDER

AND

ARTHUR WATSON THOMSON

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PREFACE.

IN this Second Edition we have combined the two volumes of the First, rearranging the whole, and introducing important additional matter. The work forms an elementary consecutive treatise on the subject of Internal Stress and Strain, based on the late Professor Rankine's treatment of the subject in his *Applied Mechanics* and *Civil Engineering*. The end kept in view is the Scientific and Practical Design of Earthworks, of Linkwork Structures, and of Blockwork Structures. The whole is illustrated by a systematic and graduated set of Examples. At every point graphical methods are combined with the analytical, and a feature of the work is, that the diagrams are to scale, and, besides illustrating the text, each diagram suits some of the numerical examples, having printed on its face both the data and results. For the Student with little time to draw, the full-page diagrams should prove useful models, furnishing concisely the data and checking the results of his constructions to a bold scale without delay.

In Chapter III. a Moving Model of Rankine's Ellipse of Stress is shown at page 63. It was exhibited to the Royal Irish Academy. The Scientific Design of Masonry Retaining Walls and of their foundations in Chapter V. is an extension of a paper by the authors in *Industries* of 14th Sept., 1888.

The Rules given in Chapter XII., page 237, for the Maxima Bending Moments caused by a locomotive crossing a bridge were published in *Engineering* on January 10th and July 25th, 1879, in response to queries in Du Bois' early treatise on *Graphic Statics*, and are quoted by him in his more recent work, *Strains in Framed Girders.* Mention is made of them also in the preface to Lévy's *La Graphique Statique*, second edition. Hele Shaw's *Report on Graphic Methods* to the Edinburgh Meeting of the British Association for the Advancement of Science refers to the use of the parabolic set-square for the rapid construction of bending moment diagrams, as introduced in our first edition. We now reduce the solution of this important problem to the construction of Diagram of Square Roots of Bending Moments, with arcs of circles only (see p. 226, and for a cut of our Moving Model of a locomotive on a girder see p. 214. Both of those are figured in the correspondence on Mr. Farr's paper on *Moving Loads on Bridges* published in the *Proc. of the Inst. of C. E.*, vol. cxli., 1900). The arcs of circles replacing the parabolic arcs are also to be seen at page 173 for the more general cases of fixed loads.

At the end of Chapter XVI., on Stress at an Internal Point of a Beam, we have reprinted in full, by the kind permission of the Council of the Phil. Soc. of Glasgow, the late Professor Peter Alexander's paper on *The Uses of the Polariscope in the Practical Determination of Internal Stress and Strain.* The lines of stress for a bent-glass prism as drawn by his mechanical pen in conjunction with the polariscope lantern is shown beside the same lines drawn by Rankine in his *Shipbuilding* as a vindication of the accuracy of the Bernoulli-Eulerian approximate theory of bending.

In Chapter XVIII. the solutions given for the uniform girder fixed horizontally at the ends, and subjected to the transit of a concentrated rolling load in one case, and of a moving uniform load in the other case, will be found interesting (see pages 368 and 374). The similarity between these two diagrams of maxima bending moments, and, as well, between the two analytical expressions for them is remarkable. The most recent graphical treatment is followed, and the symmetrical form, at least, in which the results are shown is new. The analytical result for the rolling load agrees with the unsymmetrical results given by Lévy and Du Bois, but we have not seen anywhere an attempt at the solution of the second and more practically important case of the transit of the uniform load shown at p. 374.

In Chapter XIX. we have greatly extended the part on Long Steel Struts, bringing it up to the most recent practice. We quote two of Fidler's tables for their design. On page 411 we give a formula (71*b*) for the immediate design of the economical double-tee section of required strength and required local stiffness. Also on the table, p. 414, at IV., we give a close approximate expression independent, like the others, of the thickness of the metal, for the square of the radius of gyration of the tee-section, or angle-iron constrained to bend like it, as it usually is. This is an important addition to Rankine's list, as these sections are of every-day occurrence and more especially as that marked V., on page 414, and the only one given for angle-irons by Rankine and other writers, as far as we know, is altogether misleading.

The Steel Arched Girder in Chapter XX. is treated by Lévy's graphical methods. Two numerical examples, one hinged at the ends and another fixed at the ends, are worked out in detail, and the scaled results written on the diagrams (see pp. 430 and 434). In Chapter XXI. we follow, on the other hand, his beautiful analysis of the Triangular Trussing, involving the three variables, the number of subdivisions in the span, the form of the triangles, and the ratio of the depth to span. Besides some corrections acknowledged by Lévy in the preface to *La Statique Graphique*, second edition, we have now made some extension (Table, p. 454) of the discussion of the economical shape of triangle, and some further corrections on the volume of the Fink truss with the load rolling below it, a form of truss, however, of little practical importance. The Table of Volumes of Trusses we show at one opening of the book, and the minimum volumes are placed among the others with a heavy-faced type in such a way that it can at once be seen whether they occur at depths that can be adopted consistent with the more important requirement of stiffness.

In Chapter XXII. we have much pleasure in developing the method of conjugate load areas, given by Rankine in mathematical form difficult of practical application to the equilibrium of arches. We now substitute a semigraphical method of constructing the load areas, reducing the mathematics and extending the whole to the complete design of segmental, semicircular, and semi-elliptic arches, with their abutments, spandrils, and piers. Incidentally, the design of sewers, inverts, shafts, and tunnels illustrate the full scope of the method. We have to thank R. J. SULLIVAN, C. V. G. SCOTT, and C. F. DRAPER, Graduates in Engineering of the University of Dublin, for assistance in the preparation of the examples and scaled drawings. In a paper to the Royal Irish Academy, referred to at bottom of page 510, we demonstrated the true shape of the equilibrium curve, dividing the family into two groups, the more important of which we venture to call two-nosed catenaries. We then show that these two varieties offer a philosophical explanation of the two distinct ways, in which it has been found by experiment, that masonry arches break up when the abutments are gradually removed. Elaborate tables were added for the immediate design of segmental masonry arches. We have now embodied those tables reduced to the smallest compass, and having printed on their face all the information necessary for their application. We are indebted to the Council of the Academy for the use of the blocks.

Some of the leading writers in America upon Engineering expressed approbation of our method, and Professor HOWE, in his treatise, referred to at bottom of page 510, has done us the honour of adopting it, and calls it the best method yet published, when the arch bears only *vertical loads.* In our present treatment we have both vertical and horizontal loads duly considered, but have insisted upon a central elastic portion of the arch-ring being wholly free from other than vertical loads, except when the live load covers only half the arch when the horizontal reaction of the light elastic spandrils of the opposite side come into play. Now, with the catenary tables, the segmental arches were limited to this elastic part only, and so the assumption of vertical loads only is justified, and the more especially as the horizontal load indicated in this case is necessarily *outwards,* and cannot be introduced in any practical way. It is remarkable that the two designs, shown to a common scale on pages 500, 516, arrived at by two different methods, namely fig. 267 by the catenary tables, and fig. 272, by the conjugate-load-areas, should so almost exactly coincide.

In Appendix I. we give the solution of a Roof Truss by the method of Reciprocal Figures, covering the whole question of the wind on anchor or free end when only one end in fixed, and including the complete solution of the case with both ends fixed, a problem the undeterminateness of which is usually slurred over.

In Appendix II. is shown the cross-section of a built-earth Reservoir Dam and of the contraction of the stream between two piers.

THOMAS ALEXANDER.
ARTHUR WATSON THOMSON.

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ELEMENTARY APPLIED MECHANICS.

CHAPTER I.

LINEAL STRESS AND STRAIN.

Elasticity is a property of matter. When dealing with the equilibrium of a body under the action of external forces, in order to find the relations among those external forces, the matter of the body is considered to be perfectly rigid, or, in other words, to have no such property as elasticity. When external forces, the simplest of which are stresses acting really on a part of the surface of a body, are considered to act at points on the surface, it is taken for granted that the matter of the body is infinitely strong at such points. But after considering the equilibrium of the body as a whole, we may consider the equilibrium of all or any of its parts. If we take a part on which an external stress directly acts, equilibrium is maintained between that external stress acting on the free surface and the components parallel to it, of stresses which the cut surface of the remaining part exerts on its cut surface.

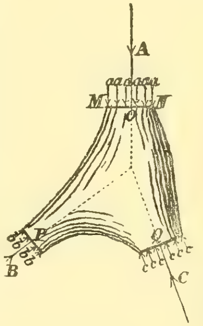


Fig. 1.

Let $MONQP$ be a solid in equilibrium under the action of the three external uniform stresses acting on planes of its surface at O, P, and Q. Let MN be the trace of the plane at O under the uniform stress A. The stresses $aaa \ldots bbb \ldots ccc$ may be represented in amount and direction by the single forces A, B, and C acting at the points O, P, and Q, rigidly connected. We know that, by the triangle of forces, A, B, and C are proportional to the sides of a triangle DFE drawn with its sides parallel to their directions. Also that they are in one plane and meet at one point. Hence we infer that the stresses which they represent are all parallel to the plane of the paper, and that the planes of action of *b* and *c* are at right angles to the plane of the paper as well as that of *a*. Thus we find the relation among the external forces.

Let a plane *m n* divide the solid into two parts (fig. 2). Consider the equilibrium of the part *M N m n*. $s_1, s_2, s_3 \ldots$ are the stresses exerted at all points of the cut surface of *M N m n* by the cut surface of the other part. *S* is the sum of their components parallel to the direction of *A*, acting through *P*, the centre of pressure. Because there is equilibrium, *S* is equal and opposite to *A*; and they act in one straight line. Also the remaining rectangular components of s_1, s_2, s_3 are themselves in equilibrium. Thus we see there is a stress on the plane *m n* and know the amount of it in one direction.

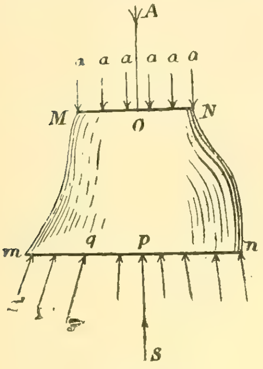


Fig. 2.

Had we been considering the equilibrium of the other part of the solid, the stresses on *m n* (fig. 3) would have been acting on the other surface as $t_1, t_2, t_3, \ldots$ in opposite directions to $s_1, s_2, s_3, \ldots$ and of equal intensities. Thus on the plane *m n* there are pairs of actions, acting at all points of it, as s_3, t_3, at *q*. These vary in intensity and obliquity to *m n* at different points of the plane. If another plane, as *gh*, dividing the solid, pass through *q*, there will be, similarly, pairs of actions at all points of it, and *a pair* of definite intensity and direction at the point *q*. If we know the stress at the point *q* in intensity and direction on all planes passing through *q*, we are said to know *the internal stress* at the point *q* of the solid. Similarly for all points of the solid.

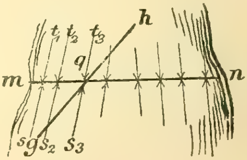


Fig. 3.

The pairs of actions as s_3, t_3 act respectively on the cut surfaces of the upper and under parts of the solid; but *m n* may be considered to be a *thin layer* of the solid with s_3 and t_3 acting on its under and upper surfaces. This layer of the solid must be, however, infinitely thin; otherwise its two surfaces would be different sections of the solid, and s_3 and t_3 not necessarily equal and opposite. If *gh* be also considered a thin layer, and *H* and *K* be the pair of actions on it at the point *q* on the two sides of it, then will the point q be a solid, in figure a parallelepiped, with a pair of stresses acting upon its opposite pairs of faces. s_3 and t_3 being equal, S is now put for each, and H is put for both H and K; and since q, instead of being a point in both planes, has small surfaces in both, though so infinitely small that the stresses over them do not vary from the intensities at the point q, yet surfaces, the stresses spread over which it is more convenient to represent by *sets of equal arrows* $SSS\ldots$, $HHH\ldots$.

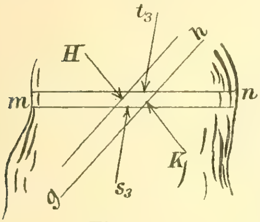


Fig. 4.

There are two convenient ways of representing by a diagram the *internal stress* at q, a point within a solid. One, as in figure 5, in which the indefinitely small parallelepiped q is all of the solid to be immediately considered; and the other, as in figure 6, in which sheaves of equal arrows stand on small portions of the planes mn and gh in the neighbourhood of q.

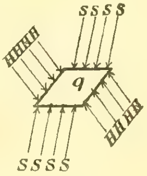


Fig. 5.

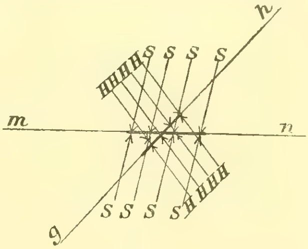


Fig. 6.

STATE OF SIMPLE STRAIN.

Thus we see that, in a solid acted upon by external forces, every particle exerts stress upon all those surrounding it. Such a body is said to be in *a state of strain.* In solids the phenomenon is marked by an alteration of shape, but not necessarily of bulk.

Let AB and CD (fig. 7) be acted upon by two equal and opposite forces P and P in the direction of their length acting in AB away from each other, and in CD towards each other. If P be uniformly distributed over the area A, the section of AB perpendicular to its length, the intensity of the stress on it is

$$p = \frac{P}{A}.$$

Let the prism be of unit thickness normal to the paper; then will the line MN be equal to the area of the section of the prism perpendicular to its axis, and

$$p = \frac{P}{MN}.$$

At any internal layer mn, perpendicular to the axis of the prism, the intensity is also p, for the equilibrium of the parts requires it; and not only is the stress of this same intensity at all points of one such section, but also upon all such sections. The solids AB and CD are said to be in *a state of simple strain*, in the case of AB of *extension*, and in that of CD of *compression*. It is usual to consider the first as positive and the second as negative.

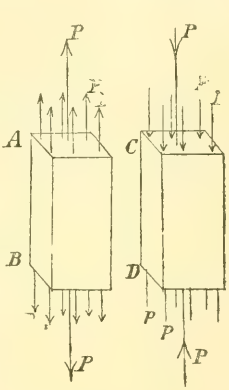


Fig. 7.

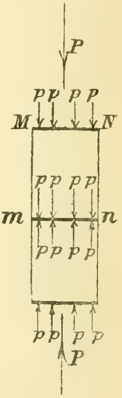


Fig. 8.

The change of dimensions due to a simple state of strain is an alteration of the length of the solid in the direction of the stress with or without an accompanying alteration of its other dimensions. Thus a piece of cork in a state of simple compression has become shorter in the direction of the thrust, yet with scarcely any, certainly without a corresponding, increase of area, normal to the thrust. Again, a piece of indiarubber grows shorter in the direction of the thrust with an almost exactly proportionate increase of area normal to it.

The increase of length in the case of an extension is the *augmentation*, in that of a compression it is a negative augmentation, and in either case it is the amount of strain. The measure of the strain is the ratio of the augmentation to the original unstrained length.

DEFINITION.—Longitudinal strain $= \dfrac{\text{augmentation of length}}{\text{length}}$

where both are in the same name, that is, both in inches or feet, &c. The foot being the unit of length, it is most convenient to take both in feet; then

$$\text{longitudinal strain} = \frac{\text{augmentation in feet}}{\text{length in feet}}.$$

Suppose the denominator on the right-hand side of the equation to be unity, then

$$\text{longitudinal strain} = \text{augmentation in feet of 1 foot of the substance}$$

$$= \text{augmentation per foot of length, expressed in feet.}$$

Hence the total augmentation or *amount of strain* in feet equals the length in feet multiplied by the strain.

If the augmentation equal the length, that is, if the piece be stretched to double its original length or compressed to nothing, then from the definition

$$\text{strain} = \text{unity}.$$

EXAMPLES.

In the following questions the weight of the material is neglected:—

1. A tie-rod in a roof, whose length is 142 feet, stretches 1 inch when bearing its proper stress. What strain is it subjected to?

$$\text{augmentation} = 1 \text{ in.}$$

$$\text{unstrained length} = 1704 \text{ in.}$$

$$\text{strain} = \frac{\text{augmentation}}{\text{length}} = \frac{1}{1704} \text{ or } \cdot 0006.$$

2. How much will a tie-rod 100 feet long stretch when subjected to ·001 of strain?

$$\frac{\text{augmentation}}{\text{length}} = \text{strain};$$

$$\therefore \text{ augmentation} = \text{strain} \times \text{length} = \cdot 001 \times 100 \text{ ft.} = \cdot 1 \text{ ft.}$$

3. A cast-iron pillar 18 feet high shrinks to 17·99 feet when loaded. What is the strain?

$$\text{augmentation of length} = - \cdot 01 \text{ ft.}$$

$$\text{strain} = \frac{\text{augmentation}}{\text{length}} = \frac{- \cdot 01 \text{ ft.}}{18 \text{ ft.}} = - \frac{1}{1800} \text{ or } - \cdot 0005.$$

4. Two wire cables, whose lengths are 100 and 90 fathoms, respectively, while mooring a ship, are stretched, the first 3 inches and the second 2·75 inches. What strains do they sustain? Which sustains the greater? Give the ratio of the strains.

For the 100-fathom cable

$$\text{strain} = \frac{\text{augmentation}}{\text{length}} = \frac{3 \text{ in.}}{7200 \text{ in.}} = \cdot 000417.$$

For the 90-fathom cable

$$\text{strain} = \frac{\text{augmentation}}{\text{length}} = \frac{2{\cdot}75 \text{ in.}}{6480 \text{ in.}} = {\cdot}000422.$$

The 90-fathom cable is the more strained.
Ratio of these strains is 417 to 422.

5. A 30-feet suspension rod stretches $\frac{1}{20}$ inch under its load. Find the strain upon it.

$$\text{strain} = {\cdot}00014.$$

6. How much does another of them, which is 23 feet long, stretch when equally strained?

$$\text{augmentation} = {\cdot}039 \text{ in.}$$

7. A submarine cable is tested with a strain of ·0002. How much did it stretch per 100 fathoms?

$$\text{aug. per 100 fathoms} = 1{\cdot}44 \text{ in.}$$

8. What is the strain upon a wooden strut 60 feet long when compressed to 55·97 feet?

$$\text{strain} = -{\cdot}0005.$$

9. A violin string 10 inches long is stretched to $10\frac{1}{4}$ inches. What is the strain upon it?

$$\text{strain} = \frac{1}{40} = {\cdot}025.$$

10. An indiarubber string 6 inches in length is stretched till it is a foot long. Find the strain.

$$\text{strain} = 1.$$

11. A cast-iron pillar bears a strain of ·001; its original length was 10 feet. Find its altered length.

$$\text{augmentation} = -{\cdot}12 \text{ inches; altered length} = 119{\cdot}88 \text{ inches.}$$

12. A pillar 40 feet high, designed to prop up a beam already supported at the ends, fits exactly into its place. If the greatest strain to which it is safe to subject the pillar be ·0004, what thickness of wedge ought to be driven between the beam and its top?

The thickness is the same as the contraction which the pillar must undergo to produce the necessary strain.

$$\text{thickness} = -\text{aug.} = {\cdot}192 \text{ in.}$$

This is upon the supposition that the beam does not rise up when the wedge is driven.

ELASTICITY.

The *elasticity* of a solid is the tendency it has when strained to regain its original size and shape. If two equal and similar prisms of different matter be strained similarly and to an equal degree, that which required the greater stress is the more elastic—*e.g.*, a copper wire 1000 inches long was stretched an inch by a weight of 680 lbs., while an iron wire of the same section and length required 1000 lbs. to stretch it an inch. Hence iron is more elastic than copper. If they be strained by equal stresses, that which is the more strained is the less elastic —*e.g.* the same copper wire is stretched as before 1 inch by a weight of 680 lbs., while the iron one is only stretched a $\frac{17}{25}$th part of an inch by 680 lbs.

Hence the elasticities of different substances are proportional to the stresses applied, and inversely proportional to the accompanying strains.

If similar rods of steel and indiarubber be subjected to the same stress, the indiarubber experiences an immensely greater strain, so that steel is very much more elastic than indiarubber.

If two similar rods of the same matter, or the one rod successively, be strained by different stresses, the corresponding strains are proportional to the stresses. Thus, if a 480 lbs. stress stretch a copper wire one inch, then a 960 lbs. stress will stretch it, or a similar rod, two inches.

Hooke's Law is "*The strain is proportional to the stress.*" It amounts to—"the effect is proportional to the cause." It is only true for solids within certain limits—*e.g.*, 2400 lbs. should stretch the copper wire mentioned above five inches by Hooke's law, but it would really tear it to pieces; and although 1920 lbs. applied very gradually will not tear it, yet it will stretch it more than four inches; and further, when that stress is removed the wire will not contract to its original length again. Strain and stress are mutually cause and effect. The effect of stress upon a solid is to produce strain; and, conversely, a body in a state of strain exerts stress. The expressions "Strain due to the stress," &c., and "Stress due to the strain," &c., are both correct.

If a solid be strained beyond a certain degree, called the proof strain, it does not regain its original length when released from the strain; in such a case the permanent alteration of length is called a *set.*

Def.—*The Proof Load* is the stress of greatest intensity which will just produce a strain having the same ratio to itself which the strains bear constantly to the stresses producing them for all stresses of less intensity.

If a stress be applied of very much greater intensity, the piece will break at once; if of moderately greater intensity, the piece will take a set; and although only of a little greater intensity, yet if applied for a long time the piece will ultimately take a set; and if it be applied and removed many times in succession the strain will increase each time and the piece ultimately break. For all stresses of intensities less than the proof load the elasticity is constant for the same substance, and the

$$\text{Def.—Modulus of elasticity} = \frac{\text{intensity of stress}}{\text{strain due to it}}$$

$$= \text{stress per unit strain.}$$

If the denominator on the right-hand side of the equation be unity, then the numerator is the stress which produces unit strain, and

Mod. of elasticity = stress which would produce unit strain

on supposition of rod not experiencing a set and Hooke's law holding.

For most substances the proof stress is a mere fraction of E, the modulus of elasticity. For steel the proof stress is scarcely $\frac{1}{100000}$th part of E. Hence in the equation above, the word *would* is employed, as it would be absurd to say that E equalled the stress that will produce unit strain, that being an impossibility with most substances; and even when possible, as in the case of indiarubber, the strains at such a pitch will have ceased to be proportional to the stresses producing them, and hence E will be no longer of a constant value. But the definition is quite accurate and definite for all substances amounting to this, that for any substance

E = 10 times the stress that will produce a strain of $\frac{1}{10}$th,

if such a pitch of strain be possible and within the limit of strain, that is, not greater than the proof strain.

But if not, then,

E = 100 times the stress that will produce a strain of $\frac{1}{100}$th,

if such a pitch of strain be possible and within the limits of strain, that is, not greater than the proof strain.

Thus for steel E equals one million times the stress which will produce a strain of one-millionth part. *Pliability* is a term applied to the property which indiarubber possesses in a higher degree than steel.

Examples.

13. A wrought-iron tie-rod has a stress of 18000 lbs. per square inch of section which produces a strain of ·0006. Find the modulus of elasticity of the iron.

$$E = \frac{\text{intensity of stress}}{\text{strain}} = \frac{18000}{\cdot 0006} = 30000000 \text{ lbs. per square inch.}$$

14. A tie-rod 100 feet long has a sectional area of 2 square inches, it bears a tension of 32,000 lbs., by which it is stretched $\frac{3}{4}$ths of an inch. Find the intensity of the stress, the strain, and modulus of elasticity.

$$\text{stress} = \frac{\text{total stress}}{\text{area}} = \frac{32000 \text{ lbs.}}{2 \text{ sq. in.}} = 16{,}000 \text{ lbs. per sq. in.}$$

$$\text{strain} = \frac{\text{aug. of length}}{\text{length}} = \frac{\cdot 75 \text{ in.}}{1200 \text{ in.}} = \cdot 000625.$$

$$E = \frac{\text{stress}}{\text{strain}} = \frac{16000}{\cdot 000625} = 25600000 \text{ lbs. per sq. in.}$$

15. A cast-iron pillar one square foot in sectional area bears a weight of 2000 tons: what strain will this produce, E for cast iron being 17,000,000 lbs.?

$$\text{total stress} = 2000 \text{ tons per sq. foot} = 2000 \times 2240 \text{ lbs. per sq. foot.}$$

$$\text{stress} = \frac{2000 \times 2240}{144} = 31111 \cdot \dot{1} \text{ lbs. per sq. in.}$$

$$E = \frac{\text{stress}}{\text{strain}}, \quad \text{or} \quad 17000000 = \frac{31111}{\text{strain}};$$

$$\therefore \text{ strain} = \frac{31111}{17000000} = \cdot 0018 \text{ ft. per ft. of length.}$$

16. The modulus of elasticity of steel is 35,000,000. How much will a steel rod 50 feet long and of $\frac{1}{8}$th inch sectional area be stretched by a weight of one ton?

$$\text{total stress} = 2240 \text{ lbs.};$$

$$\text{stress} = \frac{\text{total stress in lbs.}}{\text{area in sq. in.}} = 2240 \div \tfrac{1}{8} = 17920 \text{ lbs. per sq. inch.}$$

$$E = \frac{\text{stress}}{\text{strain}}; \quad \therefore \text{ strain} = \frac{\text{stress}}{E} = \frac{17920}{35000000} = \cdot 000512;$$

$$\frac{\text{elongation}}{\text{length}} = \text{strain};$$

$$\therefore \text{ elongation} = \text{strain} \times \text{length} = \cdot 000512 \times 50 = \cdot 0256 \text{ feet or } \tfrac{1}{3} \text{ of an inch.}$$

17. An iron wire 600 yards long and $\frac{1}{80}$th of sq. inch in section, in moving a signal, sustains a pull = 250 lbs.; how much will it stretch, assuming $E = 25000000$?

stress = 20000 lbs. per sq. inch; strain = ·0008; elongation = 1·44 feet.

18. Modulus of elasticity of copper is 17,000,000: what weight ought to stretch a copper thread, of 12 inches in length and ·004 inches in sectional area, $\frac{1}{100}$th part of an inch. If after the removal of the weight the thread remains a little stretched, what do you infer about the weight and about the strain to which the thread was subjected?

strain = $\frac{1}{1200}$th; stress = 14167 lbs. per sq. inch; weight = 56·668 lbs.

Since this weight causes a set, it is greater than the proof load.

19. A wooden tie 12 inches × 7 inches and 40 feet long was tested with a pull of 130 tons which stretched it 1·28 inches. Find the modulus of elasticity of the wood.

$$\text{stress} = \frac{291200 \text{ lbs.}}{84 \text{ sq. in.}} = 3466{\cdot}\dot{6} \text{ lbs. per sq. in.};$$

$$\text{strain} = \frac{1{\cdot}28 \text{ in.}}{480 \text{ in.}} = {\cdot}002\dot{6}; \quad \therefore\ E = 1300000 \text{ lbs.}$$

20. One rod of an hydraulic hoist is 50 feet long and 1 inch in diameter; it is attached to a plunger 4 inches in diameter upon which the pressure of the water is 800 lbs. per square inch. E being 30,000,000, how much will the rod be compressed, and what is the strain?

strain = – ·000427 ft. per ft.; compression = amount of strain = ·256 in.

21. A glass thread is $\frac{1}{1000}$th of a sq. inch in sectional area, and 15 inches long. What weight would be required to stretch it $\frac{1}{100}$th part of an inch, for glass E being 8,000,000?

stress = 5333 lbs. per sq. in.; weight = amount of stress = 5·3 lbs.

THE PRODUCTION OF STRAIN.

We have as yet only considered the *statical condition* of strain, *i.e.* of a body kept in a state of strain by external forces, these forces being balanced by the reactions of the solid at their places of application due to the elasticity, and the forces exerted on any portion of the solid being in equilibrium with the reactions of the contingent parts. Thus when we found that 32,000 lbs. produced a strain of ·00063 on a tie-rod 100 feet long and 2 square inches in area, in all stretching it $\frac{3}{4}$ths of an inch, we meant that the weight *kept it* at that strain; the rod is supposed to have arrived at that pitch of strain and to be at rest, to be stretched the $\frac{3}{4}$th inch, and so (by its elasticity or tendency to regain its original length) to balance the weight. We have taken no notice of the process by which the rod came to the strain, nor do we say it was the weight that brought it to that state, the weight being only a convenient way of giving the value of the stress on the rod when forcibly *kept* strained. In fact an actual weight of 32,000 lbs. is capable of producing greater strains on the rod in question, depending upon how it is applied to the rod as yet unstrained. The weight might be attached by a chain to the end of a rod and let drop from a height; when the chain checked its fall it would produce a strain on the rod at the instant greater the greater the height through which it dropt. Still, if that strain were not greater than the proof strain, the weight upon finally coming to rest after oscillating a while could only *keep* the rod at the strain ·00063.

We come now to consider the *kinetic relations* between the stress and the strain, that is, while the strain is being produced, the matter of the body being then in motion, we are consequently considering the relations among forces acting upon matter in motion.

If a simple stress of a specific amount be applied to a body it produces a certain strain, and in doing so the stress does work, for it acts through a space in the direction of its action equal to the total strain. But if this stress is applied gradually, so as not to produce a shock, its value increases gradually from zero to its full value, and the work it does will be equal to its mean value, multiplied by the space through which it has acted. And since the stress increases in proportion to the elongation, its average value will equal half of its full value. For example, if a stress of 30,000 lbs. be applied to a rod and produce a strain of $\frac{3}{4}$ inch, it will do $\frac{30000}{2} \times \frac{3}{4} = 11,250$ inch-lbs. of work which will be stored up as a potential energy in the stretched rod.

Suppose a scale-pan attached to the top of a strut or bottom of a tie and the other end fixed. Let a weight be put in contact with the pan, but be otherwise supported so as to exert no stress on the piece, and the next instant let it rest all its weight on the piece, then will the weight do work against the resistance offered by the straining of the piece till the weight ceases descending and comes to rest, when the piece will be for an instant at the greatest strain under the circumstances, at a strain greater than the weight can keep it at; the unstraining of the piece will therefore cause the weight to ascend again, doing work against it to the amount that the weight did in descending, and so the weight will return to its first position, then begin to descend again, and so oscillate up and down through an amplitude equal to the augmentation. Owing to other properties of the matter, whereby some of the work is dissipated during each strain and restitution, the amplitude diminishes every oscillation, and the weight will finally settle at the middle of the amplitude.

A weight applied in this manner is called *a live load.* A live load produces, the instant it is applied, an augmentation of length *double* of that which it can maintain, and therefore causes an *instantaneous strain* double the strain due to a stress of the same amount as the load.

Let now a weight W be applied in the following way:— Divide W into n equal parts of weight w each. If A be the strain due to a stress of amount W, and a the strain due to a stress w, then $W = nw$,

and by Hooke's law, $A = na$.

Let the first weight w be put into the scale-pan. It will produce a strain $2a$ at once, but the piece will settle at a strain a. Now put on the second weight w. It will produce at once an additional strain $2a$, but only of a additional after the piece settles; there being now a total strain $2a$. Add the third weight w. It also will produce at first an additional strain $2a$, but only of a after the piece settles, giving a total now of $3a$; and so, adding them one by one, there will be a strain of $(n-1)a$ when the second last one has been added and the piece has settled. Now, upon adding the nth weight w, it will at first produce an additional strain $2a$, but only of a after the piece settles, giving then a total strain na or A. Thus we have brought the piece to a pitch of strain A by means of the weight W, and only at the instant of adding the last part (w) of it was the piece strained to $(n+1)\,a$, or to a more than A. By making the parts more numerous into which we divide W, and so each part lighter and producing a lesser strain per part, we can make the strain a the extent to which the piece is strained beyond A at the instant of adding the last part, as small as we please.

By so applying the load W we can bring the piece to the corresponding strain A without at all straining it beyond that. A weight so applied is called a *dead load.*

A live load therefore produces, at the instant of its application, a strain equal to that due to a dead load of double the amount. In designing, the greatest strain is that for which provision must be made, so that live loads must be doubled in amount, and the strain then reckoned as due to that amount of dead load. The dead load, together with twice the live load, is called the *gross load.*

The weights of a structure and of its pieces are generally dead loads. Stress produced by a screw, as in tightening a tie-rod, is a dead load. The pressure of earth or water gradually filled in behind a retaining wall, and of steam got up slowly, of water upon a floating body at rest in it, &c., are all dead loads. The weight of a man, a cart, or a train coming suddenly upon a structure, is a live load; so is the pressure of steam coming suddenly into a vessel; so is a portion of the pressure of water upon a floating body which is rolling or plunging. The pressure upon a plunger used to pump water is a live load, but that on a piston when compressing gas is a dead load, the gas being so elastic itself. A load on a chain ascending or descending a pit is a dead load when moving at a constant speed or at rest, but a live load at the starting, and while the speed is increasing, partly a live and partly a dead load. The stress upon the coupling between two railway carriages is a dead load while the speed is uniform, and if the buffers keep the coupling chain tight, the stress is a live load while starting; but if the buffers do not keep it tight, but allow it to hang in a curve when at rest, then the stress upon it at starting will be greater than a live load.

EXAMPLES.

22. An iron rod in a suspension-bridge supports of the roadway 2000 lbs., and when a load of 3 tons passes over it, bears one-fourth part thereof. Find the gross load. If the rod be 20 feet long, and $\frac{3}{4}$ of a square inch in section, find the elongation, E being 29,000,000.

dead load = 2000 ; live load = 1680 lbs., equivalent to a dead load of 3360 lbs.;

$\therefore$ gross load = 5360 lbs.;

$$\text{stress} = \frac{\text{gross load}}{\text{section}} = \frac{5360}{{\cdot}75} = 7147 \text{ lbs. per sq. in.}$$

$$E = \frac{\text{stress}}{\text{strain}};$$

$$\therefore \text{strain} = \frac{\text{stress}}{E} = \frac{7147}{29000000} = {\cdot}000246 = \frac{\text{elongation}}{\text{length}};$$

$$\therefore \text{elongation} = \text{length} \times \text{strain} = 20 \times {\cdot}000246 = {\cdot}00492 \text{ ft.} = {\cdot}06 \text{ in.}$$

23. A vertical wrought-iron rod 200 feet long has to lift a weight of 2 tons. Find the area of section, first neglecting its own weight; if the greatest strain to which it is advisable to subject wrought-iron be ·0005 and $E = 30{,}000{,}000$.

Let A be the sectional area in sq. in.

live load = 4480 lbs. is equivalent to a dead load of 8960 lbs.

$$\therefore \text{stress} = \frac{8960}{A}; \quad E = \frac{\text{stress}}{\text{strain}}, \quad \text{or } 30{,}000{,}000 = \frac{8960}{A \times {\cdot}0005}.$$

$$\therefore A = \frac{8960}{{\cdot}0005 \times 30000000} = {\cdot}597 \text{ sq. in.}$$

24. Find now the necessary section at top of rod, taking the weight into account, calculated from the section found in last.

200 ft. × ·597 sq. in. gives 1433 cubic in.; reckoned at 480 lbs. per cubic foot gives 398 lbs.

Hence live load = 4480 lbs.; dead load = 398 lbs.

$$\therefore \text{gross load} = 9358; \ \text{stress} = \frac{9358}{\text{area}}; \quad E = \frac{\text{stress}}{\text{strain}}, \quad \text{or } 30000000 = \frac{9358}{\text{area} \times {\cdot}0005}.$$

$$\therefore \text{area} = \frac{9358}{{\cdot}0005 \times 30000000} = {\cdot}62 \text{ sq. in.}$$

The weight of the rod being greater when calculated at this section, a third approximation to the sectional area might be made.

25. Taking now the sectional area at ·62 sq. in. find average strain and elongation.

At lowest point

$$\text{gross load} = 8960 \text{ lbs.}\ ; \quad \text{stress} = \frac{8960}{\cdot 62}\ ; \quad \text{strain} = \frac{8960}{\cdot 62 \times 30000000} = \cdot 00048\ ;$$

while strain at highest point is ·0005;

∴ average strain = ·000495; elongation = ·00049 × 200 = ·098 ft. = 1·176 in.

26. A short hollow cast-iron pillar has a sectional area of 12 sq. in. It is advisable only to strain cast-iron to the pitch ·0015. If the pillar supports a dead load of 50 tons; being weight of floor of a railway platform, and loaded waggons pass over it, what amount should such load not exceed? $E = 20000000$.

greatest stress = 30000 lbs. per sq. in.; gross load = 360000 lbs.

deduct dead load = 112000 lbs.; gives a dead load = 248000 lbs.

The live load must not exceed one-half of this.

Note.—Other considerations limit the strength of the pillar if it be long.

Resilience.

Def.—*The Resilience* of a body is the amount of work required to produce the proof strain. A weight one-half the proof stress applied as a live load would produce the proof strain; therefore the work done is this weight multiplied by the elongation at proof strain, the distance which the weight has worked through; or

the resilience of a body

$$= \tfrac{1}{2} \text{ amount of proof stress} \times \text{elongation at proof strain.}$$

For comparison among different substances the resilience is *measured* by the resilience of one foot of the substance by one square inch in sectional area.

$$\therefore \quad R = \tfrac{1}{2} \text{ proof stress} \times \text{proof strain,}$$

R being in foot-lbs. when the stress is in lbs. per square inch and the strain in feet.

And now comparing the amount of resilience of different masses of the same substance: if two be of equal sectional area, that which is twice the length of the other has twice the amount of resilience (the elongation being double); also if two be of equal length and one have twice the sectional area of the other, then the amount of its resilience is double (the amount of stress upon it being twice that upon the other). That is, the amounts of the resilience of masses of the same substance are proportional to their volumes. This is true not only for pieces in a state of simple strain with which we are in the meantime occupied, but can be proved to be universally true for those in any state of strain however complex.

For any substance R being the amount of resilience of a prism of that substance one foot long by one square inch in sectional area, it follows from the above that the amount of resilience of a cubic inch of the substance will be $\frac{1}{12}R$ or that of any volume will be $\frac{1}{12}R \times$ volume in cubic inches.

The resilience of a piece, as defined, is the greatest amount of work which can be done against the elasticity of the piece, without injuring its material.

We can find the amounts of work done upon a piece in bringing it to pitches of strain lower than the proof strain. For brevity we will call this also resilience. Thus, for a piece 1 foot long by 1 square inch in section

$$\text{amount of resilience} = \tfrac{1}{2}\,\text{stress} \times \text{strain is pro. to (stress)}^2,$$

the strain being proportional to the stress; hence

$$\frac{\text{amount of resilience for any stress}}{\text{the resilience}} = \frac{(\text{stress})^2}{(\text{proof stress})^2};$$

$$\therefore \quad \text{amount of resilience} = R \times \left(\frac{\text{stress}}{\text{p. stress}}\right)^2.$$

For a piece of volume V cubic inches, at any stress we have either—

$$\text{amount of resilience} = \tfrac{1}{2}\,\text{stress} \times \text{strain} \times \frac{V}{12},$$

or

$$= \tfrac{1}{2}\,\text{amt. of stress} \times \text{amt. of strain}.$$

The amount of resilience of a piece, at the instant a live load is applied, will be the product of that load and the instantaneous elongation. Let W be a load the elongation due to which is A. If W be applied as a live load, the instantaneous elongation is $2A$, and the

$$\text{amount of resilience due to a live load } W = W \times 2A.$$

If W be applied as a dead load, the amount of resilience is steadily that of the piece elongated to an amount A, is the same as what it would be for an instant upon the application of a live load $\frac{W}{2}$, or

$$\text{amount of resilience due to a dead load } W = \frac{W}{2} \times A.$$

Therefore, a live load produces for an instant an amount of resilience four times that produced by an equal dead load.

EXAMPLES.

27. A rod of steel 10 feet long and ·5 of a square inch in section is kept at the proof strain by a tension of 25,000 lbs., the modulus of elasticity for steel being 35,000,000. Find the resilience of steel, also the amount of resilience of the rod.

$$\text{proof stress} = \frac{25000}{\cdot 5} = 50000 \text{ lbs. per sq. inch.}$$

$$E = \frac{\text{proof stress}}{\text{proof strain}};$$

$$\therefore \text{ proof strain} = \frac{\text{proof stress}}{E} = \frac{50000}{35000000} = \frac{1}{700}$$

$$= \cdot 00143 \text{ elongation in ft. per ft. of length.}$$

$$\text{resilience, } R = \tfrac{1}{2} \text{ proof stress} \times \text{proof strain} = \tfrac{1}{2} \times 50000 \text{ lbs.} \times \cdot 00143 \text{ ft.}$$

$$= 35{\cdot}75 \text{ ft.-lbs. of work per vol. of 1 ft. in length by 1 sq. in. in sectional area.}$$

$$\text{amt. of res. of rod} = R \times (\text{vol. expressed in number of such prisms})$$

$$= \frac{1}{12}, R \times \text{vol. in cub. in.} = \frac{1}{2} \times 35{\cdot}75 \times 120 \text{ in.} \times \cdot 5 \text{ sq. in.}$$

$$= 178{\cdot}75 \text{ foot-lbs. of work.}$$

Otherwise, to find amount of resilience directly,

$$\text{proof strain} = \frac{1}{700}; \text{ total elongation} = \frac{1}{70}\text{ft.}; \text{ amount of stress} = 25000 \text{ lbs.}$$

$$\text{amount of resilience} = \tfrac{1}{2} \text{ amount of stress} \times \text{elongation},$$

$$= \frac{25000}{2} \times \frac{1}{70}, = 178{\cdot}6 \text{ ft.-lbs. of work.}$$

28. A series of experiments were made on bars of wrought iron, and it was found that they took a set when strained to a degree greater than that produced by a stress 20,000 lbs. per square inch, but not when strained to a less degree. At that pitch the strain was ·0006. Find the resilience of this quality of iron.

$$\text{proof stress} = 20000 \text{ lbs. per sq. in.}$$

$$\text{proof strain} = \cdot 0006 \text{ ft. per ft. of length.}$$

$$R = \tfrac{1}{2} \times 20000 \times \cdot 0006 = 6 \text{ ft.-lbs.}$$

29. Find how much work it would take to bring a rod, of the above iron, 20 feet long and 2 square inches in sectional area, to the proof strain.

$$\text{volume} = 480 \text{ cub. in.}$$

R ft.-lbs. of work brings to proof strain a rod 1 foot long by 1 square inch in area: that is, of volume 12 cubic inches, and amounts of resilience being proportional to the volumes.

$$\text{work required} = \tfrac{1}{12} R \,.\, \text{vol. in cub. in.} = \tfrac{1}{12} \times 6 \times 480 = 240 \text{ ft.-lbs.}$$

or
$$\frac{\text{amount of res.}}{R} = \frac{480 \text{ cub. in.}}{12 \text{ cub. in.}} = 40.$$

$$\therefore \quad \text{amount of res.} = R \times 40 = 6 \times 40 = 240 \text{ ft.-lbs.}$$

30. A wooden strut 18 square inches in section, and 12 feet long, sustains a stress of 1000 lbs. per square inch. Find the amount of resilience of the strut, E being 1,200,000 lbs.

half of total stress = 9000 lbs ; elongation = ·01 ft. ;
amount of resilience = 90 ft.-lbs.

31. Steam at a tension of 600 lbs. on the square inch is admitted suddenly upon a piston 18 inches in diameter. If the piston rod be 2 inches in diameter and 7 feet long, what is the amount of its resilience at the instant? $E = 30000000$.

for live load stress = 97200 lbs. per sq. in.
gives instant strain = ·00324 ft. per ft. of length.
elongation = ·02268 ft.
resilience of rod = live load × elongation = 48600π lbs. × ·02268 ft.
= 1102π ft.-lbs.

32. The chain of a crane is 30 feet long and has a sectional area equivalent to $\frac{1}{2}$ of a square inch: what is the amount of its resilience when a stone of 1 ton weight resting on a wooden frame is lifted by the action of the crane? $E = 30000000$.

stress = 4480 lbs. per square inch, strain = ·000149.
amount of stress = 2240 lbs., elongation = ·00447 ft.
resilience of chain = $\frac{1}{2}$ amount of stress × elongation = 5 ft.-lbs.

33. If the chain be just tight, but supporting none of the weight of stone, and if now the wooden frame suddenly gives way, what is the amount of resilience of the chain at the instant?

Being now a live load, there is an instantaneous strain of double the former amount.

instantaneous strain = ·000298. instantaneous elongation = ·00894 ft.
resilience of chain = live load × elongation = 2240 × ·00894 = 20 ft.-lbs.

34. The wire for moving a signal 600 yards distant has, when the signal is down, a tension upon it of 250 lbs., which is maintained by the back weight of the hand lever; under the circumstances the wire is stretched 1·44 feet (see example 17), and the back weight of the signal, which is 280 lbs., rests portion of its weight upon its bed. The hand lever is suddenly pulled back and locks: the wire being more intensely strained, the signal is raised by the elasticity of the wire partially unstraining. If the point where the wire is attached to the signal moves through ·2 feet, find the range of the point where the wire is attached to the hand lever, also the force which must be exerted there.

When the signal settles up, the amount of stress on the wire is 280 lbs.

$$\frac{\text{elongation for 280 lbs.}}{\text{elongation for 250 lbs.}} = \frac{280}{250},$$

$$\text{elongation for 280 lbs.} = \frac{280}{250} \times 1{\cdot}44 = 1{\cdot}613\,; \text{ additional elongation} = {\cdot}173 \text{ ft.},$$

$\therefore$ range of point at lever = range of point at signal plus this additional elongation
= ·2 + ·173 = ·373 feet.

Thus, when the lever is put back there is upon the wire for an instant before the signal rises an additional elongation of ·373 feet. Hence the tension on the wire the instant the lever is put back will be that due to an elongation of

$$(1{\cdot}44 + {\cdot}373)\text{ ft.} = \frac{1{\cdot}44 + {\cdot}373}{1{\cdot}44}\,250\text{ lbs.} = 314{\cdot}8\text{ lbs.}$$

This is the force which must be exerted at the point where the wire is attached to the hand lever. That is, the instantaneous value of the force used to raise the signal is 34·8 lbs. greater than its weight.

35. On a chain 30 feet long $\frac{3}{4}$ of a square inch sectional area and having a modulus of elasticity of 25,000,000 lbs. there is a dead load of 3900 lbs. and a live load of 900 lbs. Find the amount of resilience of chain when dead load only is on, also at instant live load comes on.

$$\text{as strain} = \frac{\text{stress}}{E} = \frac{5200}{25000000} = {\cdot}0002,$$

$$\text{elongation} = {\cdot}006\text{ ft.},$$

$$\text{amount of resilience for dead load} = \tfrac{1}{2}\text{ amount of stress} \times \text{elongation}$$

$$= \tfrac{1}{2} \times 3900 \times {\cdot}006 = 11{\cdot}7\text{ ft.-lbs.}$$

Live load gives an additional elongation equal to that for a dead load of 1800 lbs.

$$\frac{\text{elongation (due to 1800 lbs.)}}{{\cdot}006\text{ ft.}} = \frac{1800}{3900}.$$

$$\text{inst. elongation for live load} = \frac{1800}{3900} \times {\cdot}006 = {\cdot}00277\text{ ft.}$$

Now both the 3900 lbs. and the 900 lbs. worked through this ·00277 ft.

$$\therefore\ \text{additional resilience} = 4800\text{ lbs.} \times {\cdot}00277\text{ ft.} = 13{\cdot}3\text{ ft.-lbs.}$$

amount of resilience at instant live load comes on = 25 ft.-lbs.

36. A rod 20 feet long and $\frac{1}{2}$ inch in sectional area bears a dead load of 5000 lbs. Find the live load which would produce an instantaneous elongation of another $\frac{1}{10}$th inch. $E = 30000000$. *Ans.* 3125 lbs.

37. A rod of iron 1 square inch section and 24 feet long checks a weight of 36 lbs. which drops through 10 feet before beginning to strain it. If $E = 25000000$, find greatest strain.

Let p = the stress at instant of greatest strain; then

$$\text{strain} = \frac{p}{E}, \qquad \text{elongation} = \frac{24p}{E},$$

$$\text{amount of resilience} = \tfrac{1}{2}\text{ amount of stress} \times \text{elongation}$$

$$= \frac{p}{2} \times \frac{24p}{E} = 12\,\frac{p^2}{E}\text{ ft.-lbs.}$$

$$\text{Work done by weight in falling} = 36\text{ lbs.} \times \left(10 + \frac{24p}{E}\right)\text{ft.} = 360 + \frac{864p}{E}\text{ ft.-lbs.}$$

Equating,

$$\frac{12p^2}{E} = \frac{864p}{E} + 360,$$

$$p^2 - 72p = 30E;\ p^2 - 72p + (36)^2 = 750001296,$$

$$p - 36 = 27386;\ p = 27422\text{ lbs. per square inch,}$$

$$\therefore\ \text{strain} = {\cdot}001097\text{ ft. per ft. of length.}$$

38. If the weight in Ex. 37 had fallen through the 10 feet by the time it came first to rest and $E = 30000000$ lbs., what is the greatest strain?

$$\text{amount of resilience} = 360 \text{ ft.-lbs.,} \quad \text{or} \quad \frac{12p^2}{E} = 360,$$

$$\therefore \quad p = 30000 \text{ lbs. per square inch,}$$

$$\text{strain} = \cdot 001 \text{ ft. per ft. length,}$$

39. If the proof strain of iron be ·001, what is the shortest length of the rod of one square inch in sectional area which will not take a set when subjected to the shock caused by checking a weight of 36 lbs. dropped through 10 feet?

[$E = 30000000$ lbs. per square inch.]

Let x = length in feet.

By hypothesis it comes to the proof strain; hence

$$\text{elongation} = \cdot 001 \times x \text{ ft.}$$

$$\text{proof stress} = E \times \text{proof strain} = 30000 \text{ lbs. per sq. inch.}$$

$$\text{amount of stress} = 30000 \text{ lbs.}$$

$$\text{(inst.) amount of res.} = \tfrac{1}{2} \text{ amount of stress} \times \text{elongation}$$

$$= 15000 \times \frac{x}{1000} = 15x \text{ ft.-lbs.}$$

Equating to work done by weight,

$$15x = 360; \quad x = 24 \text{ ft.}$$

Note.—This is the shortest rod of iron one square inch in sectional area which will bear the shock. The volume of this rod is 288 cubic inches, and a rod of iron which has 288 cubic inches of volume will just bear the shock; as 48 feet long by ½ square inch in area or 12 feet long by 2 square inches sectional area.

The 10 feet fallen through by the weight includes the elongation of rod. When the question is to find the shortest rod to sustain the shock in the case where the weight falls through 10 feet *before it begins to strain* the rod, the volumes of the rods would not be exactly equal for different sectional areas; for a long thin rod will sustain a greater elongation than a short thick one, and as the falling weight works through this elongation over and above the 10 feet, the first rod will require a greater cubical volume than the second.

40. Find the shortest length of a rod of steel which will just bear without injury the shock caused by checking a weight of 60 lbs. which falls through 12 feet before beginning to strain the rod. First for a rod of sectional area 2 square inches, and then for a rod of ¼ square inch sectional area. Given that for steel $E = 30000000$ lbs. and $R = 15$ ft.-lbs.

Let A = sectional area in square inches and x = length in feet.

$$\text{proof stress} \times \text{proof strain} = 2R \qquad \text{def.}$$

$$\frac{\text{proof stress}}{\text{proof strain}} = E \qquad \text{def.}$$

Dividing,

$$(\text{proof strain})^2 = \frac{2R}{E}.$$

$$\text{proof strain} = \sqrt{\frac{2R}{E}} = \frac{1}{1000}, \quad \text{elongation} = \frac{x}{1000} \text{ ft.}$$

Work done by the weight in falling

$$= 60 \text{ lbs.} \times \left(12 + \frac{x}{1000}\right) \text{ft.} = 720 + \frac{3}{50} x \text{ ft.-lbs.}$$

Amount of resilience of rod at proof strain

$$= R \times \left(\frac{1}{12} \text{ vol. in cub. in.}\right)$$

$$= R \times (\text{length in ft.} \times \text{sec. area in sq. in.}).$$

$$= 15 \times x \times 2 = 30x \text{ ft.-lbs. } \textit{first.}$$

and $$15 \times x \times \tfrac{1}{4} = \frac{15}{4} x \text{ ft.-lbs. } \textit{second.}$$

Equating for first case,

$$30x = 720 + \frac{3}{50}x, \qquad 1497x = 36000,$$

$$x = 24{\cdot}05 \text{ ft. length of rod.}$$

Equating for second case,

$$\frac{15}{4}x = 720 + \frac{3}{50}x, \qquad 738x = 144000,$$

$$x = 195{\cdot}12 \text{ ft. length of rod.}$$

For the first case length is 288·6 inches, and sectional area 2 square inches gives

$$\text{volume} = 577{\cdot}2 \text{ cub. inches.}$$

While for second case length is 2341·44 inches, and sectional area $\frac{1}{4}$ square inch, giving

$$\text{volume} = 585{\cdot}3 \text{ cub. inches,}$$

which is a little greater. See Note to Ex. 39.

This may be exhibited generally; putting W, h, and A for the weight, the distance dropped through, and the sectional area,

$$\text{work done by weight} = W\left(h + x\sqrt{\frac{2R}{E}}\right).$$

$$\text{amount of resilience} = R \,.\, x \,.\, A.$$

Equating, $$RxA = Wh + Wx\sqrt{\frac{2R}{E}}.$$

$$\therefore \quad x\left(RA - W\sqrt{\frac{2R}{E}}\right) = Wh. \qquad \therefore \quad x = \frac{Wh}{RA - W\sqrt{\frac{2R}{E}}} \text{ ft.}$$

$$\text{vol.} = A \times 12x \text{ cub. inches.}$$

$$= \frac{12AWh}{RA - W\sqrt{\frac{2R}{E}}} = \frac{12Wh}{R - \frac{W}{A}\sqrt{\frac{2R}{E}}},$$

which increases as A decreases.

41. The greatest (*working*) stress which it is safe to apply repeatedly to iron being 10,000 lbs. per square inch, what is the shortest rod of one square inch in section that may be employed to check the fall of a 36 lb. weight through 3 feet raised and let fall constantly?

Length, 64 feet.

42. Certain rods of wrought iron are for the same structure; those which will not bear a proof strain of ·001 are to be rejected. The testing machine consists of a ring weighing 120 lbs., which, falling through distances marked upward on the instrument, catches on a collar screwed on the lower end of the rod to be tested, whose upper end is fixed. The shock producing the instantaneous strain ·001 is repeated several times, when if there be no set produced the rod is passed. From what height on the scale ought the ring to be dropped when testing rods 15 feet long by $\frac{1}{2}$ square inch sectional area; also, how high should the collar be above zero? $E = 30000000$.

$$\text{proof stress} = \text{proof strain} \times E = {\cdot}001 \times E = 30000 \text{ lbs. per sq. in.},$$

$$\text{amount of stress} = 15000 \text{ lbs.},$$

$$\text{elongation} = 15 \times {\cdot}001 = \frac{3}{200} \text{ ft.} = {\cdot}015 \text{ ft.},$$

$$\text{amount of resilience} = \tfrac{1}{2} \text{ amount of stress} \times \text{elongation} = 112{\cdot}5 \text{ ft.-lbs.}$$

Let x be reading in ft. on scale; then

$$\text{work in ft.-lbs.} = 120x, \text{ and, equating,}$$

$$120x = 112{\cdot}5, \quad x = 0{\cdot}94 \text{ ft.}$$

The collar ought to be above zero the amount of the elongation, so that the weight may only have descended through the 0·94 ft. when it first comes to rest.

$$\text{height of collar} = \text{elongation} = {\cdot}015 \text{ ft.}$$

43. Find readings on scale from which ring should be dropped, and at which collar should be adjusted when testing 20 feet rods $1\frac{1}{2}$ square inches in area.

reading for ring, 3·75 ft. reading for collar, 0·02 ft.

44. Wooden piles, 2 square feet in section, are being driven by a weight of 7680 lbs. When 20 feet of a pile is above ground, what is the greatest height the weight should be dropped from, if 4000 lbs. per square inch be the greatest stress it is desirable to put upon the timber? $E = 1200000$. Neglect the fact that the weight falls through the diminution of length of strut.

height = 5 ft.

45. It is found that the pile sinks 0·2 feet, so that the straining of the timber acts through ·2 feet besides the elongation. From what height may the weight be dropped without fear of injury, considering this?

$$\text{stress} = 4000 \text{ lbs. per sq. in.}$$

$$\tfrac{1}{2} \text{ amt. of stress} = 576000 \text{ lbs.} \quad \text{strain} = -\frac{1}{300}\cdot$$

$$\text{augmentation} = -\frac{1}{15} \times {\cdot}067.$$

$$\text{amt. of resilience} = \tfrac{1}{2} \text{ amt. of stress} \times ({\cdot}2 + {\cdot}067) = 153792 \text{ ft.-lbs.}$$

$$\text{work} = 7680\,(x + {\cdot}267) = 153792 \text{ ft.-lbs.}$$

$$\therefore\ 7680x = 153792 - 2050 = 151742. \quad \therefore\ x = 19{\cdot}3 \text{ ft.}$$

From observing now the distance the pile sinks from this stroke, another greater height may be calculated from which it would be safe to let the rammer fall. Should the pile, however, come upon a large stone, it might splinter when the rammer fell from a greater height than 5 feet.

46. A mass of W lbs. moving horizontally with a velocity of V feet per second is stopped by a chain (l feet long and a square inches effective sectional area), whose other end is securely anchored. Find the greatest pull on the chain, E being the modulus of elasticity, and g the acceleration of a falling body.

$h = \frac{V^2}{2g}$ is the height through which W must fall from rest to acquire the velocity V. Hence kinetic energy of W is

$$W \text{ lbs.} \times h \text{ ft.} = \frac{WV^2}{2g} \text{ ft.-lbs.}$$

This work must be done upon W to bring it to rest; hence it is the work W will do upon the chain.

Let P = greatest pull on chain in lbs. at the instant W first comes to rest.

$$\text{stress} = \frac{P}{a} \text{ lbs. per sq. in.} \qquad E = \frac{\text{stress}}{\text{strain}},$$

$$\text{or strain} = \frac{\text{stress}}{E} = \frac{P}{aE}, \qquad \text{elongation} = \text{strain} \times \text{length} = \frac{Pl}{aE},$$

$$\text{amt. of resilience} = \tfrac{1}{2} \text{ amt. of stress} \times \text{elongation}$$

$$= \frac{P}{2} \times \frac{Pl}{aE} = P^2 \cdot \frac{l}{2aE}.$$

$$\text{Equating,} \quad P^2 \cdot \frac{l}{2aE} = \frac{WV^2}{2g}. \quad \therefore P^2 = \frac{aEWV^2}{gl}. \quad \therefore P = V\sqrt{\frac{aEW}{lg}}.$$

47. A waggon going down an incline is attached to a rope worked by a stationary engine. Its weight is five tons, and its velocity four miles per hour, when it is suddenly stopped by the accidental stopping or reversing of the engine. Find the greatest instantaneous pull upon the rope, which is 4 square inches in sectional area, there being 600 feet of free rope between waggon and engine. $E = 25000000$ lbs. and $g = 32$ feet per second per second.

$$\text{amount of res.} = \tfrac{1}{2} \text{ amount of stress} \times \text{elongation}$$

$$= \frac{P}{2} \times \frac{6P}{1000000} = \frac{3}{1000000} P^2 \text{ft.-lbs.}$$

$$\text{kinetic energy of waggon} = 6023{\cdot}5 \text{ ft.-lbs.}$$

$$\text{Equating,} \qquad P^2 = 2007833333. \quad \therefore P = 44808 \text{ lbs.} = 20 \text{ tons.}$$

48. A body of mass W lbs., moving with a velocity V feet per second, is connected by means of a chain l feet long, and of a square inches sectional area, to another body of mass w at rest. W and w being great compared to the mass of chain, its inertia is neglected. Find greatest pull upon chain.

As the chain begins to stretch W loses and w gains velocity, till both together with chain arrive for the first time at a common velocity, at which instant the tension is the greatest possible upon the chain, which will now begin to contract, causing w to move quicker than W, again extending as W increases till common velocity is arrived at; finally, after many such stretchings and contractings, the chain will settle at a fixed pitch of strain, and all will continue to move at common velocity above.

Let $U =$ common velocity,

$P =$ pull on chain in lbs. (at instant),

$E =$ modulus of elasticity of chain,

$WV =$ momentum when w was at rest,

$(W + w)\,U =$ momentum when at com. vel.

Equating, $U = \dfrac{WV}{W + w}$

Owing to W's velocity being reduced, it has done a certain amount of work upon the chain, and since w has acquired velocity the chain has done a certain amount of work upon it, and the difference of these amounts of work equals the amount of resilience of chain.

From equation $h = \dfrac{v^2}{2g}$, where v is velocity acquired by a body falling from rest through h feet, we have

$$\frac{V^2}{2g} = \text{ht. } W \text{ must fall to acquire vel. } V,$$

$$\text{and}\quad \frac{U^2}{2g} = \text{ht. } W \text{ must fall to acquire vel. } U.$$

$$\therefore\quad \frac{V^2 - U^2}{2g} = \text{ht. } W \text{ must fall to increase from vel. } U \text{ to vel. } V,$$

$$\text{and}\quad W \cdot \frac{V^2 - U^2}{2g} = \text{work lost by } W, \quad w \cdot \frac{U^2}{2g} = \text{work gained by } w.$$

$$\text{diff. of work} = \frac{1}{2g}\left\{W(V^2 - U^2) - wU^2\right\} = \frac{1}{2g}\left\{WV^2 - (W + w)\,U^2\right\}.$$

Putting for U its value,

$$U = \frac{1}{2g}\left\{WV^2 - (W + w)\left(\frac{WV}{W + w}\right)^2\right\}$$

$$= \frac{1}{2g}\left(WV^2 - \frac{W^2V^2}{W + w}\right) = \frac{V^2}{2g} \cdot \frac{Ww}{W + w} \text{ ft.-lbs.}$$

Now upon chain

$$\text{stress} = \frac{P}{a} \text{ and } E = \frac{\text{stress}}{\text{strain}}, \quad \text{or strain} = \frac{\text{stress}}{E} = \frac{P}{aE},$$

$$\text{elongation} = \text{strain} \times \text{length} = \frac{Pl}{aE},$$

$$\text{amt. of res.} = \tfrac{1}{2} \text{ amt. of stress} \times \text{elongation} = \frac{P}{2} \text{lbs.} \times \frac{Pl}{aE} \text{ft.} = P^2 \frac{l}{2aE} \text{ft.-lbs.}$$

Equating to difference of work,

$$P^2 \cdot \frac{l}{2aE} = \frac{V^2}{2g} \cdot \frac{Ww}{(W + w)} = V^2 \cdot \frac{aE}{lg} \cdot \frac{Ww}{(W + w)}$$

$$\therefore\quad P = V \cdot \sqrt{\frac{aEWw}{lg(W + w)}} \text{ lbs.}$$

48*a*. Same problem by method of the Calculus. Let x_2 and x_1 be the spaces passed through by W and w respectively, and P the pull upon chain at any time t; hence at that instant

$$\text{elongation} = x_2 - x_1. \quad \text{strain} = \frac{x_2 - x_1}{l} \text{ def.}$$

$$\text{stress} = \frac{P}{a}. \qquad E = \frac{\text{stress}}{\text{strain}} = \frac{Pl}{a(x_2 - x_1)}.$$

$$\therefore \; P = \frac{Ea}{l}(x_2 - x_1) \qquad (1).$$

Since P accelerates w we have its acceleration

$$\frac{d^2x_1}{dt^2} = \frac{P}{w}g = \frac{Eag}{wl}(x_2 - x_1),$$

and since P retards W its acceleration

$$\frac{d^2x_2}{dt^2} = -\frac{P}{W}g = -\frac{Eag}{Wl}(x_2 - x_1),$$

subtracting,

$$\frac{d^2(x_2 - x_1)}{dt^2} = -\frac{Eag}{l}\left(\frac{1}{W} + \frac{1}{w}\right)(x_2 - x_1).$$

A differential equation from which $(x_2 - x_1)$ is to be determined.

Put $(x_2 - x_1) = c\cos(nt - e)$, then $\dfrac{d^2(x_2 - x_1)}{dt^2} = -n^2(x_2 - x_1)$,

Equating $n^2 = \dfrac{Eag}{l}\left(\dfrac{1}{W} + \dfrac{1}{w}\right)$, $n = \sqrt{\dfrac{Eag(W + w)}{lWw}}$.

When $t = 0$, $x_2 - x_1 = 0$. $\therefore \; 0 = c\cos(nt - e)$.

$$\therefore \; \cos(-e) = 0, \text{ or } e = \frac{\pi}{2}. \quad \therefore \; (x_2 - x_1) = c\cos\left(nt - \frac{\pi}{2}\right) = c\sin nt.$$

Differentiating,

$$\frac{dx_2}{dt} - \frac{dx_1}{dt} = nc\cos nt.$$

Since $\dfrac{dx_2}{dt}$ and $\dfrac{dx_1}{dt}$ are the velocities of W and w, their difference is zero at instant of greatest strain.

$$\therefore \; 0 = nc\cos nt. \quad \therefore \; nt = (2m + 1)\frac{\pi}{2}, \text{ where } m \text{ is any integer.}$$

$\therefore \; t = \dfrac{2m + 1}{2n}\pi$, gives times at which W and w come to a common velocity.

When $t = 0$, the difference of W and w's velocities is V.

$$\therefore \quad \frac{dx_2}{dt} - \frac{dx_1}{dt} = V$$

$$\therefore \quad V = nc \cos 0 = nc, \quad \therefore \quad c = \frac{V}{n}.$$

Sub. into
$$P = \frac{Ea}{l}(x_2 - x_1) = \frac{Ea}{l} c \sin nt = \frac{Ea}{l} \frac{V}{n} \sin (2m+1)\frac{\pi}{2}$$

$$= \frac{Ea}{l} \cdot \frac{V}{n} = \frac{Ea}{l} \cdot V \div \sqrt{\frac{Eag(W+w)}{lWw}}$$

$$= V \cdot \frac{Ea}{l} \sqrt{\frac{lWw}{Eag(W+w)}} = V \cdot \sqrt{\frac{EaWw}{lg(W+w)}}. \qquad (2)$$

Other things being constant, this is greatest when $W = w$, for $(W + w)$ constant. Note also that the stress

$$p = \frac{P}{a} = V\sqrt{\frac{EWw}{alg(W+w)}}, \qquad (3)$$

increases if a be decreased, although the amount of stress P (2) decreases. So that the only way to lessen P without increasing $\frac{P}{a}$ is to increase l the length of chain.

49. A mass of 5 tons, intended to act as a drag upon a ship being launched, is connected by means of a chain 100 feet long and 8 square inches effective sectional area, and $E = 25000000$. Find the greatest pull on the chain if the ship, when floating, have inertia equivalent to a mass of 400 tons, and it be estimated that her velocity will be 20 feet per second when the drag comes into play.

$$P = V \cdot \sqrt{\frac{aEWw}{lg(W+w)}}$$

$$= 20\sqrt{\frac{8 \times 25000000 \times 896000 \times 11200}{100 \times 32 \times (896000 + 11200)}}$$

$$= 20\sqrt{691358025}$$

$$= 525874 \text{ lbs.} = 235 \text{ tons total pull.}$$

$$\therefore \quad p = \left(\frac{P}{a}\right) = \frac{235}{8} = 29 \text{ tons per square inch,}$$

which is about the utmost stress that wrought iron can bear.

CHAPTER II.

INTERNAL STRESS AND STRAIN, SIMPLE AND COMPOUND.

In the following Chapter, except specially stated, we premise that—

(*a*) All forces and stresses are parallel to one plane.

(*b*) That plane is the plane of the paper in all diagrams. Hence planes subjected to the stresses we are considering are shown in diagrams by strong lines, their *traces.*

(*c*) The diagrams represent slices of solid, of unit thickness normal to the paper; hence, the lengths of the strong lines are the areas of the planes.

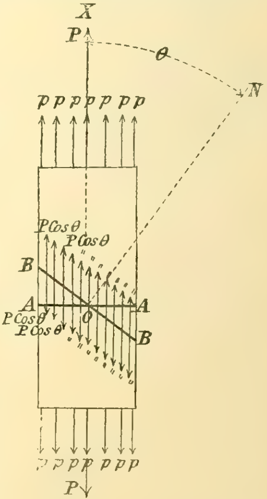


Fig. 9.

(*d*) The stresses which are normal to the paper are supposed constant both in direction and intensity, or every point on a diagram is in the same circumstances with respect to stress normal to the paper.

(*e*) The relative position of two planes is measured by the angle between their normals.

(*f*) The obliquity of the stress to the plane upon which it acts is the angle its direction makes with the normal to the plane.

Internal stress at a point in a solid in a simple state of strain.

Let the axis OX (fig. 9) be drawn in the direction of the stress P. Let AA be any section normal to this axis. Since the stress is uniformly distributed over AA, the intensity of the stress at all points of the plane AA is the same Consider the point O, the intensity of the stress at O on the plane normal to OX is

$$p = \frac{\text{total stress}}{\text{area of plane}} = \frac{P}{AA}.$$

Through O draw any oblique plane, BB, whose normal, ON, makes the angle θ with OX. The stress on this plane is in the direction OX, and the amount of stress upon it is P (for the equilibrium of the parts). But the intensity of the stress on BB is less than p, since P is spread over a larger area than AA.

Since

$$AOB = \theta, \quad \text{and} \quad \frac{OA}{OB} = \cos AOB,$$

$$\therefore \quad OB = \frac{OA}{\cos\theta}, \quad \text{or} \quad BB = \frac{AA}{\cos\theta}.$$

Intensity of stress on

$$BB = \frac{\text{total stress}}{\text{area of plane}} = \frac{P}{BB} = \frac{P}{\left(\dfrac{AA}{\cos\theta}\right)}$$

$$= \frac{P}{AA} \cdot \cos\theta = p \cdot \cos\theta.$$

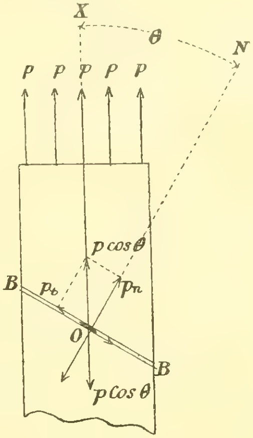


Fig. 10.

Hence the internal stress at all points within a solid, in a state of simple strain, is parallel to the direction of that stress —is greatest in intensity on the plane normal to that direction— on any other plane inclined at an angle θ to last, the intensity is one (cosine θ)th part of that intensity, and zero on any plane parallel to the direction of the stress.

The stress p cos θ on BB being oblique to BB, it is convenient to resolve it into components normal and tangential to BB respectively.

The arrow p cos θ (fig. 10) represents the stress at the point O on the plane BB; from its extremity perpendiculars are dropped on ON and BB, which, by parallelogram of forces, give p_n and p_t, the intensities of the stresses upon BB, normal and tangential respectively.

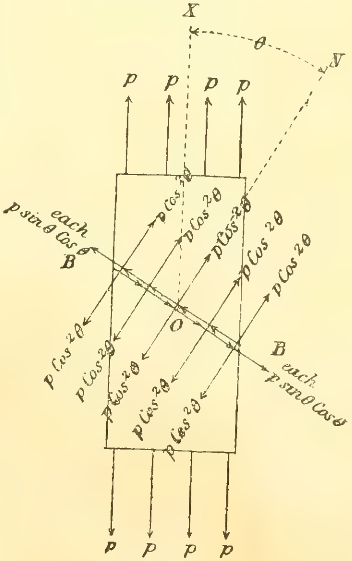


Fig. 11.

Now $\frac{p_n}{p \cos \theta} = \cos \theta$ def. $\therefore \quad p_n = p \cos^2 \theta$; (fig. 11)

also $\frac{p_t}{p \cos \theta} = \sin \theta. \quad \therefore \quad p_t = p \sin \theta \cos \theta.$

From the superposition of forces these two sets of forces may be considered independently of each other. For some cases in designing it might only be necessary to consider one set, if it were manifest that in providing for it there would be more than sufficient provision made for the other.

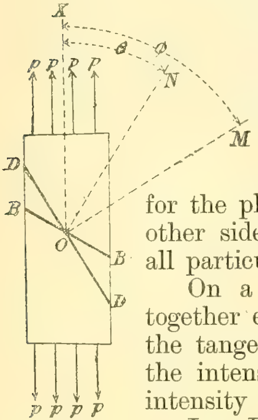


Fig. 12.

It is apparent from symmetry that for the plane *CC* inclined at the angle θ on the other side of the axis the stress is the same in all particulars as that on *BB*.

On a pair of planes whose obliquities are together equal to a right angle, the intensities of the tangential stresses are equal, and the sum of the intensities of the normal stresses equals the intensity of the initial stress.

Let *BB* (fig. 12) be inclined at the angle θ, and *DD* at the angle ϕ, where

$$\theta + \phi = \frac{\pi}{2}.$$

On *BB* $\quad p_n = p \cos^2 \theta, \quad p_t = p \sin \theta \cos \theta.$

On *DD* $\quad p'_n = p \cos^2 \phi, \quad p_t = p \sin \phi \cos \phi.$

But $\quad \sin \theta = \cos \phi,$ and $\cos \theta = \sin \phi$;

therefore $\quad p_t = p'_t,$ or the tangential component stresses have the same intensity on both planes.

Also $\quad p_n + p'_n = p (\cos^2 \theta + \cos^2 \phi)$

$$= p (\cos^2 \theta + \sin^2 \theta) = p;$$

or the sum of the intensities of the normal component stresses equals the intensity of the primary stress.

There are therefore at one point O (fig. 13) four planes BB, DD, CC, and EE, two inclined on each side of OX, upon which the tangential stress has the same intensity.

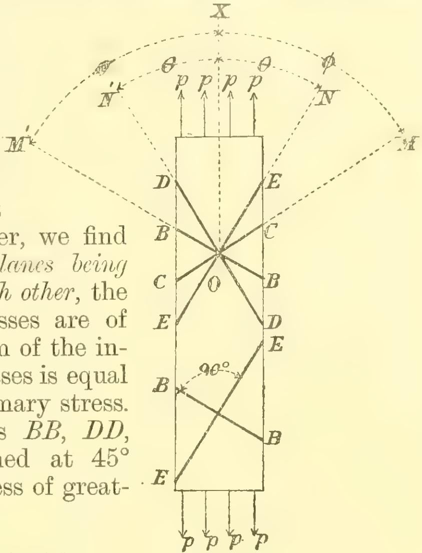


Fig. 13.

Grouping together the pair of planes BB and EE, the one inclined at θ on the one side of OX, and the other at ϕ upon the opposite side, and therefore at $\theta + \phi$, or 90° to each other, we find that *at any point two planes being chosen at right angles to each other*, the *tangential* or *shearing* stresses are of equal intensity, and the sum of the intensities of the *normal* stresses is equal to the intensity of the primary stress.

For all planes such as BB, DD, &c., that which is inclined at 45° sustains the tangential stress of greatest intensity, for

$$p_t = p \sin\theta \cos\theta = \frac{p}{2} \sin 2\theta.$$

Therefore p_t is greatest when $\sin 2\theta$ is greatest,

when $\sin 2\theta = 1$, when $2\theta = 90°$, or $\theta = 45°$.

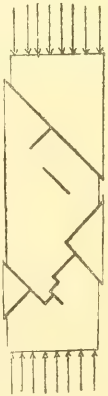


Fig. 14.

The tangential stress on a plane such as BB is called a *shearing stress.* Many substances fracture under a shearing stress very readily. Notably cast iron under a strain of compression fractures by shearing along an oblique plane, the one portion sliding upon the other (fig. 14). The resistance then which cast iron offers to shearing is that which must be considered in designing short pillars to bear great loads. The planes upon which the intensity of the shearing stress is greatest, that is, planes inclined at 45° to the direct thrust, are those upon which it will shear. As the texture of the material is never homogeneous it may shear along planes more or less inclined than 45°, also the toughness of the skin will cause great irregularity.

Brick stalks give way by the mortar shearing, and the upper portion sliding down an oblique section like a splice.

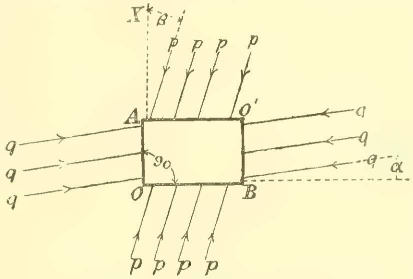
INTERNAL STRESS AT A POINT IN A SOLID IN A COMPOUND STATE OF STRAIN.

A solid is in *a compound state of strain* when subjected to two or more simple stresses in different directions simultaneously. We proceed to consider a solid in such a state of strain without inquiring how it was brought into that state ; all its parts being supposed to be at rest, and all the parts into which it may be divided in equilibrium under the stresses exerted among each other, due to their elasticity, and those exerted at the external surface: but at the outset we do not regard those external stresses.

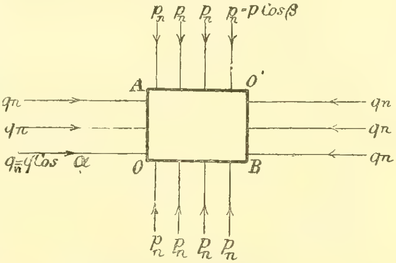
Upon any plane passing through a point within the solid, the stress at that point is definite in intensity and direction ; for if along that plane the solid were divided into two parts, the mutual pressures between the cut surfaces at that point (no matter how complicated) can be compounded into one force, definite in amount and direction. Along this plane the intensity and direction of the stress varies, and at the point will only be constant over a very small part of the surface round it. If the stress be stated in lbs. per square inch, the total stress on this small surface which we are considering would be a mere fraction of the intensity. It will be convenient to consider the intensities of these stresses to be expressed, say, in lbs. per millionth part of a square in., so that in the diagrams *two or three* arrows (*each* representing the *intensity*) may be drawn to represent the total amount of stress upon such small planes, without leading us to the supposition that they are of a few square inches in extent. And yet whatever results we arrive at are equally true for intensities expressed in the usual units, for the intensity at a point on a plane, upon which the intensity varies, can be expressed to any degree of accuracy in lbs. per square inch. Thus, at the point, the intensity of the stress in lbs. per square inch equals roughly, nearly, more nearly, &c.

Amount of stress on the square inch surrounding point, roughly.
10 times amount of stress on the $\frac{1}{10}$th of a square inch surrounding point.
100 times amount of stress on the $\frac{1}{100}$th of a square inch surrounding point.
1,000,000 times amount of stress on the $\frac{1}{1000000}$th of a square inch surrounding point, &c., &c.

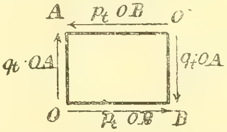
Let $OAO'B$ (fig. 15) be a small rectangular parallelepiped at the point O in a solid in a state of strain.



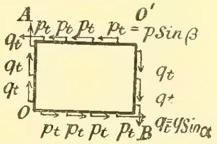
intensities and obliquities at O of stresses upon OA and OB



Normal components



Amounts of tangential stress



Intensities of tangential components

Fig. 15.

Let q = intensity of stress on the faces OA and $O'B$ at an obliquity α.

Let p = intensity of stress on faces OB and $O'A$ at an obliquity β.

The normal components are—

$$p_n = p \cos \beta, \quad q_n = q \cos \alpha.$$

The two sets of forces p_n directly balance each other, and may be removed, and also the two sets q_n, leaving the parallelepiped in equilibrium under the action of the tangential components.

$$p_t = p \sin \beta, \quad q_t = q \sin \alpha.$$

The amount of tangential stress on each of the faces OA and $O'B$ is $q_t . OA$. Also the amount on each of the faces OB and $O'A$ is $p_t . OB$.

The two forces $q_t . OA$ form a couple with a leverage OB tending to turn the parallelepiped in the direction in which the hands of a watch turn, while the two forces $p_t . OB$ form a couple with a leverage OA tending to turn it in the opposite direction. Since the parallelepiped is in equilibrium under these two actions alone, the moments of these two couples must be equal.

$$\begin{array}{ccccccc} \text{Force.} & & \text{Leverage.} & & \text{Force.} & & \text{Leverage.} \\ q_t . OA & \times & OB & = & p_t . OB & \times & OA. \end{array}$$

Now the area OA multiplied by the length OB gives the volume of the parallelepiped, and the area OB multiplied by the length OA also gives the volume.

$$\therefore \quad q_t \,.\, V = p_t \,.\, V, \qquad \therefore \quad q_t = p_t.$$

Hence, at a point within a solid in a state of strain, the tangential components of the stresses upon any two planes through it at right angles to each other are of an equal intensity.

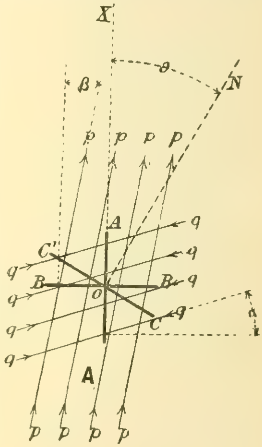


Fig. 16.

Cor.—If it be possible that the stress upon any plane through a point be wholly normal, then will the stress be wholly normal upon another plane at right angles to that plane.

At a point within a solid in a state of strain the directions and intensities of the stresses upon two planes at right angles to each other being given, to find the direction and intensity of the stress upon any third plane (fig. 16).

Let q, the intensity of the stress at O on the plane AA', be inclined to the normal at the angle α.

Let p, the intensity of the stress at O on the plane BB be inclined to the normal at the angle β.

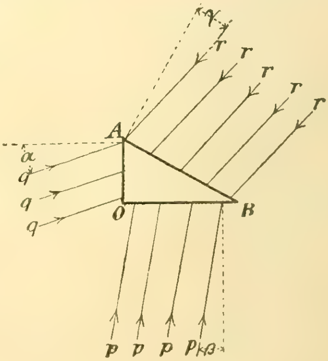


Fig. 17.

These two planes being inclined to each other at a right angle, it is required to find the intensity and direction of the stress on a third plane CC' through O inclined at any angle θ to AA.

Consider a small triangular prism OAB at O (fig. 17) bounded by portions of these three planes. It is in equilibrium under the stresses q on the face OA, p on the face OB, and the required stress r on the face AB.

It is to be borne in mind that P the total stress (fig. 17) taken as acting at the middle of OB, Q on OA, and R on AB, intersect at a point; and in constructing the parallelogram of forces, later on, this point of intersection is taken as if it were at O.

Choosing OX and OY along OA and OB as rectangular axes, and resolving p and q into components normal and tangential to the planes they act upon, we have

$p_n = p \cos \beta$, normal to OB, $\quad q_n = q \cos \alpha$, normal to OA,

and

$p_t = p \sin \beta$, tangential to OB, $\quad q_t = q \sin \alpha$, tangential to OA.

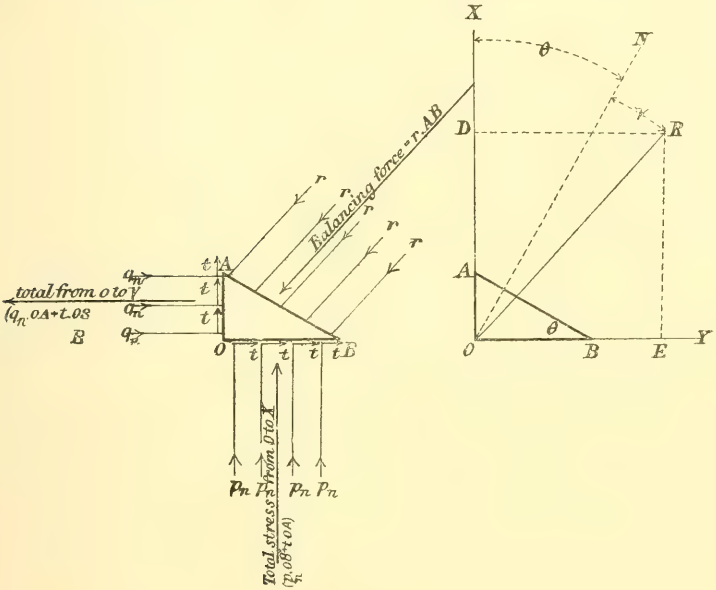


Fig. 18.

Since the intensities of the tangential component stresses upon the two planes OA and OB must be equal, one symbol t is put for both upon the diagram (fig. 18)

$$t = p_t = q_t = p \sin \beta = q \sin \alpha.$$

Note that $OA = AB \sin \theta$, and $OB = AB \cos \theta$.

Lay off $\quad OD$ = force parallel to the axis OX

= amount of normal stress on OB + amount of tangential stress on OA

$= p_n \,.\, OB + t \,.\, OA = p_n \,.\, AB \cos \theta + t \,.\, AB \sin \theta$

$= AB\,(p_n \cos \theta + t \sin \theta)$.

Lay off OE = force parallel to OY

= amount of normal stress on OA + amount of tangential stress on OB

$= q_n . OA + t . OB = q_n . AB \sin\theta + t . AB \cos\theta$

$= AB(q_n \sin\theta + t\cos\theta)$.

And completing the parallelogram, we have for equilibrium

RO = amount of stress on AB in direction and magnitude,

since
$$RO^2 = OD^2 + OE^2$$
$$= AB^2\{(p_n \cos\theta + t\sin\theta)^2 + (q_n \sin\theta + t\cos\theta)^2\}$$
$$= AB^2\{p_n^2 \cos^2\theta + 2p_n t \sin\theta\cos\theta + t^2\sin^2\theta$$
$$+ q_n^2 \sin^2\theta + 2q_n t\sin\theta\cos\theta + t^2\cos^2\theta\}.$$

Adding 1st and 4th terms, 2nd and 5th, 3rd and 6th within the brackets, and noticing in adding last pair that $\sin^2\theta + \cos^2\theta = 1$, we have

$$RO^2 = AB^2\{p_n^2\cos^2\theta + q_n^2\sin^2\theta + 2t(p_n + q_n)\sin\theta\cos\theta + t^2\}.$$

Now the intensity of the stress upon AB equals the total stress upon it divided by the area of the plane AB.

$$\therefore \quad r = \frac{RO}{AB} = \sqrt{\{p_n^2\cos^2\theta + q_n^2\sin^2\theta + 2t(p_n+q_n)\sin\theta\cos\theta + t^2\}};$$

also $$\tan XOR = \frac{RD}{DO} = \frac{OE}{OD} = \frac{q_n\sin\theta + t\cos\theta}{p_n\cos\theta + t\sin\theta} \text{ gives } XOR.$$

Hence the obliquity of the stress r to the plane AB is

$$\gamma = NOR = XOR - \theta.$$

Thus we have found r and γ, the intensity and direction of the stress upon CC' in terms of p, q, α, and β. Hence, for a body in a possible state of strain, if at any point the stresses be given upon a pair of rectangular planes through it, the *internal stress* at that point is known. For the same point the internal stress might be known from having the stresses on different such pairs of rectangular planes. These are called *equivalent sets* of stresses. By a more elaborate process we might show that the internal stress at a point is known if the stresses on any pair of planes, not coincident, be given.

For some position of the plane CC' the stress OR will coincide with the normal ON, and we shall have

$$\theta = XON = XOR.$$

$$\therefore \quad \tan\theta = \frac{q_n \sin\theta + t\cos\theta}{p_n\cos\theta + t\sin\theta}, \quad \text{or} \quad \frac{\sin\theta}{\cos\theta} = \frac{q_n\sin\theta + t\cos\theta}{p_n\cos\theta + t\sin\theta};$$

and clearing of fractions

$$p_n \sin\theta\cos\theta + t\sin^2\theta = q_n\sin\theta\cos\theta + t\cos^2\theta.$$

$$\therefore \quad (p_n - q_n)\sin\theta\cos\theta = t(\cos^2\theta - \sin^2\theta).$$

$$\therefore \quad \frac{p_n - q_n}{2}\, 2\sin\theta\cos\theta = t(\cos^2\theta - \sin^2\theta).$$

$$\therefore \quad \frac{p_n - q_n}{2}\sin 2\theta = t\cos 2\theta.$$

$$\therefore \quad \frac{\sin 2\theta}{\cos 2\theta} = t\,.\,\frac{2}{p_n - q_n}, \quad \text{or} \quad \tan 2\theta = \frac{2t}{p_n - q_n}.$$

Two values of θ differing by a right angle will satisfy this equation; that is, there are two planes CC' and FF' inclined at an acute angle θ, and an angle $90 + \theta$ respectively to AA', and therefore at right angles to each other, upon which the stresses are wholly normal, the value of θ being such that the tangent of twice θ equals the ratio of sum of the common intensity of the tangential stresses upon AA' and BB' to the difference of the intensities of the normal stresses thereon (fig. 20).

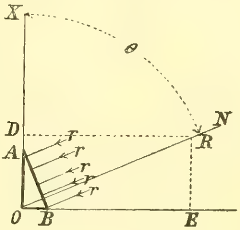


Fig. 19.

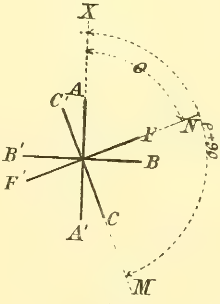


Fig. 20.

Now if we know the internal stress at O from having the stresses upon the pair of rectangular planes AA' and BB' given, we may calculate the stresses upon the pair of rectangular planes CC' and FF', and so express the internal stress at O by means of this equivalent set. Of all the equivalent sets at O this is the simplest by means of which to express the internal stress, as the intensities only of the stresses upon CC' and FF' require to be specified, being normal, and the position specified of one only of the planes since they are rectangular.

Let AA' and BB' be the pair of rectangular planes through O upon which the stresses are wholly normal, they are called the *planes of principal stress*, the stresses themselves the *principal stresses* at the point O, and the axes OX and OY the *axes of principal stress* at that point (fig. 21).

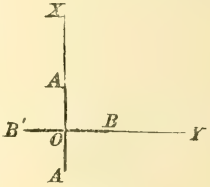


Fig. 21.

If the stress be oblique upon a plane through a point within a solid in a state of strain, and another plane be drawn through the point parallel to the stress thereon, then will the stress upon the second plane be parallel to the first plane.

At the point O (fig. 22), let p be the stress on the plane BB'. Draw AA' parallel to p; then will q be parallel to BB'.

Consider the equilibrium of the parallelepiped $OAO'B$ at O.

P and P' (the amount of the stress on the two faces OB and AO') are equal and in one straight line, being drawn parallel to OA through E and F, the middle points of those faces. Therefore they are in equilibrium, and may be removed, leaving Q and Q' in equilibrium themselves. Hence Q and Q' are in one straight line, and as this straight line passes through C and D, the middle points of the faces OA and BO', it is parallel to OB; that is, q is parallel to BB'.

These are called *a pair of conjugate stresses*. Being in equilibrium independently of each other, these stresses can be considered separately.

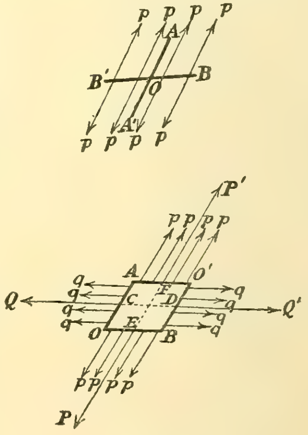


Fig. 22.

Cor.—The principal stresses are also conjugate.

In the following examples the stresses are positive.

Examples.

50. A short pillar 2 square feet in area bears a load of 36 tons: find the intensity of the stress upon a plane section of it inclined at 20° to the axis, also the intensities of the normal and tangential component stresses on it.

$$p = \frac{80640}{288} = 280 \text{ lbs. per square inch on oblique plane.}$$

$$\text{intensity} = p \cos\theta = 263{\cdot}1 \text{ lbs. per sq. inch,}$$
$$p_n = p\cos^2\theta = 247{\cdot}3 \quad ,, \quad ,,$$
$$p_t = p\sin\theta\cos\theta = 90 \quad ,, \quad ,,$$

51. Find greatest intensity of shearing force on a plane section of pillar in last. It will be upon a plane inclined at 45°.

$$p_t = p \sin 45^\circ \cos 45^\circ = p \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{p}{2} = 140 \text{ lbs. per sq. inch.}$$

52. A stone obelisk weighing 200 tons, and covering an area of 10 square feet, stands above a slate formation inclined to the horizontal 10°: find the intensity of the stress normal and tangential to the beds.

$$p_n = 301{\cdot}8 \text{ lbs. per sq. inch,} \qquad p_t = 53{\cdot}2 \text{ lbs. per sq. inch.}$$

53. At a point within a solid the stress on a plane BB' through it is 90 lbs. per square inch, and inclined to the normal at an angle of 20°, while the normal component of the stress on another plane through it (AA' at right angles to BB) is 60 lbs. per square inch. Find the total stress upon this other plane (fig. 23).

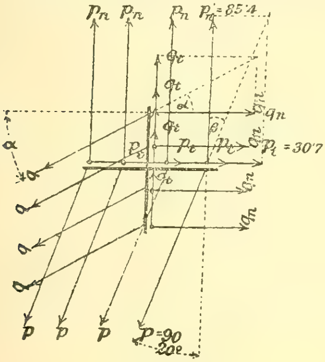


Fig. 23.

Given stress on BB'

$$p = 90 \quad \text{and} \quad \beta = 20^\circ,$$

normal component

$$p_n = 90 \,.\, \cos 20^\circ = 84{\cdot}41,$$

tangent component

$$p_t = 90 \,.\, \sin 20^\circ = 30{\cdot}78.$$

Let q be the required stress on AA' and α its obliquity,

its normal component $\quad q_n = 60, \quad$ and $\quad q_t = p_t = 30{\cdot}78.$

Hence $\quad q^2 = q_n^2 + q_t^2 = 4547{\cdot}4, \qquad \therefore \quad q = 67{\cdot}4$ lbs. per sq. inch,

also $\quad \cos \alpha = \dfrac{q_n}{q} = {\cdot}89, \qquad \therefore \quad \alpha = 27^\circ 7.$

The total stress on AA' is 67·4 lbs. per square inch, and is inclined at 27° 7′ to the normal.

54. AA' and BB' are a pair of rectangular planes through a point within a solid, the stress on BB' is 3280 lbs. per square inch, and its obliquity is 10°. The normal component stress on AA' being 2000 lbs. per square inch, find the intensity and obliquity of the total stress upon it.

$$q = 2080 \text{ lbs. per sq. inch.} \qquad \alpha = 15^\circ 57'.$$

55. In last find the stress on CC' a third plane through the point inclined at an angle of 30° to BB'. See diagram to general proposition, page 33, which is drawn to scale for this example (fig. 18).

$$p_n = 3280 \cos 10^\circ, \quad q_n = 2000, \quad t = 3280 \sin 10^\circ \quad \text{or} \quad 2080 \sin 16^\circ$$
$$= 3230{\cdot}1 \qquad\qquad\qquad\qquad = 569{\cdot}5.$$

Lay off along OX the total amount of stress parallel to its direction.

$$\begin{aligned} OD &= \text{amt. of normal stress on } OB + \text{amt. of tangent stress on } OA \\ &= p_n \,.\, OB + t \,.\, OA = p_n \,.\, AB \cos\theta + t \,.\, AB \sin\theta \\ &= AB\,(p_n \cos\theta + t \sin\theta) = AB\,(3230{\cdot}1 \times \cos 30^\circ + 569{\cdot}5 \sin 30^\circ) \\ &= AB\,(2797{\cdot}4 + 284{\cdot}8) = 3082{\cdot}2\ AB. \end{aligned}$$

Lay off

$$\begin{aligned} OE &= \text{total stress parallel to axis } OY \\ &= \text{amt. of normal stress on } OA + \text{amt. of tangent stress on } OB \\ &= q_n \,.\, OA + t \,.\, OB = q_n \,.\, AB \sin\theta + t \,.\, AB \cos\theta \\ &= AB\,(q_n \sin\theta + t \cos\theta) = AB\,(2000 \sin 30^\circ + 569{\cdot}5 \cos 30^\circ) \\ &= AB\,(1000 + 404{\cdot}6) = 1404{\cdot}6\ AB\,; \end{aligned}$$

now

$$OR^2 = OD^2 + OE^2 = 11472700\ AB^2.$$

$$\therefore\ OR = 3387\ AB \text{ total stress on } AB,$$

and

$$r = \frac{\text{total stress on } AB}{\text{area of } AB} = \frac{OR}{AB} = 3387 \text{ lbs. per sq. inch.}$$

Also

$$\tan XOR = \frac{OE}{OD} = \frac{1404{\cdot}6}{3082{\cdot}2} = {\cdot}4557.$$

$$\therefore\ XOR = 24^\circ\,30' \quad\text{and}\quad \gamma = XOR - 30 = -5^\circ\,30'.$$

The intensity of the stress on CC' is 3387 lbs. per square inch, and is inclined at an obliquity of 5° 30′.

56. In example 53, find the intensity and obliquity of the stress on a third plane through the point inclined to BB' at 5°.

$$r = 93{\cdot}8 \text{ lbs. per sq. inch.} \quad \gamma = 17^\circ\,28'.$$

57. The internal stress at a point within a solid in a state of strain being given 240 lbs. per square inch, of obliquity 8° on one plane BB' through it, and 193 lbs. per square inch, of obliquity 10°, on the rectangular plane AA'. Find the position of the planes of principal stress.

$$t = \begin{cases} p_t = 240 \sin 8^\circ = 33{\cdot}4, \\ q_t = 193 \sin 10^\circ = 33{\cdot}4. \end{cases}$$

These must be equal or the stresses as given are impossible.

$$p_n = 240 \cos 8^\circ = 237{\cdot}6, \qquad q_n = 193 \cos 10^\circ = 190.$$

Let θ be the inclination of the planes of principal stress to the plane BB'.

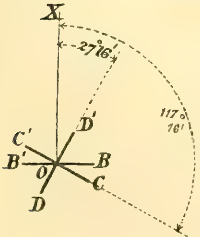


Fig. 24.

$$\tan 2\theta = \frac{2t}{p_n - q_n} = 1{\cdot}4034.$$

$$\therefore\ 2\theta = 54^\circ\,32' \quad\text{or}\quad 234^\circ\,32',$$

$$\therefore\ \theta = 27^\circ\,16' \quad\text{or}\quad 117^\circ\,16'.$$

If BB' be the plane upon which the given stress is 240, and OX the normal to it, then CC' and DD' are the planes of principal stress whose normals ON and OM make 27° 16′ and 117° 16′ with OX respectively.

58. Find the principal stress in example 57; that is, find the stress upon a third plane CC', inclined at 27° 16′ to BB'.

$$OD = p_n \, . \, OB + t \, . \, OA = AB \, (p_n \cos\theta + t \sin\theta)$$

$$= AB \, (237{\cdot}6 \cos 27^\circ 16' + 33{\cdot}4 \sin 27^\circ 16') = 226{\cdot}5 \, AB.$$

$$OE = q_n \, . \, OA + t \, . \, OB = AB \, (q_n \sin\theta + t \cos\theta) = 116{\cdot}7 \, AB.$$

$$OR^2 = OD^2 + OE^2 = 64910 \, AB^2. \quad OR = 254{\cdot}8 \, AR.$$

$$r = \frac{OR}{AB} = 254{\cdot}8 \text{ lbs. per sq. in.,}$$

and is of course normal.

In finding the other principal stress—that is, on third plane DD', inclined at 117° 16′ to BB', we may use the functions of this angle and proceed as above, observing signs; but it is better to take the functions of

$$DOK = 62^\circ 44',$$

when

$$OA = AB \sin 62^\circ 44',$$

and

$$OE = AB \cos 62^\circ 44'.$$

$$OD = p_n \, . \, OB - t \, . \, OA, \text{ since } t \text{ acts against } p_n$$

$$= AB \, (p_n \cos 62^\circ 44' - t \sin 62^\circ 44')$$

$$= AB \, (108{\cdot}86 - 29{\cdot}69) = 79{\cdot}17 \, AB.$$

$$OE = q_n \, . \, OA - t \, . \, OB, \text{ since } t \text{ acts against } q_n$$

$$= AB \, (q_n \sin 62^\circ 44' - t \cos 62^\circ 44')$$

$$= AB \, (168{\cdot}89 - 15{\cdot}3) = 153{\cdot}6 \, AB.$$

$$SO^2 = OD^2 + OE^2 = 29858 \, AB^2.$$

$$SO = 172{\cdot}8 \, AB.$$

$$s = \frac{SO}{AB} = 172{\cdot}8.$$

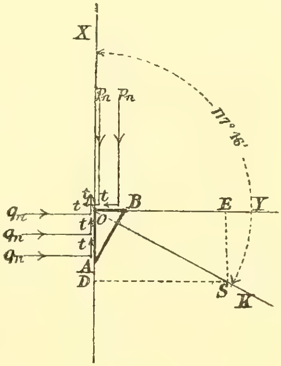


Fig. 25.

The principal stresses are 254·8 and 172·8 lbs. per square inch.

59. In Ex. 57 find the stress on a plane GG inclined at 30° 39′ to BB'. Directly from data as given in 57,

$$OD = p_n \, . \, OB + t \, . \, OA = AB \, (p_n \cos\theta + t \sin\theta)$$

$$= AB \, (237{\cdot}6 \cos 30^\circ 39' + 33{\cdot}4 \sin 30^\circ 39') = 221 \, AB.$$

$$OE = q_n \, . \, OA + t \, . \, OB = AB \, (q_n \sin\theta + t \cos\theta)$$

$$= AB \, (190 \sin 30^\circ 39' + 33{\cdot}4 \cos 30^\circ 39') = 125{\cdot}59 \, AB.$$

$$OR^2 = OD^2 + OE^2 = 64792 \, AB^2. \quad OR = 254{\cdot}5 \, AB.$$

$$r = \frac{OR}{AB} = 254{\cdot}5 \text{ lbs. per square inch.}$$

Also $\tan XOR = \dfrac{OE}{OD} = {\cdot}5673. \quad \therefore \; XOR = 29^\circ 34'$

$$\gamma = \theta - XOR = 30^\circ 39' - 29^\circ 34' = 1^\circ 5', \text{ obliquity of } r \text{ upon } GG'.$$

To find the stress upon GG by finding the principal stresses, first as in 58, and then finding the stress upon GG from these.

The plane of greatest principal stress CC' is inclined to BB', the given plane, at 27° 16′; hence GG' will be inclined to CC' at 3° 23′. Hence we have the principal stresses 254·8 and 172·8 to find the stress upon GG' inclined at 3° 23′.

Since there are no tangential stresses on CC' and DD' we have

$$OD = p \,.\, OB = p \,.\, AB \cos\theta = AB \,.\, 254{\cdot}8 \cos 3^\circ 23' = 254{\cdot}3 \,.\, AB.$$

$$OE = q \,.\, OA = q \,.\, AB \sin\theta = AB\; 172{\cdot}8 \sin 3^\circ 23' = 10{\cdot}2 \,.\, AB.$$

$$OR^2 = OD^2 + OE^2 = 64764\; AB^2. \quad OR = 254{\cdot}5\; AB.$$

$$r = \frac{OR}{AB} = 254{\cdot}5 \text{ lbs. per square inch.}$$

$$\tan XOR = \frac{OE}{OD} = {\cdot}04011. \quad XOR = 2^\circ 18'.$$

$$\gamma = \theta - XOR = 3^\circ 23' - 2^\circ 18' = 1^\circ 5', \text{ obliquity of } r \text{ on } GG'.$$

60. At a point within a solid in a state of strain, the stresses upon a pair of rectangular planes through it are given—on BB' the intensity of the normal component stress is 40, on AA' the intensity of the normal component stress is 30, and the tangential component stresses are each of intensity 10. Find the obliquity to BB' of the planes of principal stress, and find the principal stresses.

$$\tan 2\theta = 2. \quad \therefore \quad \theta = 31^\circ 43' \text{ and } 121^\circ 43';$$

for $$\theta = 31^\circ 43' \left\{\begin{array}{l} OD = 39{\cdot}281\; AB \\ OE = 24{\cdot}277\; AB \end{array}\right\} \quad \therefore \quad r = 46{\cdot}2,$$

for $$\theta = 121^\circ 43' \left\{\begin{array}{l} OD = 12{\cdot}522\; AB \\ OE = 20{\cdot}26\; AB \end{array}\right\} \quad \therefore \quad s = 23{\cdot}8.$$

61. In 60 find the stress on a plane inclined at 46° 43′ to BB'. Deducting 31° 43′, we find the plane to be inclined at 15° to the plane of greatest principal stress.

Using results of 60, and t being then zero, $r = 45$ and $\gamma = 7^\circ 8'$.

CHAPTER III.

RANKINE'S METHOD OF THE ELLIPSE OF STRESS.

The preceding method of finding the stress upon any particular plane through a point at which the state of strain is known, is too tedious to be readily remembered or applied, and becomes intricate when the stresses are some thrusts and others tensions. We now proceed to a general method. Having already proved that there is a pair of principal stresses at a point, we proceed, upon the supposition that these are given, to find the stress on a third plane through the point.

Direct Problem, the Principal Stresses given.

Equal-like principal stresses (fig. 26).—If the pair of principal stresses at a point be like (both thrusts or both tensions), and be of equal intensity, the stress on any third plane through the point is of that same intensity, and is normal to the plane.

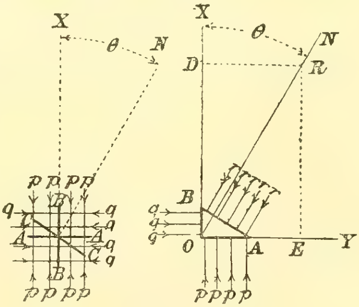


Fig. 26.

Let AA' and BB' (fig. 26) be the planes of principal stress at the point O, and let the intensities of the principal stresses, p and q, be equal and alike (both thrusts).

CC' is any third plane through O, inclined at θ to AA'.

OAB is a small triangular prism at O, having its faces in those planes. This prism is in equilibrium under the three forces—the total thrusts upon OA, OB, and AB.

The three total pressures P, Q, R on the faces, if represented by straight lines, intersect at the middle point of AB. The parallelogram of forces is however drawn from the point O.

Lay off OD = total stress parallel to $OX = p \, . \, OA$,

and OE = total stress parallel to $OY = q \, . \, OB$.

Complete the parallelogram.

Then RO represents the total stress on AB in direction and amount.

$$\tan ROD = \frac{OE}{OD} = \frac{q\,.\,OB}{p\,.\,OA} = \frac{OB}{OA}, \quad \text{since} \quad p = q.$$

$$\therefore \quad ROD = OAB = \theta\,; \qquad \therefore \quad OR \text{ is upon } ON.$$

Hence RO is normal to AB.

$$\text{And} \qquad RO^2 = OD^2 + OE^2 = p^2\,.\,OA^2 + q^2\,.\,OB^2$$

$$= p^2\,(OA^2 + OB^2) = p^2\,.\,AB^2, \text{ as } p = q.$$

$$\therefore \quad RO = p\,.\,AB.$$

$$\text{Now} \qquad r = \frac{\text{amount of stress on } AB}{\text{area of } AB} = \frac{RO}{AB} = p \text{ or } q.$$

Cor.—Every plane through O is a plane of principal stress Each point in a fluid is in this state of strain.

Equal-unlike principal stresses (fig. 27).—If the pair of principal stresses at a point be unlike (one a thrust and the other a tension), and be of equal intensity, the stress on any third plane through the point is of that same intensity, and is inclined at an angle to the normal to the plane of principal stress, equal to that which the normal to this third plane makes therewith, but upon the *opposite side.*

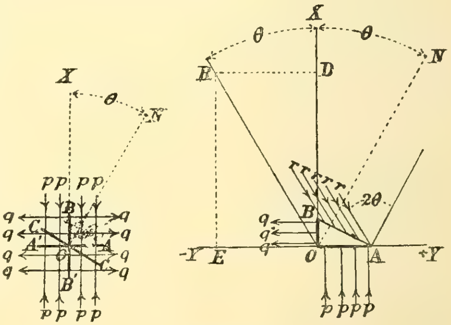


Fig. 27.

Let AA' and BB' be the planes of principal stress at the point O; p and q the intensities of the principal stresses of equal value, p being a thrust and q a tension; and CC' any third plane through O inclined at θ to AA'.

OAB, a small triangular prism at O bounded by these three planes, is in equilibrium under the three forces, viz., the amounts of stress on its faces OA, OB, and AB.

Lay off OD = total stress parallel to $OX = p\,.\,OA$,

and OE = total stress parallel to OY in the direction of q $= q\,.\,OB.$

Complete the parallelogram *ODRE*. Then *RO* represents the total stress on *AB* in direction and amount.

$$\therefore \quad \tan ROD = \frac{OE}{OD} = \frac{q\,.\,OB}{p\,.\,OA} = \frac{OB}{OA} = \tan OAB;$$

$$\therefore \qquad ROD = OAB = \theta.$$

That is, *RO* is inclined at the same angle to the axis *OX* as *ON* is, but on the opposite side. Hence the inclination of *RO* to the normal *ON* is 2θ.

Again
$$RO^2 = OD^2 + OE^2 = p^2\,.\,OA^2 + q^2\,.\,OB^2$$
$$= p^2\,(OA^2 + OB^2) = p^2\,.\,AB^2;$$

$$\therefore \quad RO = p\,.\,AB, \quad r = \frac{\text{amount of stress on } AB}{\text{area of } AB} = \frac{RO}{AB} = p \text{ or } q.$$

Consider the triangle of forces *OER*, we have *OE* drawn from *O* in the direction of q, then *ER* drawn from *E* in the direction of p; hence *RO*, taken in the same order, is the direction of r.

If θ be greater than 45°, r is like q.

If θ equal 45°, r is entirely tangential to *AB*.

If θ be less than 45°, r is like p.

Hence, if the principal stresses at a point be equal and unlike the stress on a third plane, is of that same intensity, is like the stress on the plane it is least inclined to, and its direction is inclined to the axis at the same angle as the normal is, but upon the opposite side. If the new plane be inclined at 45°, the stress is entirely tangential.

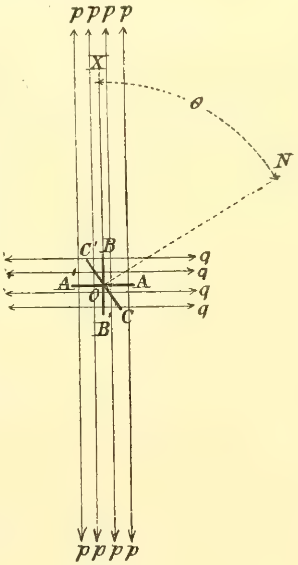


Fig. 28.

Unequal principal stresses (fig. 28).—The principal stresses at a point within a solid in a state of strain being given, to find the intensity and obliquity of the stress at that point on a third plane through it.

AA′ and *BB′* are the planes of principal stress at *O*; p and q are the principal stresses. Let p be the greater, and let them be both positive, say both tensions. It is required to find r, the intensity of the stress upon *CC′*, and γ, the angle it makes with *ON*, the normal to *CC′*. θ is the inclination of *CC′* to *AA′*, the plane of greatest principal stress.

Of two unequal quantities the greater is equal to the sum of their *half sum* and their *half difference*, while the lesser equals their difference.

Therefore $$p = \frac{p+q}{2} + \frac{p-q}{2}, \text{ an identity,}$$

and $$q = \frac{p+q}{2} - \frac{p-q}{2}, \text{ an identity.}$$

We may look upon the plane *AA′* (fig. 29) as bearing two separate tensions of intensities $\frac{p+q}{2}$ and $\frac{p-q}{2}$ in lieu of the tension of intensity p; and on the plane *BB′* as bearing a tension of intensity $\frac{p+q}{2}$ and a thrust of intensity $\frac{p-q}{2}$ in lieu of the tension of intensity q. We may now group these together in pairs, thus: the tension on *AA′* of intensity $\frac{p+q}{2}$ along with the tension on *BB′* of intensity $\frac{p+q}{2}$, and the tension on *AA′* of intensity $\frac{p-q}{2}$ along with the thrust

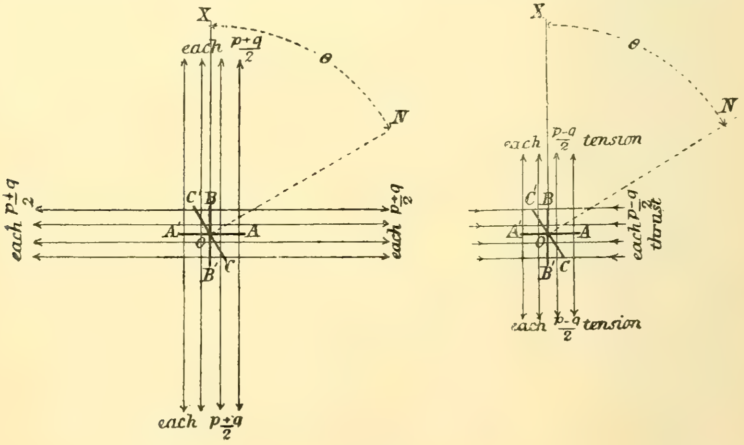


Fig. 29.

on *BB′* of intensity $\frac{p-q}{2}$. Then find separately for each pair the stress upon *CC′*, and finally compound these two stresses on *CC′* by means of the triangle of forces. The first pair is a pair of *equal-like* principal stresses $\left(\text{both tensions of intensity } \frac{p+q}{2}\right)$. So the consequent stress on *CC′* will be a tension of intensity $\frac{p+q}{2}$, and normal to *CC′* (fig. 30). The second pair is a pair of *equal-unlike* principal stresses of intensity $\frac{p-q}{2}$ (a tension on *AA′* and a thrust on *BB′*), so the consequent stress on *CC′* will be of intensity $\frac{p-q}{2}$ and inclined at an angle θ upon the side of *OX* opposite from that upon which *ON* lies.

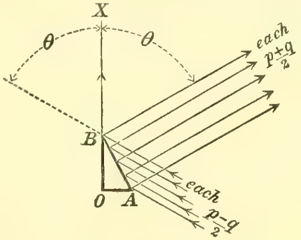


Fig. 30.

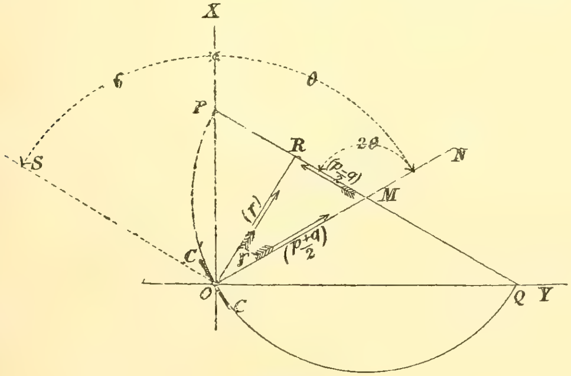


Fig. 31.

The diagram shows these partial resultant stresses on *AB*, a very small part of *CC′*, at *O*. To find the total resultant stress upon *CC′*, it remains to compound these by the triangle of forces. From *O* (fig. 31) lay off $OM = \frac{p+q}{2}$ = the intensity of the first partial stress and in the direction thereof, *i.e.*, along *ON*. From *M* draw $MR = \frac{p-q}{2}$ = the intensity of the second partial stress and in the direction thereof, *i.e.* parallel to *OS*, which direction is most conveniently found by describing from M as centre with radius MO a semicircle QOP, and joining QMP.

Then will OR, the third side of the triangle OMR, taken in the *opposite* order (see arrows) be the direction and intensity of the resultant stress r on CC'.

The preceding construction, as shown on last figure, is geometrically all that is required, p and q being given to find r; the text and figures given before being the development and proof.

From the construction note that

$$MP = MQ = OM = \frac{p+q}{2};$$

also $$QR = MQ + MR = \frac{p+q}{2} + \frac{p-q}{2} = p,$$

and $$PR = MP - MR = \frac{p+q}{2} - \frac{p-q}{2} = q;$$

$$RMN = 2\theta;\ ROM = \gamma, \text{ the obliquity of } r.$$

Normal and tangential components of r, the stress on the third plane CC'.

Drop RT perpendicular to ON. The tangential component of r is

$$\begin{aligned} r_t &= RT \\ &= MR \sin RMT \\ &= \frac{p-q}{2} \sin 2\theta \\ &= (p-q) \sin\theta \cos\theta, \end{aligned}$$

since $\sin 2\theta = 2 \sin\theta \cos\theta$.

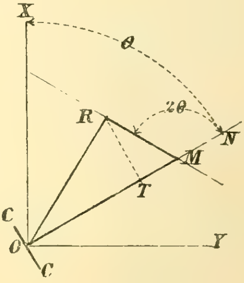


Fig. 32.

Cor.—If DD' be the plane at right angles to CC', its inclination to the axis OX being $\theta' = (\theta + 90°)$, the sine of which equals $\cos\theta$, and the cosine of which equals $-\sin\theta$; the value of r_t for DD' will be the same as above, that is, the tangential components of the stresses on any pair of rectangular planes is the same. This we arrived at by the general method at page 32, which compare. See also fig. 13.

The normal component of r is

$$r_n = OT = OM - MT$$

$$= OM - MR \cos RMT = OM + MR \cos RMN$$

(These two angles have the same cosine, but of opposite sign.)

$$= OM + MR \cos 2\theta$$

$$= \frac{p+q}{2} (\cos^2 \theta + \sin^2 \theta) + \frac{p-q}{2} (\cos^2 \theta - \sin^2 \theta)$$

$$= \cos^2 \theta \left(\frac{p+q}{2} + \frac{p-q}{2}\right) + \sin^2 \theta \left(\frac{p+q}{2} - \frac{p-q}{2}\right)$$

$$= p \,.\, \cos^2 \theta + q \,.\, \sin^2 \theta.$$

Cor.—If s_n be the normal component of stress on DD' the plane at right angles to CC', whose inclination to OX is $\theta' = (\theta + 90^\circ)$, then

$$s_n = p \cos^2 \theta' + q \sin^2 \theta'.$$

But $\cos \theta' = - \sin \theta$, and $\sin \theta' = \cos \theta$;

$$\therefore \quad s_n = p \sin^2 \theta + q \cos^3 \theta.$$

Now, $r_n = p \cos^2 \theta + q \sin^2 \theta,$

and adding, we get

$$s_n + r_n = p (\sin^2 \theta + \cos^2 \theta) + q (\sin^2 \theta + \cos^2 \theta) = p + q.$$

That is, the sum of the normal components of the stresses on any pair of rectangular planes is equal to the sum of the principal stresses. From these two corollaries verify Ex. 58, where principal stresses are found; and Ex. 57, where the normal components on a pair of rectangular planes are found. Verify Ex. 60, which could have been solved by these two corollaries arithmetically.

As CC' moves through all positions, M moves in a circle round O, and R moves in a circle round M, OM and MR keeping equally inclined to the vertical on opposite sides of it. The diagram shows their positions for eight positions of the plane CC'. The locus of R is an ellipse, the major semi-axis being

$$OR_4 = OM_4 + M_4R_4$$
$$= \frac{p+q}{2} + \frac{p-q}{2} = p;$$

and the minor semi-axis is

$$OR_1 = OM_1 - M_1R_1$$
$$= \frac{p+q}{2} - \frac{p-q}{2} = q.$$

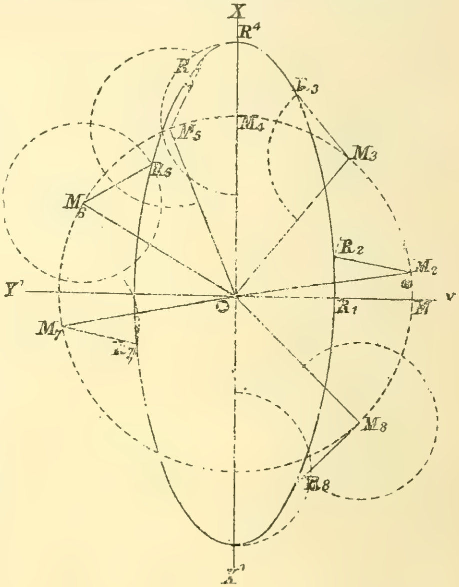


Fig. 33.

The moving model, figure 49, shows these positions nicely.

This is called *the ellipse of stress* for the point O within a solid in a state of strain. Its principal axes are the normals to the planes of principal stress, the principal semi-axes being equal to the intensities of the principal stresses. The radius-vectors OR_2, OR_3, &c., are the stresses in direction and intensity upon the planes at O to which OM_2, OM_3, &c., are respectively the normals.

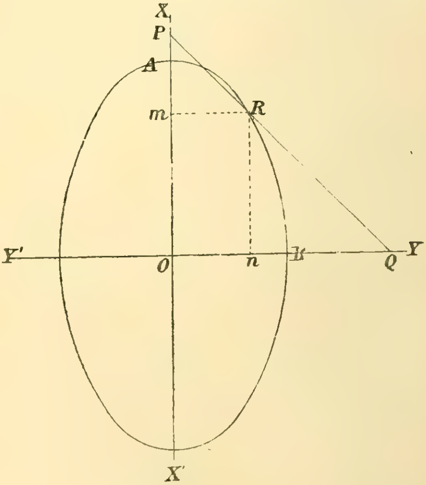


Fig. 34.

The ordinary trammel (fig. 34) for constructing ellipses consists of a piece like PRQ, whose extremities P and Q slide in two grooves, XOX' and YOY', at right angles to each other, while the point R traces an ellipse whose semi-axes are PR and QR.

When Q arrives at O, R is at A and $OA = QR = p$; when P arrives at O, R is at B and $OB = PR = q$ (page 46).

Taking O as origin, the coordinates of R are

$$x = Om\,; \quad y = On\,; \quad \therefore\ x = nR = QR \,.\cos\theta = p\,.\cos\theta,$$

and $$y = mR = PR\sin\theta = q\sin\theta\,;\ \therefore\ \frac{x}{p} = \cos\theta, \text{ and } \frac{y}{q} = \sin\theta\,;$$

$$\therefore\ \frac{x^2}{p^2} + \frac{y^2}{q^2} = \cos^2\theta + \sin^2\theta = 1,$$

the ordinary equation to an ellipse in terms of the semi-axes p and q.

If p and q are both thrusts, it is convenient to consider a thrust positive, and the proof is exactly the same, all the sides of OMR representing the opposite kind of stress from what they did in the last case.

When p and q are unlike, the kind of stress of which the greater p consists, is to be considered positive.

Thus, if $p > q$, and p a tension while q is a thrust, the preceding proof will hold if q be considered to include its negative sign; but in this case if $(-q)$ be substituted for q, we have *arithmetically*

$$OM = \frac{p-q}{2}, \quad \text{and} \quad MR = \frac{p+q}{2}$$

so that now $OM < MR$ (see fig. 36).

It is important to notice that although OM is always positive, that is like p the greater principal stress, yet MR is now positive or negative according as θ is less or greater than 45°, and $r = OM$ is positive or negative according as θ is of a lesser or greater value than that shown on figure 36. Hence, the proposition is proved generally.

An advantage of this geometrical method, the ellipse of stress, is that we are now in a position to examine the value and sign of r, the stress upon a third plane CC', and of its normal and tangential components for special positions of that plane. OM is always normal to CC', while MR generally is resolvable into two components, one tangential to CC' and the other normal, which last has to be either added to, or subtracted from, OM to give the total normal component according as OMR is an obtuse or an acute angle.

(a) *Positions of CC' for which r the stress upon it will have the greatest or least value.*

Since OM and MR are constant, OR increases as the angle OMR increases, is greatest when $OMR = 180°$, and OM and MR are in one straight line, and a continuation one of the other, when

$$OR = OM + MR; \quad r = \frac{p+q}{2} + \frac{p-q}{2} = p;$$

and OR is least when $\angle\ OMR$ is zero, and OM and MR again in one straight line, but MR lapping back on OM, when

$$OR = OM - MR; \quad r = \frac{p+q}{2} - \frac{p-q}{2} = q.$$

Hence the planes of principal stress are themselves the planes of greatest and least stress.

(b) *Position of CC' for which the intensity of the shearing stress has the greatest value.*

As OM is always normal to CC' it does not give any tangential component, whereas MR assumes all positions as CC' changes, and will give a component tangential to CC', which will be the greatest possible when MR is altogether tangential to CC'. Hence the position of CC', which makes MR parallel to CC', is that for which the shearing stress has the greatest possible intensity (fig. 35).

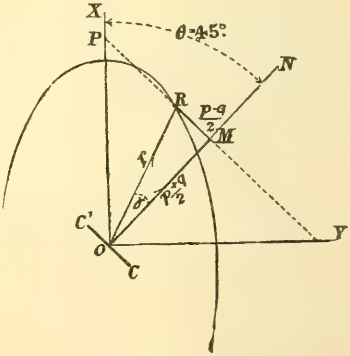


Fig. 35.

Hence intensity of shearing stress $= MR = \dfrac{p-q}{2}$.

And since MR is parallel to CC' and OM normal to it,

$$\therefore \quad OMR = 90°,$$

and the triangle MOP being isosceles, we have

$$\theta = \text{inclination of } CC' = MOP = 45°.$$

And we know that the tangential stress is the same on the section perpendicular to CC'; that is, the planes of greatest tangential stress are the two planes inclined at 45° to the axes.

We saw (fig. 14) that cast iron subjected to a simple thrust would give way by shearing along the plane inclined at 45° to the thrust. We now infer that if it be in a compound state of strain it will most readily give way by shearing along a plane inclined at 45° to the planes of principal stress.

(c) *Position of CC' for which the total stress r upon it will be entirely tangential.*

When q is like p, it is impossible for the stress to be entirely tangential to CC', because $OM > MR$, and however acute OMR may be, the normal component of MR, which has to be subtracted from OM to give the total normal stress, cannot be greater than MR itself, and consequently is always less than OM, and so there will always be a remainder; that is, for all positions of CC' there is a normal component stress, and the total stress can never be entirely tangential.

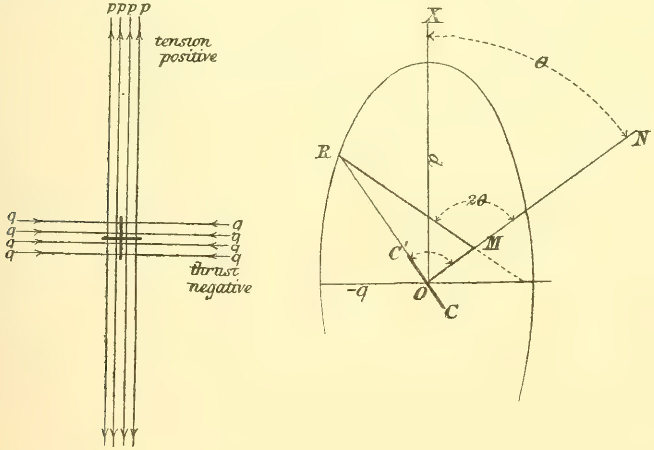


Fig. 36.

But when q is unlike p, then $OM < MR$, and for the particular position of CC' (fig. 36), when the angle OMR is of such an acuteness that the normal component of MR, which has to be subtracted from OM to give the total normal stress, is exactly of the same length as OM; then the total normal stress will be zero, and the total stress r entirely tangential.

This occurs when R is in one straight line with CC'. ROM is then a right angle, making MO the normal component of MR, equal and opposite to OM, which it destroys, leaving the total stress OR tangential to CC'; its magnitude is found thus:

$$OR^2 = MR^2 - OM^2 \qquad \text{(Euc. I. 47),}$$

or

$$r^2 = \left(\frac{p+q}{2}\right)^2 - \left(\frac{p-q}{2}\right)^2 = p \cdot q,$$

$$\therefore \quad r = \sqrt{pq},$$

i.e., the stress on CC' is the geometrical mean of the principal stresses. Also

$$2\theta = RMN; \quad \therefore \quad \cos 2\theta = \cos RMN,$$

$$\cos 2\theta = -\cos RMO = -\frac{MO}{MR} = -\frac{p-q}{p+q},$$

which determine θ, the position of CC' for which the total stress is tangential.

Here we must guard against supposing that the above is the position of CC' for which the tangential stress has the greatest intensity; for case (b) holds for all conditions of p and q; that is, the tangential stress on CC' when inclined at 45°, although only a component of the total stress, will be of greater intensity than the total tangential stress in case (c).

(d) *Position of CC' for which γ, the obliquity of the stress thereon, is the greatest possible.*

When q is unlike p, case (c) is the solution, for in it $\gamma = RON = 90°$, the greatest possible.

When q is like p, $OM > MR$; and the obliquity of OR, the stress on CC' is greatest when $\gamma = ROM$ is the greatest possible of all triangles constructed with OM and MR for two of their sides. This occurs when ORM is a right angle.

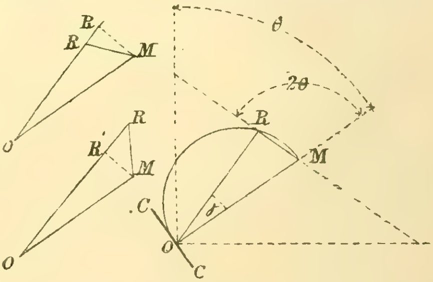


Fig. 37.

For suppose the triangle OMR constructed with ORM not a right angle; then drop MR' at right angles to OR. It is evident that MR' is less than MR.

Now $\sin ROM = \dfrac{MR'}{MO}$ is greatest when MR' is greatest; that is, when $MR' = MR$; that is, when ORM is a right angle, and ROM is greatest when its sine is greatest.

In this case the intensity of the stress is

$$OR^2 = OM^2 - MR^2;$$

$$r^2 = \left(\frac{p+q}{2}\right)^2 - \left(\frac{p-q}{2}\right)^2 = pq,$$

and $$r = \sqrt{pq},$$

a geometrical mean between the principal stresses.

Also $$2\theta = RMN;\ \cos 2\theta = \cos RMN$$

$$\cos 2\theta = -\cos ROM = -\frac{MR}{OM} = -\frac{p-q}{p+q},$$

which determines θ, the position of CC' for which the stress has the greatest obliquity possible.

Note that these values of r and $\cos 2\theta$ are the same as those of (c), and that whether p and q are like or unlike. But this is not the case with γ, the obliquity, which is 90° when p and q are unlike, and has $\frac{p-q}{p+q}$ for its sine when p and q are alike.

Examples.

62. At a point within a solid in a state of strain the principal stresses are tensions of 255 lbs. and 171 lbs. per square inch. Find the stress on a plane inclined at 27° to the plane of greatest principal stress (converse of Ex. 58) (fig. 38).

Data.

$$p = 255,\quad q = 171,\quad \text{and}\quad \theta = 27°;$$

hence

$$\frac{p+q}{2} = 213,\quad \text{and}\quad \frac{p-q}{2} = 42.$$

Construction.—OX and OY, the axes of principal stresses; draw ON the normal to CC', making $XON = \theta = 27°$. Lay off along it $OM = \frac{p+q}{2} = 213$. From M as centre with radius MO, describe semicircle POQ and join PMQ; lay off from M towards P, $MR = \frac{p-q}{2} = 42$. This construction makes MR to be inclined to OX at an angle $\theta = 27°$, but upon the opposite side of it from OM.

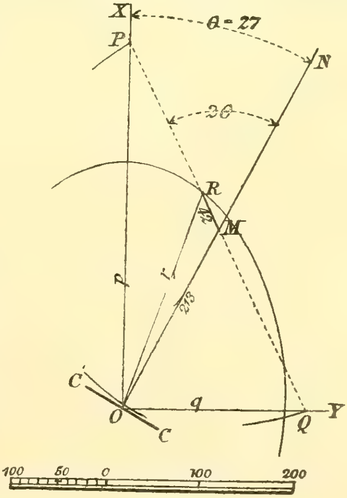


Fig. 38.

Looking upon the principal stresses as a pair of like principal stresses, tensions of intensities 213, together with a pair of unlike principal stresses, a tension and a thrust of intensities 42. Then OM represents a tension 213 upon plane CC' due to first group, and MR the tension 42 upon CC' due to second group; hence OR, the third side of the triangle, taken in the opposite direction, represents the total stress upon CC' in direction and intensity.

$$OR^2 = OM^2 + MR^2 - 2OM\,.\,MR \cos OMR,$$

but $$\cos OMR = -\cos RMN = -\cos 2\theta;$$

$$\therefore\quad OR^2 = OM^2 + MR^2 + 2OM\,.\,MR \cos 2\theta;$$

$$\therefore\quad r^2 = 45369 + 1764 + 17892 \cos 54^\circ = 57649;$$

$$r = 240 \text{ lbs.}$$

Also $$\frac{\sin\gamma}{\sin 2\theta} = \frac{\sin ROM}{\sin OMR} = \frac{MR}{r};$$

$$\therefore\quad \sin\gamma = \frac{42}{240} \times \sin 54^\circ = \cdot 14158; \quad \therefore\quad \gamma = 8^\circ\, 8';$$

and figure shows that r is upon the same side of the normal as OX. Also r is a tension, since OR is like OM.

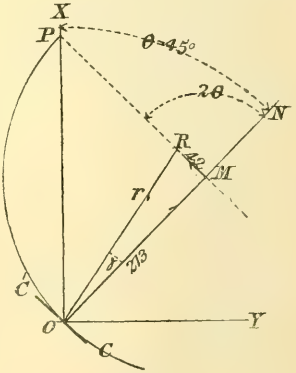


Fig. 39.

63. In Ex. 62 find the intensity of the tangential stress on that plane through the point upon which the tangential stress is of greatest intensity (fig. 39).

The plane is that which is inclined at 45° to the axes of principal stress.

Since OMR is 90°, MR is the tangential component of OR,

$$r_t = MR = \frac{p-q}{2} = 42 \text{ lbs. per square inch.}$$

64. In Ex. 62 find the obliquity to the plane of greatest principal stress of that plane, through the point, upon which the stress is more oblique than upon any other; also find the stress (fig. 40).

OM and MR being constant, the angle MOR has its greatest value when MRO is a right angle.

Construction.—Upon OM describe a semicircle; from M as centre, with radius MR, describe an arc cutting the semicircle in R; join OR.

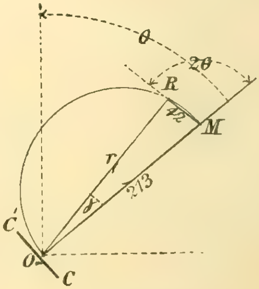


Fig. 40.

$$\cos 2\theta = \cos RMN = -\cos OMR$$

$$= -\frac{MR}{OM} = -\frac{p-q}{p+q} = -\frac{42}{213}$$

$$= -\cdot 19718;$$

$\therefore\quad 2\theta =$ 101° 22′ obtuse, cosine being negative;

$\therefore\quad \theta =$ 50° 41′, obliquity of CC'.

$$r^2 = OR^2 = OM^2 - MR^2$$

$$= \left(\frac{p+q}{2}\right)^2 - \left(\frac{p-q}{2}\right)^2 = p\,.\,q;$$

$\therefore\quad r = \sqrt{p\,.\,q} = \sqrt{43605} = 208\cdot 8$ lbs. per square inch of tension like OM.

And $\sin\gamma = \sin RON = \dfrac{MR}{MO} = \cdot 19718$; $\therefore\ \gamma = 11^\circ\, 22'$, obliquity of r.

65. The principal stresses at a point being a tension of 300 lbs. and a thrust of 160 lbs. per square inch.

Find (a) the intensity, obliquity, and kind of stress on a plane through the point, inclined at 30° to the plane of greatest principal stress; (b) find the intensity of tangential stress on the plane upon which that stress is greatest; and (c) find the inclination to the plane of greatest principal stress of that plane upon which the stress is entirely tangential and the intensity thereof.

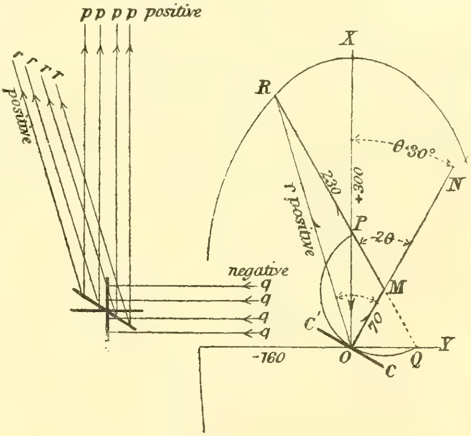


Fig. 41.

Data.

$p = 300$; $q = -160$, considering a tension positive ;

$\therefore\ \dfrac{p+q}{2} = 70$ tension like p ;

and $\dfrac{p-q}{2} = 230$ tension like p.

(a) *Construction.*—Draw ON at 30° to OX (fig. 41). Lay off $OM = 70$. From M as centre, with radius MO, describe semicircle POQ. Lay off $MPR = 230$. Then OR, the third side of the triangle OMR, taken in the opposite order, is the stress on CC' in direction and intensity.

$$OR^2 = OM^2 + MR^2 - 2OM\,.\,MR\cos OMR$$

$$= OM^2 + MR^2 + 2OM\,.\,MR\cos 2\theta,$$

$$r^2 = 4900 + 52900 + 16100 = 73900,$$

$$r = 272 \text{ lbs. per square inch;}$$

and $\dfrac{\sin\gamma}{\sin 2\theta} = \dfrac{\sin\ ROM}{\sin\ RMO} = \dfrac{MR}{OR}$.

$\therefore\ \sin\gamma = \dfrac{230}{272}\sin 60^\circ = \cdot 7323$.

$\therefore\ \gamma = 47^\circ\, 5'$, being acute, OR is like OM, a tension.

(b) Take

$$\theta = 45^\circ;\ r_t = MR = 230 \text{ lbs.}$$

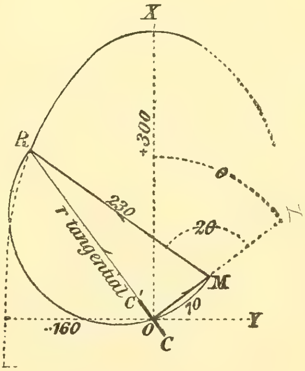


Fig. 42.

(c) On MR (fig. 42) describe a semicircle, and from M, with radius MO, describe arc cutting it at O.

$$RMN = 2\theta, \quad \cos 2\theta = \cos RMN = -\cos RMO = -\frac{OM}{MR} = -\frac{70}{230} = -\cdot 3044\,;$$

$2\theta = 107° \; 44'$; $\theta = 53° \; 52'$, obliquity of plane upon which the stress is entirely tangential.

$$r^2 = OR^2 = MR^2 - OM^2 = 52900 - 4900 = 48000, \; r = 219,$$

or $r = \sqrt{p \,.\, q} = \sqrt{(300 \times 160)} = 219$ lbs. per square inch.

Note that, though r is entirely tangential, it is less than r_t was in (*b*).

66. At same point as in Ex. 65, find intensity, kind, and obliquity of a stress on a plane inclined at 85° to the plane of greatest principal stress (fig. 43).

Since $\theta > 53° \; 52'$, $>$ the obliquity of plane upon which the stress was wholly tangential, *OR* will make with *ON* an angle greater than 90°, and *OR* will be unlike *OM*, and therefore a thrust.

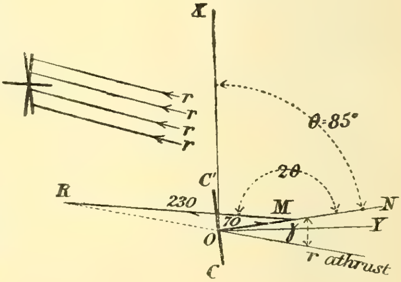


Fig. 43.

Ans. $r = 161\cdot 5$ lbs. per sq. in. ;

$\gamma = 165° \; 41'$.

67. The principal stresses on *AA′* and *BB′* are thrusts of 60 lbs. per square inch. Find direction and intensity of the stress on a third plane *CC′* inclined at 65° to *AA′*.

Ans. A thrust of 60 lbs. per square inch normal to *CC′*.

68. The principal stresses on *AA′* and *BB′* are of the equal intensity of 34 lbs. per square inch, being a thrust on *AA′* and a tension on *BB′*. Find the direction and intensity of the stress on a third plane *CC′* inclined at 65° to *AA′*.

Ans. A tension of 34 lbs. per square inch, its direction being inclined at 65° upon the other side of *OX* from that to which *ON* is inclined.

69. The principal stresses on *AA′* and *BB′* at a point *O* are a thrust of 94 lbs. and a thrust of 26 lbs. Find kind, intensity, and obliquity of a stress on a third plane *CC′* inclined at 65° to *AA′*. Using results of Exs. 67 and 68,

$r = 46\cdot 2$ lbs. per square inch thrust, $\gamma = 34° \; 19'$.

70. Two unlike principal stresses are: on *AA′* a thrust of 146 and on *BB′* a tension of 96 lbs. per square inch. Find the stress on *CC′* a third plane inclined to *AA′* at 50°.

$$p = 146, \quad \text{and} \quad q = -96.$$

Half sum $\dfrac{p+q}{2}$ is a thrust like p,

half diff. $\dfrac{p-q}{2}$ is a tension like q, since $\theta > 45°$ (see fig. 27).

Ans. $r = 119\cdot 22$ lbs. per square inch thrust, $\gamma = 88° \; 5'$.

71. At a point within a solid the principal stresses are thrusts of 248 lbs. and

172 lbs. per square inch. Find the normal and tangential component stresses on a plane inclined at 15° to the plane of greatest principal stress.

$$r_t = (p - q) \sin\theta \cos\theta = 19 \text{ lbs. per square inch,}$$

$$r_n = p \,.\, \cos^2\theta + q \sin^2\theta = 243 \text{ lbs. per square inch.}$$

These two results may be obtained with less labour from the formulæ

$$r_t = \frac{p-q}{2} \sin 2\theta\,; \quad r_n = \frac{p+q}{2} + \frac{p-q}{2} \cos 2\theta.$$

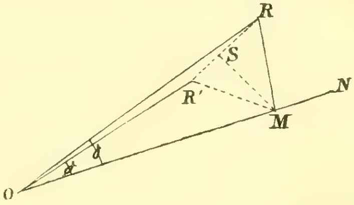
INVERSE PROBLEM, TO FIND THE PRINCIPAL STRESSES.

Given the intensities, obliquities, and kinds of the stresses upon any two planes at a point within a solid, find the principal stresses and their planes.

In the general problem we know of the triangle OMR (fig. 31), the parts OR and γ for two separate positions of the plane CC', and we also know that OM and MR are the *same* for both.

If the two given stresses be *alike* and *unequal* (fig. 44). Let r and r' be their intensities, and γ and γ' their obliquities upon their respective planes CC' and DD'. Let r be greater than r'. Note that it is not necessary to have given the inclination to each other of CC' and DD'.

Choose any line ON and draw $OR = r$, and making the angle $NOR = \gamma$, also draw $OR' = r'$, and making the angle $NOR' = \gamma'$. Join RR', and from S the middle point of RR' draw, at right angles to it, SM meeting ON at M. Then will $MR = MR'$.



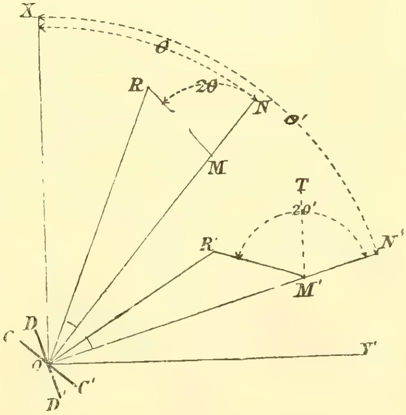


Fig. 44.

Thus we have found OM and MR to suit both data, and comparing the construction of the direct problem (figs. 28, 29), we have

$$OM = \frac{p+q}{2}, \quad \text{and} \quad MR = \frac{p-q}{2},$$

and therefore

$$p = OM + MR\,; \quad q = OM - MR.$$

Consider the triangle $OM'R'$ alone, and consider ON' the normal to DD': then $R'M'N' = 2\theta'$; hence OX, drawn parallel to $M'T$ (the bisector of $R'M'N'$), is the axis of greatest principal stress. Thus we have found the principal stresses p and q, and the position of their axes OX and OY relative to DD', one of the given planes.

Since

$$R'MR = R'MN - RMN = 2\theta' - 2\theta; \quad \therefore \quad RMS = \theta' - \theta,$$

the inclination to each other of CC' and DD'; hence if the other triangle OMR be moved round O through this angle, it and consequently CC', to which ON is the normal, will also be in their proper positions with respect to the axes OX and OY.

This triangle might be further turned round O till ON is inclined at an angle $XON = \theta$ on the other side of OX, when CC' would again be in a position for which the stress would be the same as given. This would increase the relative inclination of DD' and CC' by twice XON or by 2θ. Adding this to $\theta' - \theta$ gives $\theta' + \theta$. That is, the inclination of CC' and DD' to each other is

$$(\theta' - \theta) = RMS \text{ on diagram,}$$

or

$$(\theta' + \theta) = NMS \text{ on diagram,}$$

according as they lie on the same or on opposite sides of OX, the axis of principal stress.

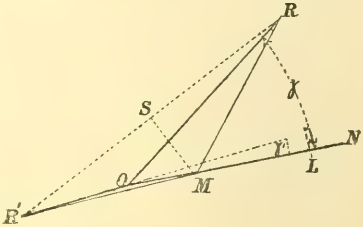


Fig. 45.

If the two given stresses be *unlike* and *unequal* (fig. 45), considering r the greater as positive, r' will be negative. Follow the same construction, only $OR' = r'$ must be laid off from O in the *opposite* direction. Complete the figure as before, and we have from either figure—

Trigonometrically

$$MR^2 = OM^2 + OR^2 - 2OM \,.\, OR \cos MOR$$

$$= OM^2 + r^2 - 2OM \,.\, r \cos \gamma.$$

Similarly,

$$MR'^2 = OM^2 + r'^2 \mp 2OM \,.\, r' \cos \gamma'$$

from figs. 44 and 45 respectively.

Subtracting, $0 = r^2 - r'^2 - 2OM\,(r \cos \gamma \mp r' \cos \gamma')$,

and therefore $$\frac{p+q}{2} = OM = \frac{r^2 - r'^2}{2\,(r \cos \gamma - r' \cos \gamma')}, \qquad \text{(A)}$$

r' to include its sign;

also $$\left.\begin{aligned} \frac{p-q}{2} = MR &= \sqrt{(OM^2 + r^2 - 2OM\,.\,r \cos \gamma)} \\ \text{or,} \qquad &= \sqrt{(OM^2 + r'^2 - 2OM\,.\,r' \cos \gamma')} \end{aligned}\right\} \qquad \text{(B)}$$

a known quantity when the value of OM is substituted from equation (A).

p and q are now obtained by adding and subtracting equations (A) and (B).

From R drop RL perpendicular to ON, then

$$ML = OL - OM; \quad MR\,.\cos NMR = OR\,.\cos ROM - OM,$$

$$\frac{p-q}{2} \cos 2\theta = r \cos \gamma - \frac{p+q}{2};$$

$$\therefore \quad \cos 2\theta = \frac{2r \cos \gamma - p - q}{p - q}. \qquad \text{(C)}$$

This gives twice the obliquity of the axis of greatest principal stress to the given plane CC', and similarly for DD'

$$\cos 2\theta' = \frac{2r' \cos \gamma' - p - q}{p - q}.$$

These three equations (A), (B), and (C) are the general solution of the inverse problem of the ellipse of stress. (A) and (B) give the intensities of the principal stresses, which will come out with signs showing whether they are like or unlike r, the greater of the given stresses.

In some particular cases the construction gives a much simpler figure from which the equations (A), (B), and (C) in their modified form are readily calculated.

Particular case (a) (figure 46). Given the intensities and common obliquity of a pair of *conjugate stresses* at a point; find the principal stresses, and position of the axes of principal stress. (Note—There are more than sufficient data.)

In this case $\gamma = \gamma'$, and R, S, and R' are in one straight line with O.

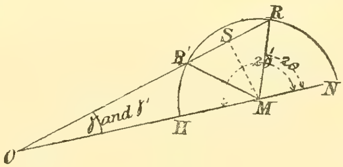


Fig. 46.

Draw any line ON; draw OR, making $NOR = \gamma = \gamma'$; and lay off $OR = r$ and $OR' = r'$ in the same or opposite directions according as it is like or unlike r; and from S, the middle point of RR', draw SM at right angles to it, meeting ON at M. Join M to R and R'.

Then

$$\frac{OS}{OM} = \cos MOS\,; \quad OM = \frac{OS}{\cos MOS} = \frac{\frac{1}{2}(OR + OR')}{\cos MOS},$$

or
$$\frac{p+q}{2} = \frac{r+r'}{2\cos\gamma}. \tag{A}$$

Again
$$MR^2 = MS^2 + RS^2 = (OM^2 - OS^2) + RS^2$$
$$= OM^2 - (OS^2 - RS^2) = OM^2 - \left\{\left(\frac{r+r'}{2}\right)^2 - \left(\frac{r-r'}{2}\right)^2\right\}$$
$$= OM^2 - rr' = \left(\frac{r+r'}{2\cos\gamma}\right)^2 - rr'\,;$$

$$\therefore \quad \frac{p-q}{2} = \sqrt{OM^2 - rr'} = \sqrt{\left\{\left(\frac{r+r'}{2\cos\gamma}\right)^2 - rr'\right\}}, \tag{B}$$

and
$$\cos 2\theta = \frac{2r\cos\gamma - p - q}{p-q} \tag{C}$$

as in general case,

$$NON' = \theta' + \theta = NMS = MSO + MOS = \frac{\pi}{2} + \gamma.$$

Hence, the angle between the two normals to the sections CC' and DD' (or the *obtuse* angle between CC' and DD') exceeds the obliquity by a right angle. This we know ought to be the case from the *definition* of conjugate stresses.

It may be easier to calculate $\cos 2\theta = -\cos RMO$ in terms of the sides of the triangle OMR when these have been already found.

From M as centre with radius MR, describe the semicircle $HR'RN$; then (fig. 46)

$$OH = OM - MR = q, \qquad ON = OM + MR = p.$$

But

$$ON \times OH = OR \times OR' \text{ (Euc. iii. 36)}, \quad \text{and} \quad pq = rr' \tag{B$_1$}$$

may be used instead of (B).

Particular case (*b*) (fig. 47). Given the intensities and obliquities of the stresses on a pair of *rectangular planes,* find the principal stresses and the position of the axes of principal stress. (Note—There are more than sufficient data.)

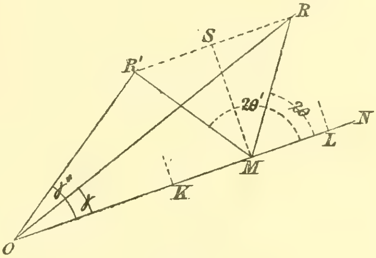


Fig. 47.

If r and r' be like stresses, draw any line ON. Draw $OR = r$, making $NOR = \gamma$, also $OR' = r'$ making $NOR' = \gamma'$.

Complete the figure as before.

The given planes being at right angles are necessarily inclined upon opposite sides of the axis of principal stress; hence

$$NMS = \text{inclination of given planes} = 90^\circ,$$

and RSR' is parallel to ON.

Therefore $$MS = RL = r \sin \gamma = R'K = r' \sin \gamma',$$

or $$r \sin \gamma = r' \sin \gamma'.$$

That is, the tangential components of r and r' are equal.

$$OM = \tfrac{1}{2}(OL + OK); \quad \therefore \quad \frac{p+q}{2} = \tfrac{1}{2}(r \cos \gamma + r' \cos \gamma'). \quad \text{(A)}$$

That is, the sum of the principal stressses is equal to the sum of the normal components of r and r'. (Compare page 47.)

$$MR^2 = RS^2 + MS^2 = \left(\frac{OL - OK}{2}\right)^2 + MS^2$$

$$= \frac{(r \cos \gamma - r' \cos \gamma')^2}{4} + r^2 \sin^2\gamma;$$

$$\therefore \quad \frac{p-q}{2} = \sqrt{\left\{\frac{(r \cos \gamma - r' \cos \gamma')^2}{4} + r^2 \sin^2\gamma\right\}}. \quad \text{(B)}$$

$$\tan 2\theta = \frac{RL}{ML} = \frac{RL}{\frac{1}{2}(OL - OK)} = \frac{r \sin \gamma}{\frac{1}{2}(r \cos \gamma - r' \cos \gamma')}$$

$$= \frac{2r \sin \gamma}{r \cos \gamma - r' \cos \gamma'}. \quad \text{(C)}$$

Putting $r_t = MS = r \sin \gamma = r' \sin \gamma' =$ the common value of the tangential components of r and r'; also

$$r_n = OL = r \cos \gamma = \text{norm. comp. of } r,$$
$$r_n' = OK = r' \cos \gamma' = \text{norm. comp. of } r',$$

the equations become

$$\frac{p+q}{2} = \frac{r_n + r'_n}{2}, \qquad (A_1)$$

$$\frac{p-q}{2} = \sqrt{\left\{\frac{(r_n - r'_n)^2}{4} + r_t^2\right\}}. \qquad (B_1)$$

and $$\tan 2\theta = \frac{2r_t}{r_n - r'_n}. \qquad (C_1)$$

(Compare (C_1) with page 35.)

When r and r' are unlike stresses, consider r, the greater, as positive; then must OR' be laid off in the opposite direction from O (fig. 48).

Now $R'K = RL = r_t$, the common tangential component of r and r'; hence RR' and KL *bisect* each other at S or M, which coincide.

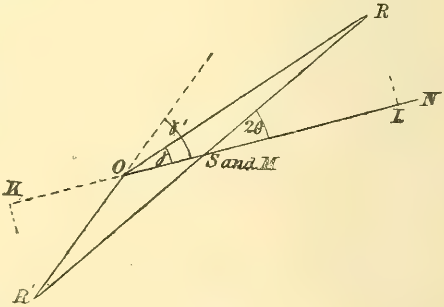


Fig. 48.

$$\therefore \quad OM = \tfrac{1}{2}(OL - OK); \quad \frac{p+q}{2} = \frac{r \cos \gamma - r' \cos \gamma'}{2}. \qquad (A_2)$$

$$MR^2 = ML^2 + RL^2 = \left(\frac{OL + OK}{2}\right)^2 + RL^2,$$

or $$\frac{p-q}{2} = \sqrt{\left\{\frac{(r \cos \gamma + r' \cos \gamma')^2}{4} + r^2 \sin^2 \gamma\right\}} \qquad (B_2)$$

$$\tan 2\theta = \frac{RL}{ML} = \frac{RL}{\frac{1}{2}LK} = \frac{RL}{\frac{1}{2}(OL + OK)}$$

$$= \frac{2r \sin \gamma}{r \cos \gamma + r' \cos \gamma'}. \qquad (C_2)$$

These three equations (A_2), (B_2), and (C_2) are identical with (A), (B), and (C) with $(-r')$ substituted for r'.

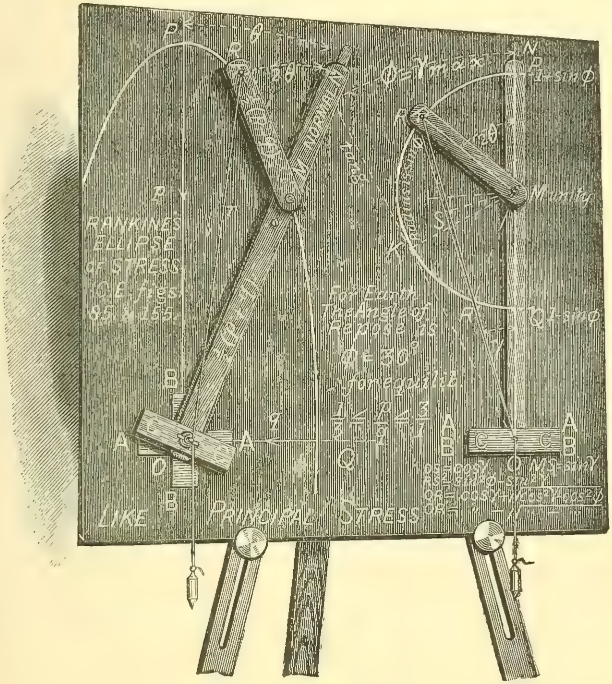


Fig. 49.

The above diagram represents the kinematical model in the Engineering Laboratory of Trinity College, Dublin, devised by Professors Alexander and Thomson for illustrating Rankine's Method of the Ellipse of Stress for Uniplanar Stress. On the left hand of the diagram, *ON* is a ⊤ square, pivoted to the blackboard at *O*, and *MR* is a pointer pivoted to the blade of the ⊤ square at *M*. On the pivot *M* a wheel is fixed at the back of the blade, and round the wheel an endless chain is wrapped, which also wraps round a wheel fixed to the board at *O*. The wheel at *O* is of a diameter double that of the wheel at *M*. Hence, when the blade of the ⊤ square is turned to the right the pointer *MR* automatically turns to the left, so that the angle *RMN* is always equal to 2θ when *PON* equals θ; or, in other words, the bisector of RMN is always parallel to OP. A cord is fastened to the pointer at R, it passes through a swivel ring at O and is kept tight by a plummet. It is easily seen that the point R describes an ellipse.

CC, the head-stock of the T square, represents the trace on the blackboard of any plane through O normal to the board, and the vector of the ellipse, consisting of the segment RO of the cord, represents the stress on the plane CC in intensity and direction. The angle $PON = \theta$ is the position of the plane CC relative to the plane of greater principal stress AA, while $RON = \gamma$ is the obliquity of the stress r upon CC.

The model* shows clearly the interesting positions of the plane CC; thus CC may be turned to coincide with AA, when the cord RO will be found to be normal to CC and to be of a maximum length. On the other hand, if CC be turned to coincide with BB the cord RO is again normal to CC, but of a minimum length. Again, when CC is so placed that the angle RMO is a right angle, the component of RO parallel to CC is a maximum; and lastly, if it be turned till MRO is a right angle, then ROM, the obliquity of the cord, is a maximum.

The auxiliary figure (see figs. 50 and 62) on the right of the diagram is for solving the general problem of uniplanar stress at a point, viz. given the stress in intensity and obliquity for two positions of CC, to find the stress for any third required position of CC. On the auxiliary figure, the T squares for all positions of CC are superimposed upon each other and are represented by one T square fixed to the board, and only the pointer MR turns.

In solving questions on the stability of earthworks, some linear dimension upon the auxiliary figure represents the known weight of a column of earth, while another dimension is the required stress on a retaining wall or on a foundation, &c. Generally two angular quantities also are known, such as $KON = \phi$ the maximum value of γ, this being the steepest possible slope of the loose earth, while $RON = \gamma$ may be the known slope of surface of earth.

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* The model was exhibited to the Royal Irish Academy early in 1891, and described for the first time in the Academy's *Transactions*. Reference may be made to Rankine's *Civil Engineering*, or *Applied Mechanics*, and to Williamson's *Treatise on Stress*. Numerical examples are here worked by this method, and in Howe's *Retaining Walls*. The model is made by Messrs. Yeates & Son, Grafton-street, Dublin.

In Williamson's *Treatise on Elasticity*, this model is shown in its proper place relative to the complete systematic treatment of elasticity.

On the other side of the board, another T square is similarly pivoted to the same pin O, the only difference being that OM is shorter than the pointer MR. It represents the case of Unlike Principal Stresses, and is useful in demonstrating the composition of stresses such as that of thrust and bending on a pillar, or of bending and twisting on a crank-pin.

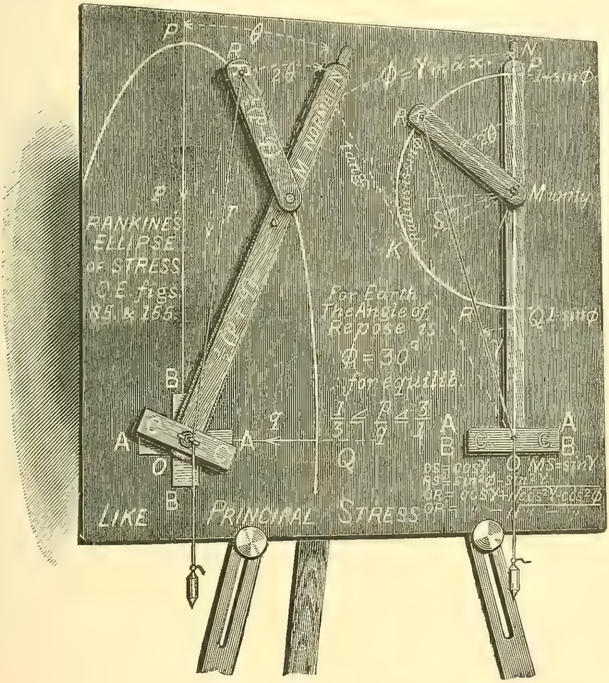

Fig. 49 (*bis*).

By means of the slots on the easel the board can be placed so that any desired vector of the ellipse may be vertical.

Examples.

72. If from external conditions it be known that the stresses on two planes at a point in a solid are thrusts of 54 and 30 lbs. per square inch, and inclined at 10° and 26° respectively to the normals to these planes—find the principal stresses at that point; the position of the axis of greater principal stress relative to the first plane; and the inclination of the two planes to each other.

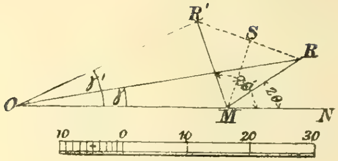

Fig. 50.

Make

$$NOR = \gamma = 10°,$$

and

$$NOR' = \gamma' = 26°.$$

Lay off

$$OR = r = 54,$$

and

$$OR' = r' = 30.$$

Join RR', bisect it in S, draw SM at right angles to RR', meeting ON at M: complete figure.

Then $$\frac{p+q}{2} = OM, \quad \text{and} \quad \frac{p-q}{2} = MR = MR',$$

or $$p = (OM + MR), \quad \text{and} \quad q = (OM - MR), \quad \text{also} \quad 2\theta = NMR.$$

Trigonometrically

$$MR^2 = OM^2 + OR^2 - 2OM \, . \, OR \cos MOR = OM^2 + r^2 - 2OM \, . \, r \cos \gamma.$$

Similarly $$MR'^2 = OM^2 + r'^2 - 2OM \, . \, r' \cos \gamma'.$$

Therefore subtracting

$$0 = r^2 - r'^2 - 2OM\,(r \cos \gamma - r' \cos \gamma')\,;$$

$$\therefore \quad OM = \frac{r^2 - r'^2}{2\,(r \cos \gamma - r' \cos \gamma')}, \quad \text{or} \quad \frac{p+q}{2} = \frac{2016}{52{\cdot}43} = 38{\cdot}45. \qquad \text{(A)}$$

$$MR^2 = (38{\cdot}45)^2 + (54)^2 - 2 \times 38{\cdot}45 \times 54 \cos 10°$$

$$= 1478{\cdot}4 + 2916 - 4088{\cdot}8 = 305{\cdot}6\,;$$

$$\therefore \quad \frac{p-q}{2} = \sqrt{305{\cdot}6} = 17{\cdot}48. \qquad \text{(B)}$$

The principal stresses are

$$p = \frac{p+q}{2} + \frac{p-q}{2} = 55{\cdot}93 \text{ lbs. per square inch thrust, like } r,$$

$$q = \frac{p+q}{2} - \frac{p-q}{2} = 20{\cdot}97 \text{ lbs. per square inch, thrust being } +.$$

Drop RL perpendicular to ON,

$$ML = OL - OM, \quad \text{or} \quad MR \cos LMR = OR \cos LOR - OM\,;$$

$$\therefore \quad \frac{p-q}{2} \cos 2\theta = r \cos \gamma - \frac{p+q}{2}. \qquad \text{(C)}$$

$$\cos 2\theta = \frac{53{\cdot}179 - 38{\cdot}45}{17{\cdot}48} = {\cdot}8426\,; \quad \therefore \quad 2\theta = 32°\ 35'\,; \quad \therefore \quad \theta = 16°\ 17\tfrac{1}{2}' = XON,$$

the inclination of OX, the axis of greatest principal stress, to ON, the normal to the plane for which r was given.

Similarly,

$$\cos 2\theta' = -\cdot 6573\,;$$

$$\therefore\quad 2\theta' = 131^\circ\ 6' \text{ (obtuse for } - \text{ sign)}\,;$$

$$\therefore\quad \theta' = 65^\circ\ 33', \text{ inclination } XON'.$$

And inclination of the two planes to each other

$$NON' = RMS = (\theta' - \theta) = 49^\circ\ 15\tfrac{1}{2}',$$

or

$$= NMS = (\theta' + \theta) = 81^\circ\ 50\tfrac{1}{2}',$$

according as they are on the same or opposite sides of OX.

73. Knowing that at a point within a solid there is, on some one plane, a thrust of 84 lbs. per square inch of obliquity 5°, and on another a tension of 24 lbs. per square inch of obliquity 20°, find the principal stresses and the angle made by the axis of greatest principal stress with the normal to first plane.

Consider a thrust positive (see fig. 45, inverse problem).

$$r = 84, \quad \text{and} \quad \gamma = 5^\circ\,; \qquad r' = -24, \quad \text{and} \quad \gamma' = 20^\circ.$$

$$\frac{p+q}{2} = OM \text{ is a thrust like } r, \quad \text{and} \quad = \frac{r^2 - r'^2}{2\,(r\cos\gamma - r'\cos\gamma')} = 30\cdot 5. \qquad \text{(A)}$$

$$\frac{p-q}{2} = MR \text{ is a thrust like } r,$$

$$MR^2 = OM^2 + r^2 - 2OM\,.\,r\cos\gamma = 930\cdot 25 + 7056 - 5104\cdot 48,$$

$$\frac{p-q}{2} = MR = \sqrt{2881\cdot 77} = 53\cdot 7. \qquad \text{(B)}$$

Therefore by adding and subtracting (A) and (B),

$$p = 84\cdot 2, \quad \text{and} \quad q = -23\cdot 2,$$

or the principal stresses are a thrust of 84·2 and a tension of 23·2 lbs. per square inch respectively.

$$\cos 2\theta = \frac{2r\cos\gamma - p - q}{p - q} = \frac{167\cdot 36 - 61}{107\cdot 4} = \cdot 9903\,;$$

$$\therefore\quad 2\theta = 7^\circ\ 59', \quad \text{and} \quad \theta = 3^\circ\ 59\tfrac{1}{2}', \text{ inclination to axis } OX.$$

74. The stresses on two planes at a point within a solid are 240 at an obliquity of 8°, and 254·5 at 1° 5′. Find the principal stresses and the obliquities of these planes to the axes of principal stress. (Note, these are the planes BB', Ex. 57, and GG', Ex. 59 ; also principal stresses are calculated Ex. 58.)

$$\left.\begin{aligned} \frac{p+q}{2} &= OM = 213\cdot 5 \\ \frac{p-q}{2} &= MR = 41\cdot 26 \end{aligned}\right\} \quad \begin{aligned} &\therefore\quad p = 254\cdot 76, \\ &\therefore\quad q = 172\cdot 24, \end{aligned}$$

$$\cos 2\theta = \cdot 9926, \quad \text{or} \quad 2\theta = 7^\circ\ 1'\,; \quad \therefore\ \theta = 3^\circ\ 30\tfrac{1}{2}.$$

75. At a point within a solid, on one plane, there is a tension of 272 lbs. per square inch, of obliquity 47° 5′, and on another a thrust of 161·5 lbs. per square

inch, of obliquity 15° 25′. Find the principal stresses and the angles which the normals to these planes make with the axis of greatest principal stress.

$$\left.\begin{aligned}\frac{p+q}{2} &= OM = 70{\cdot}2 \\ \frac{p-q}{2} &= MR = 230\end{aligned}\right\}\quad \begin{aligned}&\therefore\quad p = 300{\cdot}2 \text{ tension,} \\ &\therefore\quad q = -\,159{\cdot}8 \text{ thrust,}\end{aligned}$$

$$\cos 2\theta = {\cdot}5, \qquad \text{or} \quad 2\theta = 60^\circ; \qquad \therefore\quad \theta = 30^\circ,$$

$$\cos 2\theta' = -\,{\cdot}982, \quad \text{or} \quad 2\theta' = 169^\circ\, 8'; \qquad \therefore\quad \theta' = 84^\circ\, 34'.$$

Therefore, inclination between planes, $(\theta' - \theta) = 55°$ nearly. Compare Examples 65 and 66.

76. At a point within a solid a pair of conjugate stresses are thrusts of 40 and 30 lbs. per square inch, and their common obliquity is 10°. Find the principal stresses and the angle which normal to plane of greater conjugate stress makes with the axis of greatest principal stress.

Draw OR, making $NOR = \gamma = \gamma' = 10°$; lay off $OR = r = 40$, and $OR' = r' = 30$. Bisect RR' in S, draw SM perpendicular to RR'; complete figure. Then

$$\frac{p+q}{2} = OM, \text{ and } \frac{p-q}{2} = MR, \text{ and } 2\theta = RMN.$$

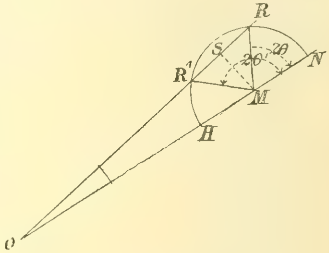

Fig. 51.

$$\frac{OS}{OM} = \cos\gamma;$$

$$\therefore\quad OM = \frac{\frac{1}{2}(r + r')}{\cos\gamma}, \quad \text{or} \quad \frac{p+q}{2} = \frac{35}{{\cdot}9848} = 35{\cdot}55. \qquad \text{(A)}$$

$$MR^2 = MS^2 + RS^2 = OM^2 - (OS^2 - RS^2)$$

$$= OM^2 - \left\{\left(\frac{r + r'}{2}\right)^2 - \left(\frac{r - r'}{2}\right)^2\right\} = OM^2 - rr';$$

$$\therefore\quad \frac{p-q}{2} = \sqrt{(1263{\cdot}8 - 1200)} = 8. \qquad \text{(B)}$$

Adding and subtracting (A) and (B),

$$p = 43{\cdot}5 \text{ a thrust,} \quad \text{and} \quad q = 27{\cdot}5 \text{ a thrust,}$$

$$\cos 2\theta = \frac{78{\cdot}8 - 71}{16} = {\cdot}49. \qquad \text{(C)}$$

Therefore $\quad 2\theta = 60°\, 40'$, and $\theta = 30°\, 20'$.

Or geometrically describe semicircle $HR'RN$ (fig. 51),

$$ON = OM + MR = p, \quad \text{and} \quad OH = OM - MR = q.$$

$$ON \,.\, OH = OR \,.\, OR' \text{ (Euc. iii. 36), or } p \,.\, q = rr' = 1200. \qquad \text{(B}'\text{)}$$

Now $\quad p + q = 71{\cdot}1;$ (A)

$$\therefore\quad p^2 + 2pq + q^2 = 5055{\cdot}2;$$

but (B), $\quad 4pq = 4800;\quad \therefore\ p^2 - 2pq + q^2 = 255{\cdot}2;\quad \therefore\ p - q = 16:$

adding to and subtracting from (A),

$$\therefore\quad 2p = 71{\cdot}1 + 16, \quad \text{and} \quad 2q = 71{\cdot}1 - 16; \quad \therefore\ p = 43{\cdot}55, \quad \text{and} \quad q = 27{\cdot}55.$$

77. At a point within a solid, a pair of conjugate stresses are 182 (tension) and 116 (thrust), common obliquity 30°. Find the principal stresses and the position of axes (fig. 52).

r' is negative

$$\frac{p+q}{2} = OM = 38 \cdot 14,$$

$$\frac{p-q}{2} = MR = 150 \cdot 3\,;$$

$$\therefore\ p = 188 \cdot 4 \text{ (thrust)},$$

and

$$\therefore\ q = -\,112 \cdot 2 \text{ (tension)},$$

$$\cos 2\theta = \cdot 7947\,;$$

$$\therefore\ \theta = 18^\circ\, 41'.$$

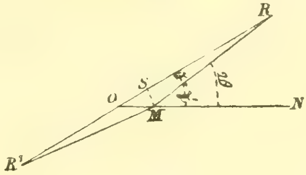

Fig. 52.

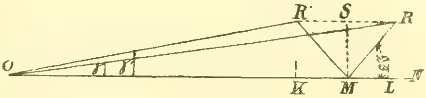

Fig. 53.

78. The stresses on two planes at right angles to each other being thrusts of 240 and 193 lbs. per square inch of obliquities, respectively, 8° and 10°. Find the principal stresses and their axes (fig. 53).

$$r_n = r\cos\gamma, \quad \text{and} \quad r'_n = r'\cos\gamma'\,; \quad \text{also} \quad r_t = r\sin\gamma = r'\sin\gamma'\,;$$

$$= 237 \cdot 6 \qquad\qquad = 190 \qquad\qquad = 33 \cdot 4$$

$$\frac{p+q}{2} = OM = \frac{r_n + r'_n}{2} = 213 \cdot 8,$$

$$\frac{p-q}{2} = MR = \sqrt{\left\{\left(\frac{r_n - r'_n}{2}\right)^2 + r_t^2\right\}} = \sqrt{(566 + 1115)} = 41\,;$$

$$\therefore\ p = 254 \cdot 8, \quad \text{and} \quad q = 172 \cdot 8\,; \quad \tan 2\theta = \frac{2r_t}{r_n - r'_n} = 1 \cdot 4034\,;$$

$$\therefore\ 2\theta = 54^\circ\, 32'\,; \quad \therefore\ \theta = 27^\circ\, 16'. \quad \text{(See Examples 74 and 57.)}$$

79. In Ex. 78 given that the 193 is a tension, find the principal stresses.

$$r = 240, \quad \text{and} \quad r' = -\,193,$$

$$r_n = 237 \cdot 6, \qquad r'_n = -\,190,$$

also $(r_t)^2$ is always positive,

$$\frac{p+q}{2} = OM = \frac{r_n + r'_n}{2} = 23 \cdot 8,$$

$$\frac{p-q}{2} = MR = \sqrt{\left\{\left(\frac{r_n - r'_n}{2}\right)^2 + r_t^2\right\}}$$

$$= \sqrt{\{(213 \cdot 8)^2 + (33 \cdot 4)^2\}} = 216 \cdot 3\,;$$

$$\therefore\ p = 240 \cdot 1, \text{ a thrust}, \quad \text{and} \quad q = -\,192 \cdot 5, \text{ a tension}\,;$$

$$\tan 2\theta = \frac{2r_t}{r_n - r'_n} = \frac{66 \cdot 8}{427 \cdot 6} = \cdot 1562\,; \quad \therefore\ 2\theta = 8^\circ\, 53'\,; \quad \therefore\ \theta = 4^\circ\, 26'.$$

CHAPTER IV.

APPLICATION OF THE ELLIPSE OF STRESS TO THE STABILITY OF EARTHWORK.

LOOSE earth, built up into a mass on a horizontal plane, will only remain in equilibrium with its faces at slopes, whose inclinations to the horizontal plane are less than an angle ϕ. If the earth be heaped up till the slope is greater, it will *run* till the slope is at greatest ϕ. Moist and compressed masses of earth can be massed up into a heap with slopes greater than ϕ, and will remain in equilibrium for some time, but will ultimately crumble down till the slopes do not exceed ϕ. The surface soil, which is in a compressed state, may be cut away, leaving banks with slopes much greater than ϕ. These banks will only remain in equilibrium for a time. Slips will occur till ultimately the slopes are not greater than ϕ.

This angle ϕ, which is the greatest inclination (of the slopes to the horizontal plane) at which a mass of earth will remain in equilibrium, is called the *angle of repose.* It has different values for different kinds of earth, and also different values for the same earth kept at different degrees of moistness. Average values of ϕ for different kinds of earth have been ascertained by experiment and observation, and are tabulated.

If two particles of earth are pressed together by a pair of equal thrusts p and p' normal to their surface of contact, it requires a pair of equal thrusts q and q' tangential to that surface to make them slide upon each other. For the same material, when q is just sufficient to make them slide, it is a constant fraction of p. The fraction which q requires to be of p just to cause slipping is called the *coefficient of friction* for that material. Hence the coefficient of friction

$$\mu = \frac{q}{p}.$$

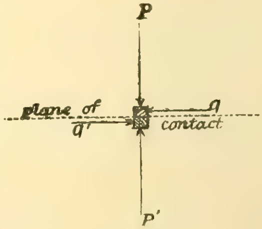

Fig. 54.

Figure 55 is a section of two troughs enclosing earth, and pressed together with a thrust of intensity p normal to MN, the plane where the troughs are just not in contact, and P is the amount of this thrust. A thrust of intensity q tangential to the plane MN tends to cause the earth to slide in two parts along MN, also Q is the amount of this thrust. If Q be just sufficient to cause slipping along MN, then the coefficient of friction of the earth is

$$\mu = \frac{Q}{P}.$$

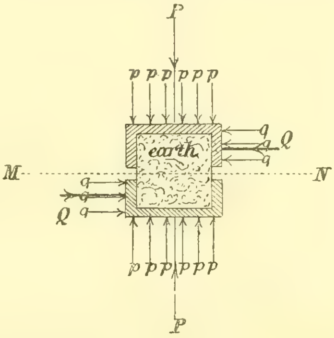

Fig. 55.

If on AB and CD there be a thrust of intensity p inclined at an angle ϕ to the normal, we know that for equilibrium of the prism $ABCD$ there must be a stress q upon the faces AC and BD, whose tangential component equals that of p; but as far as stability along the plane MN is concerned, we may neglect q, whose normal components destroy each other through the material of the trough, and the tangential ones are at right angles to MN. Considering the components of P, the amount of p, we have $P \cos \phi$ normal to MN. If slipping is just about to take place, then

$$\mu = \frac{P \sin \phi}{P \cos \phi} = \tan \phi.$$

It is apparent that ϕ is the same angle we were before considering; for, if P be due to the weight of the material, the figure ought to be turned till the direction of P is vertical, when MN, the plane of slipping, will be inclined at ϕ to the horizontal. The relation between the coefficient of friction and the angle of repose is

$$\mu = \tan \phi.$$

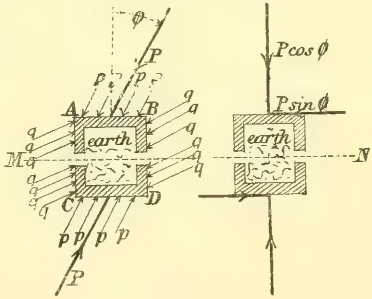

Fig. 56.

NOTE.—If it were not upon the supposition that the two troughs (being very rigid compared to the earth) transmitted the equal and opposite forces tangential to MN without causing lateral compression of the earth, we could not neglect q. From this result we learn that the tendency to slip along the plane MN, due to p, depends entirely upon the obliquity of p, and not at all upon its intensity. Thus, if p be inclined at an angle less than ϕ, slipping will not occur though p be ever so great; but, if p be inclined at an angle greater than ϕ, slipping will take place, though p be ever so small.

Consider now the equilibrium of a small prism at a point within a mass of earth in a compound state of strain. The earth will have a tendency to slip along any plane through the point (as there is no artificial envelope), except along the planes of principal stress at the point; and the tendency to slip will be greater along the plane upon which the resultant stress is more oblique, and greatest along the pair of planes upon which the *resultant stress is most oblique*, it being of no consequence how intense the stresses upon these various planes may be, but only how oblique. If the stresses upon the pair of planes, for which the resultant stress is more oblique than that upon any other plane through the point, be themselves less oblique than ϕ, no slipping will occur upon any plane through that point; but if more oblique than ϕ, slipping will take place along one or both of those planes.

The condition of equilibrium of a mass of earth in a compound state of strain is that, at every point, the obliquity of the stress on the plane upon which, of all others through the point, the resultant stress is most oblique, shall itself not be greater than ϕ.

Since earth can only sustain thrusts, the principal stresses at a point will be both thrusts which excludes case (c), and if γ be the obliquity of the resultant stress upon the plane through the point upon which the stress is most oblique, then by case (d) (fig. 37),

$$\sin \gamma = \frac{p-q}{p+q}; \quad \therefore \quad \frac{p}{q} = \frac{1+\sin \gamma}{1-\sin \gamma}.$$

By increasing γ, the numerator of the term on right-hand side of equation increases, while the denominator decreases, and so the ratio $\frac{p}{q}$ increases. But ϕ is the greatest value of γ for which equilibrium is just possible.

$$\therefore \quad \frac{p}{q} = \frac{1+\sin \phi}{1-\sin \phi}$$

is the greatest ratio of p to q consistent with equilibrium; hence

The condition of equilibrium of a mass of earth is most conveniently stated thus: that at every point the ratio of the greatest to the least principal stress shall not exceed that of $(1 + \sin\phi)$ *to* $(1 - \sin\phi)$.

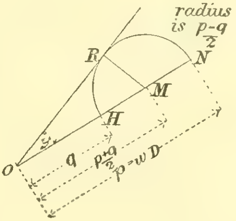

Fig. 57.

Or geometrically,

let $OM = \frac{p+q}{2}$, make $MOR = \phi$.

Drop MR perpendicular to OR.

Describe the semicircle HRN.

Because MOR = obliquity of thrust on plane which sustains most oblique strain,

and $ORM = 90°$.

$$MR = \frac{p-q}{2}. \quad \text{(See case (d), page 52.)}$$

Therefore $ON = (OM + MR) = p$,

and $OH = (OM - MR) = q$;

$$\therefore \frac{p}{q} = \frac{ON}{OH} = \frac{OM + MR}{OM - MR} = \frac{OM + OM\sin\phi}{OM - OM\sin\phi} = \frac{1+\sin\phi}{1-\sin\phi}.$$

For earth whose upper surface is horizontal, the vertical stress due to the weight and the horizontal stress are for *all* points the principal stresses, and their intensities are the same for all points on the same horizontal plane. Generally the vertical is the greater principal stress in any ratio not exceeding the above; whenever it exceeds the horizontal thrust by a greater ratio the earth *spreads*. But the horizontal thrust may be artificially increased till it exceeds the vertical in any ratio not exceeding the above. Whenever it exceeds the vertical by a greater ratio, the earth *heaves up*.

The third axis of principal stress, which we are all along neglecting, is also horizontal. When the earth is in horizontal layers with a horizontal surface, all vertical planes are symmetrical, and the three planes of principal stress are *any two* vertical planes at right angles to each other and the horizontal plane. The stress on the two vertical planes being equal, the *ellipsoid of stress becomes a spheroid.* When, however, the horizontal thrust on one vertical plane is artificially increased, that plane becomes *one* of the planes of principal stress, and the stress may be different on all three. See the definition of a *geostatic* load in Chapter XXII., the numerical examples there, and especially the quotations from Simms on *Tunnelling.*

Earth in horizontal layers loaded with its own weight to find the pressure against a retaining wall with vertical back.

Let

w = weight in lbs. of a cub. ft. of earth,

ϕ = its angle of repose,

D = depth of cutting.

Consider a layer 1 foot thick normal to paper, and choose a small rectangular prism at depth x feet.

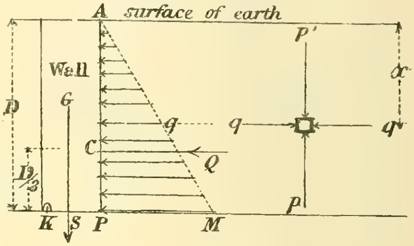

Fig. 58.

Let p = intensity of vertical pressure at depth x feet in lbs. per square foot

= weight of a volume of earth one square foot in section, x feet high

= wx lbs. per square foot.

If q = least horizontal stress which will give equilibrium, we have

$$\frac{p}{q} = \frac{1 + \sin\phi}{1 - \sin\phi}; \quad \therefore \quad q = \frac{1 - \sin\phi}{1 + \sin\phi} \cdot p.$$

$$q = \frac{1 - \sin\phi}{1 + \sin\phi}\, w \, . \, x \text{ lbs. per square foot}$$

= intensity of horizontal pressure on wall at depth x.

On the right side of equation all is constant but x; hence q is proportional to x, is zero at the top, and uniformly increases to

$$q = \frac{1 - \sin\phi}{1 + \sin\phi}\, w \, . \, D \text{ at the bottom,}$$

and therefore

$$\frac{1 - \sin\phi}{1 + \sin\phi} \cdot \frac{wD}{2} = \text{average intensity of pressure upon wall.}$$

And the area exposed to this pressure is D square feet. Hence the total pressure on wall is

$$Q = \text{average intensity of pressure} \times \text{area}$$
$$= \frac{1 - \sin \phi}{1 + \sin \phi} \cdot \frac{wD^2}{2} \text{ lbs.}$$

This tends to make the wall slide as a whole along MP; for equilibrium the weight of the wall, multiplied by the coefficient of friction at the bed-joint there, must be greater than Q.

If PM be laid off to represent the horizontal pressure at P, and M be joined to A, then MA gives the horizontal thrusts at all points as shown by arrows; Q, the resultant of all these, is horizontal, and passes through the centre of gravity of the triangle APM: it therefore acts at a point C called the centre of pressure, and

$$PC = \tfrac{1}{3} PA = \frac{D}{3}.$$

Q tends to overturn the wall with a moment

$$M = Q \times \text{leverage about } P$$
$$= Q \cdot \frac{D}{3} = \frac{1 - \sin \phi}{1 + \sin \phi} \cdot \frac{wD^3}{6} \text{ foot-lbs.}$$

Let K be the centre of the vertical pressure due to the weight of the wall and horizontal pressure of earth at the bed-joint at M; also let the vertical line, drawn through G the centre of gravity of the wall, cut the joint at S; then for equilibrium the moment, weight of wall × leverage KS, must be greater than M the overturning moment.

It is generally sufficient to ascertain if this lowest bed-joint be stable: but for some forms of wall it is necessary to go through all calculations for each bed-joint considered in turn as bottom of wall.

In a wall of uniform thickness throughout its height, the weight increases as D; whereas the force Q increases as D^2, and the lowest bed-joint is most severely taxed. Similarly, for overturning, KS being constant, the product, KS × weight of wall, increases as D while M increases as D^3. K would be the extreme outside of the wall if the material were perfectly strong; for stone retaining walls SK the distance from the middle of base of wall to the centre of pressure at that base is $\frac{3}{8}$ths or $\frac{3}{10}$ths of the thickness.

Depth to which the foundation of a wall must, at least, be sunk in earth laid in horizontal layers consistent with the equilibrium of earth.

Consider one lineal foot of wall, normal to paper.

V = vol. of wall in cub. ft.,
W = weight of wall per „
h = height of wall in feet,
b = breadth of wall in feet,
d = required depth of foundation,
w = weight per cubic feet of earth,
ϕ = its angle of repose.

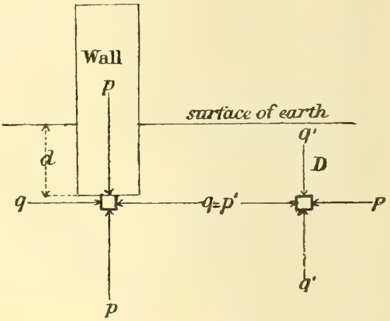

Fig. 59.

When the wall has just stopped subsiding, the earth on each side is on the point of heaving up, so at the horizontal layer at the depth of d, for points in contact with the bottom of foundation—p exceeds q in the greatest possible limit, that earth being on the point of spreading,

or
$$\frac{p}{q} = \frac{1 + \sin\phi}{1 - \sin\phi},$$

while, for points just clear of it, p' exceeds q' in that limit;

$$\therefore \quad \frac{p'}{q'} = \frac{1 + \sin\phi}{1 - \sin\phi}, \quad \text{and} \quad \frac{pp'}{qq'} = \left(\frac{1 + \sin\phi}{1 - \sin\phi}\right)^2.$$

Now $p' = q$, being horizontal thrust on same horizontal layer, cancel these and substitute the values

$$p = \frac{\text{weight of wall}}{\text{area exposed to } p} = \frac{WV}{b},$$

$$q' = \text{weight of column of earth} = wd; \text{ hence}$$

$$\frac{WV}{bwd} = \left(\frac{1 + \sin\phi}{1 - \sin\phi}\right)^2; \quad \therefore \quad d = \frac{WV}{wb}\left(\frac{1 - \sin\phi}{1 + \sin\phi}\right)^2 \text{ feet.}$$

Earth spread in layers at a uniform slope, and loaded with its own weight, to find the pressure against a retaining wall with a vertical back. Analytical Solution.

The simplest (commonest in practice) case is when the vertical face of wall is at right angles to the section showing greatest declivity of free surface. Let the paper be that section; then AB is the trace of the upper surface, and γ is its *greatest* inclination to the horizon.

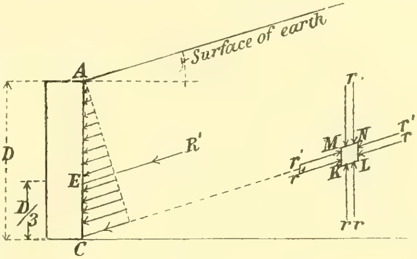

Fig. 60.

This inclination must be less than the angle of repose, or the earth would run over the wall. In an *extreme* case they may be equal.

Generally $\gamma < \phi$.

Take a slice one foot normal to paper; suppose the earth to be spread behind the wall in layers sloping at the angle γ, consider a small parallelepiped in the layer of depth D having vertical faces. At this depth D, the intensity in lbs. per square foot of the vertical pressure due to the weight of earth above, on *a horizontal surface*, would be the weight of a cubic foot of earth multiplied by the depth D in feet. Hence

$$w \,.\, D \text{ lbs. per square foot}$$

= intensity of vertical pressure on parallelepiped had its surface been horizontal; but the sloping surface MN is greater than the corresponding horizontal surface that supports the same earth; so the vertical stress thereon will be less than wD, (fig. 10, page 27), and will be

$$r = wD \cos \gamma \text{ lbs. per square foot.}$$

This is the intensity of the pressure upon the faces MN and KL, and its direction is vertical and therefore parallel to any pair of vertical faces of the parallelepiped; hence the pressure on any pair of vertical faces is in its turn parallel to the face MN; that is, *every vertical plane is conjugate to the free surface.*

Now, as we have selected the faces of *MNLK*, the pressure on the faces parallel to the paper when drawn parallel to the free surface will be horizontal, so that the stress normal to the paper is *a principal stress*, and the plane of the paper is the plane of the other two principal stresses. We can apply therefore our preceding results.

Let r' be the stress on the vertical faces *MK* and *NL*: it must be parallel to the free surface, and so its direction is that of the sloping layer, so that every point in that layer is in the same state of strain, and r' is transmitted along the layer to act on the wall.

To find out the ratio of the *pair of conjugate stresses* r and r' whose common obliquity is γ; from particular case (a) of the inverse problem (fig. 46, page 59), we have

$$\sqrt{\left\{\frac{(r+r')^2}{4\cos^2\gamma} - rr'\right\}} = \frac{p-q}{2}, \tag{B}$$

and

$$\frac{r+r'}{2\cos\gamma} = \frac{p+q}{2}; \tag{A}$$

squaring both, we have

$$\frac{(r+r')^2}{4\cos^2\gamma} - rr' = \frac{(p-q)^2}{4}, \quad \text{and} \quad \frac{(r+r')^2}{4\cos^2\gamma} = \frac{(p+q)^2}{4};$$

dividing, we have

$$1 - \frac{4rr'\cos^2\gamma}{(r+r')^2} = \left(\frac{p-q}{p+q}\right)^2.$$

But when earth is just in equilibrium,

$$\frac{p}{q} = \frac{1+\sin\phi}{1-\sin\phi}, \quad \text{or} \quad \frac{p-q}{p+q} = \sin\phi;$$

$$\therefore\ 1 - \frac{4rr'\cos^2\gamma}{(r+r')^2} = \sin^2\phi; \quad \therefore\ \frac{4rr'\cos^2\gamma}{(r+r')^2} = 1 - \sin^2\phi = \cos^2\phi;$$

$$\therefore\ \frac{4rr'}{(r+r')^2} = \frac{\cos^2\phi}{\cos^2\gamma}, \quad \text{or} \quad (r+r')^2 = 4rr'\frac{\cos^2\gamma}{\cos^2\phi}. \tag{I.}$$

Now identically $4rr' = 4rr'$.

Subtracting,

$$(r - r')^2 = 4rr'\left(\frac{\cos^2\gamma}{\cos^2\phi} - 1\right) = 4rr'\,\frac{\cos^2\gamma - \cos^2\phi}{\cos^2\phi}. \qquad \text{II.}$$

Dividing II. by I.,

$$\left(\frac{r - r'}{r + r'}\right)^2 = \frac{\cos^2\gamma - \cos^2\phi}{\cos^2\phi}\cdot\frac{\cos^2\phi}{\cos^2\gamma} = \frac{\cos^2\gamma - \cos^2\phi}{\cos^2\gamma};$$

$$\therefore \quad \frac{r - r'}{r + r'} = \frac{\pm\sqrt{(\cos^2\gamma - \cos^2\phi)}}{\cos\gamma};$$

$$\therefore \quad \frac{r}{r'} = \frac{\cos\gamma \pm \sqrt{(\cos^2\gamma - \cos^2\phi)}}{\cos\gamma \mp \sqrt{(\cos^2\gamma - \cos^2\phi)}}.$$

That is, r may be greater than r' in the ratio taken with the upper signs, when the earth is on the point of spreading, and the wall is subjected to the least possible value of r'; and again r may be less than r' in the ratio taken with the lower signs when the wall is artificially pressed against the earth till the earth is on the point of heaving up and the wall subjected to the greatest possible value of r'.

For equilibrium of retaining wall take upper signs, and reverse the proportion,

$$\frac{r'}{r} = \frac{\cos\gamma - \sqrt{(\cos^2\gamma - \cos^2\phi)}}{\cos\gamma + \sqrt{(\cos^2\gamma - \cos^2\phi)}}.$$

Cor.—In extreme case $\gamma = \phi$, and therefore $\frac{r'}{r} = 1$, or $r' = r$, *i.e.*, the conjugate thrusts are equal.

Substituting the value of r, we have the least intensity of the conjugate thrust at the depth D,

$$r' = wD\cos\gamma\,\frac{\cos\gamma - \sqrt{(\cos^2\gamma - \cos^2\phi)}}{\cos\gamma + \sqrt{(\cos^2\gamma - \cos^2\phi)}},$$

and its direction is parallel to the upper free surface.

On right-hand side of equation everything is constant except D, so that r' varies as the depth.

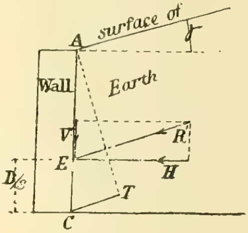

Fig. 61.

Let D be depth of vertical face of wall. Lay off CT to to represent r'. Join AT, and this locus will represent the thrust on the wall. The average intensity is $\frac{r'}{2}$, and the total thrust is

$$R' = \text{average intensity} \times \text{area exposed}$$

$$= \frac{r'}{2} \text{ lbs. per sq. ft.} \times D \text{ sq. ft.}$$

$$= \frac{wD^2}{2} \cos\gamma \frac{\cos\gamma - \sqrt{(\cos^2\gamma - \cos^2\phi)}}{\cos\gamma + \sqrt{(\cos^2\gamma - \cos^2\phi)}} \text{ lbs.},$$

and it is parallel to free surface and passes through the centre of gravity of the triangle ATC so that $EC = \frac{D}{3}$.

Resolving R' into horizontal and vertical components,

$$H = R' \cos\gamma, \qquad V = R' \sin\gamma.$$

H tends to make the wall slide as a whole along the bed-joint at C; and for equilibrium of the wall, weight of wall × coefficient of friction at bed-joint must be greater than H.

H tends to overturn the wall with a moment

$$M = H\left(\frac{D}{3}\right) = \frac{wD^3}{6} \cos^2\gamma \frac{\cos\gamma - \sqrt{(\cos^2\gamma - \cos^2\phi)}}{\cos\gamma + \sqrt{(\cos^2\gamma - \cos^2\phi)}} \text{ ft.-lbs.}$$

For equilibrium of wall, its weight multiplied by KC feet must exceed M. For position of K see fig. 58 and foot of page 75.

NOTE.—V, the tangential component of the pressure of earth on the back of wall multiplied by KC, tends to resist M and to increase effective weight of wall, but the friction of the earth there is liable to be destroyed by water lodging, and it is not always safe to rely on it.

Geometrical Solution.—$r = wD\cos\gamma$, being the vertical conjugate thrust, on a layer at depth D, due to the weight of the earth, to find in terms of r,

r', the conjugate thrust parallel to layer.
p and q, the principal stresses in the plane of paper.
$\theta - \gamma$, the inclination to the direction of r (*i.e.*, the vertical), of the axis of p.

And the third principal stress normal to plane of paper.

Construction of the Auxiliary Figure to Ellipse of Stress.

Let (fig. 62)

$$OM = \frac{p+q}{2}; \quad \text{make} \quad MOK = \phi,$$

the angle of repose of earth. Drop MK perpendicular on OK; then

$$MR = \frac{p-q}{2}. \qquad \text{(Case (d), p. 52.)}$$

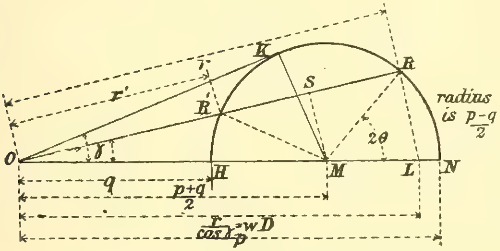

Fig. 62.

Draw semicircle

$$OH = OM - MH = q, \qquad ON = OM + MN = p.$$

Draw $OR'R$, making $NOR = \gamma$, the common obliquity of the conjugate thrusts r and r', and

$$OR = r, \qquad OR' = r'. \qquad \text{(Case } (a)\text{, p. 59.)}$$

The relations among those are easily expressed trigonometrically by supposing OM proportional to unity, when

OM prop. to 1; radius ρ prop. to $\sin\phi$;

OS prop. to $\cos\gamma$; MS prop. to $\sin\gamma$.

$RS = \sqrt{(MR^2 - MS^2)}$, or $\sqrt{(\rho^2 - MS^2)}$ (Euc. i. 47.)

prop. to $\sqrt{(\sin^2\phi - \sin^2\gamma)}$, or $\sqrt{(\cos^2\gamma - \cos^2\phi)}$.

p or $ON = OM + \rho$, prop. to $(1 + \sin\phi)$,

q or $OH = OM - \rho$, prop. to $(1 - \sin\phi)$,

$$r' \text{ or } OR' = OS - RS, \quad \text{prop. to } \{\cos\gamma - \sqrt{(\cos^2\gamma - \cos^2\phi)}\},$$

$$\text{and} \quad r \text{ or } OR = OS + RS, \quad \text{prop. to } \{\cos\gamma + \sqrt{(\cos^2\gamma - \cos^2\phi)}\}.$$

$$\therefore \begin{cases} \dfrac{r'}{r} = \dfrac{\cos\gamma - \sqrt{(\cos^2\gamma - \cos^2\phi)}}{\cos\gamma + \sqrt{(\cos^2\gamma - \cos^2\phi)}}, \\[2ex] \dfrac{p}{r} = \dfrac{1 + \sin\phi}{\cos\gamma + \sqrt{(\cos^2\gamma - \cos^2\phi)}}, \\[2ex] \dfrac{q}{r} = \dfrac{1 - \sin\phi}{\cos\gamma + \sqrt{(\cos^2\gamma - \cos^2\phi)}}. \end{cases}$$

The axis of p makes an angle $\theta = \frac{1}{2}RMN$, with ON the normal to the (sloping layer) plane upon which r acts and on the same side.

Also $$\cos 2\theta = \frac{2r\cos\gamma - p - q}{p - q}. \qquad \text{(Case } (a)\text{, p. 60.)}$$

Since the earth is upon the point of spreading, the principal stress normal to the paper will be the least possible, that is, it will be equal to q.

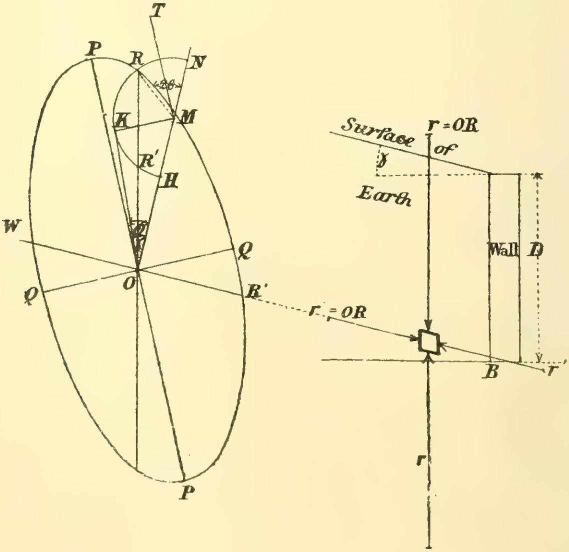

Fig. 63.

Hence this is the horizontal stress on vertical face of a wall running up the steepest declivity: that it is greater than the horizontal thrust for horizontal layers may be seen by supposing fig. 57 on page 73 to be drawn to such a scale that the line *ON* (which there represents the weight of vertical column of earth) will be of the same length as *OL* (which in fig. 62 represents same), and superimposing it upon this figure, *OH* there, will be seen to be shorter than *OH* here.

From a point *O* on layer at depth *D* draw *ON* (fig. 63) the normal to layer. Lay off *OM*, &c., complete construction as in last, but now in its proper position, properly oriented and to scale.*

Draw *OP* parallel to *MT* the bisector of *RMN*; this and *OQ* are the axes of the ellipse of stress parallel to plane of paper.

Lay off $OP = ON$ and $OQ = OH$, and draw ellipse; since *MR* is always less than *OM* for like principal stresses, $MRO > \gamma$, $\therefore NMR > 2\gamma$, $\therefore \theta > \gamma$, and *OP* is always in the acute angle *ROW* between the vertical and the line of greatest declivity, and making $(\theta - \gamma)$ with vertical.

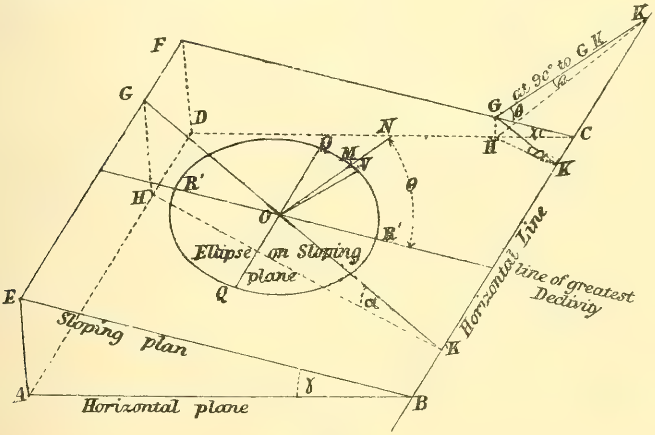

Fig. 64.

Since the third principal stress normal to paper is also *OQ*, then if the ellipse revolves about *POP* it will sweep out a *spheroid.* Its trace on a plane parallel to the free surface is the ellipse *R′QR′* (fig. 64), some vector of which is the stress on any vertical plane.

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* In Professor Malvarn A. Howe's *Treatise on Retaining Walls*, in which he adopts this method of Rankine's as developed by us, he compares Bauschinger's construction with that of fig. 63, p. 82, and shows that they are identical. Bauschinger's construction is very arbitrary, and fails to recommend itself as Rankine's does by rational steps readily remembered.

EXAMPLES.

80. The weight of a certain earth is 120 lbs. per cubic foot, its angle of repose 25°. It is spread in horizontal layers. Find the average intensity of the pressure against a retaining wall with vertical face and 4 feet in depth. Also, find total pressure against a slice of wall 1 foot in the direction of the length of the wall and the overturning moment of the earth about the lowest point.

$$p = 4w = 480 \text{ lbs. per square foot,}$$

$$q = \frac{1 - \sin\phi}{1 + \sin\phi} \cdot p = 194{\cdot}8 \text{ lbs. per square foot,}$$

$$\text{average pressure} = \tfrac{1}{2}q = 97{\cdot}4 \text{ lbs. per square foot,}$$

$$\text{total pressure } Q = 97{\cdot}4 \text{ lbs. per square foot} \times 4 \text{ square feet} = 389{\cdot}6 \text{ lbs.,}$$

$$\text{overturning moment } M = Q \text{ lbs.} \times \tfrac{4}{3} \text{ ft.} = 519{\cdot}5 \text{ ft.-lbs.}$$

81. Gravel is heaped against a vertical wall to a height of 3 feet; weight of gravel 94 lbs. per cubic foot; angle of repose 38°. Find horizontal thrust per lineal foot of wall, also overturning moment.

$$Q = 100{\cdot}5 \text{ lbs.}; \qquad M = 100{\cdot}5 \text{ ft.-lbs.}$$

82. A ditch 6 feet deep is cut with vertical faces in clay. These are shored up with boards, a strut being put across from board to board 2 feet from bottom at intervals of 5 feet apart. The coefficient of friction of the moist clay is ·287, and it weighs 120 lbs. per cubic foot. Find the thrust on a strut; also find the greatest thrust which might be put upon the struts before the adjoining earth would heave up.

Since $\tan\phi = {\cdot}287$; $\therefore\ \sin\phi = {\cdot}276$;

therefore $Q = 1225{\cdot}5$ lbs. per lineal foot.

Thrust per strut = 6127·5 lbs., just to prevent earth from falling in.

Greatest thrust which might be artificially put upon each strut before earth would heave up = 19029 lbs.

83. A wall 10 ft. high and 2 ft. thick, and weighing 144 lbs. per cubic foot, is founded in earth 112 lbs. per cubic foot, and whose angle of repose is 32°. Find least depth of foundation.

p = intensity of vertical pressure below bottom of foundation

$= 144 \times 10 = 1440$ lbs. per square foot,

q' = intensity of vertical pressure at same depth clear of foundation $= 112 \,.\, d$,

but $$\frac{q'}{p} = \left(\frac{1 - \sin\phi}{1 + \sin\phi}\right)^2; \quad \therefore\ \frac{d \,.\, 112}{1440} = {\cdot}094; \quad \therefore\ d = 1{\cdot}21 \text{ ft.}$$

NOTE.—The height of wall above ground is $10 - d = 8{\cdot}79$ ft.

84. An iron column is to bear a weight of 20 tons; the foundation is a stone 3 ft. square on bed, sunk in earth weighing 120 lbs. per cubic foot, angle of repose 27°. Find least depth to which it must be sunk for equilibrium.

$$p = \frac{44800 \text{ lbs.}}{9 \text{ sq. ft.}} = 4978 \text{ lbs. per square foot}; \quad q' = 120d,$$

but $$\frac{q'}{p} = \left(\frac{1 - \sin\phi}{1 + \sin\phi}\right)^2; \quad \therefore \quad d = \frac{4978}{120} \times \cdot 141 = 6 \text{ ft.}$$

85. A brick wall, allowing for openings, weighs 4000 lbs. per rood of 36 square feet (on an average one brick and a half), and stands 42 feet above ground; the foundation is to widen to four bricks at bottom. Find depth of foundation in clay weighing 130 lbs. per cubic foot (angle of repose 27°). 1st, neglect weight of unknown foundation.

WV = weight of 1 lineal foot of wall = 4667 lbs.,

$$p = \frac{WV}{b} = \frac{\text{weight of lineal feet}}{\text{area of base}} = \frac{4667 \text{ lbs.}}{3 \text{ sq. ft.}} = 1556 \text{ lbs. per square foot},$$

$$q' = 130 \,.\, d; \quad \text{but} \quad \frac{q'}{p} = \left(\frac{1 - \sin 27°}{1 + \sin 27°}\right)^2,$$

or $$\frac{130 \,.\, d}{1556} = \cdot 141; \quad \therefore \quad d = 1 \cdot 7 \text{ feet least depth.}$$

Say 2 feet deep by an average of 3 bricks thick, *i.e.*, 4½ cubic feet per lineal foot at 125 lbs. gives extra weight of 563 lbs. Adding this,

$$\therefore \quad WV = 5230 \text{ lbs.}; \quad \therefore \quad p = 1743; \quad \text{and} \quad d = \frac{1743}{130} \times \cdot 141 = 2 \text{ feet.}$$

86. A wall 9 feet high faces the steepest declivity of earth at a slope of 20° to the horizon; weight of earth 130 lbs. per cubic foot, angle of repose 30°. Find average intensity of thrust on wall, the total thrust on wall, the horizontal component of thrust, and the overturning moment of this component.

Data: $\gamma = 20°$, greatest slope of earth, $\phi = 30°$, angle of repose of earth,

$w = 130$ lbs. weight of cubic feet of earth, $D = 9$ feet depth of wall.

r = intensity of vertical stress at depth 9 feet per square foot of sloping surface

$= wD \cos\gamma = 130 \times 9 \times \cos 20° = 1099$ lbs. per square foot.

r' is the conjugate thrust whose direction is parallel to sloping surface when earth is just about to spread.

$$\frac{r'}{r} = \frac{\cos\gamma - \sqrt{(\cos^2\gamma - \cos^2\phi)}}{\cos\gamma + \sqrt{(\cos^2\gamma - \cos^2\phi)}} = \frac{\cdot 9397 - \cdot 364}{\cdot 9397 + \cdot 364} = \cdot 442;$$

$$\therefore \quad r' = 1099 \times \cdot 442 = 486 \text{ lbs. per square foot.}$$

Average intensity of conjugate stress = 243 lbs. per square foot.
Total thrust per lineal foot of wall

$$R' = \text{average intensity} \times \text{area} = 243 \times 9 = 2187 \text{ lbs.},$$

$$H = R' \cos\gamma = 2055 \text{ lbs.}, \qquad M = H \text{ lbs.} \times \frac{D}{3} \text{ ft.} = 6165 \text{ ft.-lbs.}$$

Weight of wall multiplied by coefficient of friction at lowest bed-joint (if horizontal) must equal H multiplied by a factor of safety. Weight of wall multiplied by $\frac{3}{4}$ths of half thickness at bottom must equal M multiplied by factor of safety. (Weight in lbs., thickness in feet.)

87. The slope of a cutting being one in one and a half, weight of earth being 120 lbs. per cubic feet, and its angle of repose 36°. Find average intensity, amount of horizontal component, and overturning moment of the thrust upon a 3 feet retaining wall at bottom of slope.

$$\tan\gamma = \frac{1}{1\,.\,5} = \cdot 6666\,; \quad \therefore\ \gamma = 33^\circ\,42', \quad \phi = 36^\circ, \quad \text{and} \quad w = 120\ \text{lbs.},$$

$$D = 3\ \text{feet}\,; \quad \therefore\ r = wD\cos\gamma = 299\ \text{lbs. per square foot.}$$

$$\frac{r'}{r} = \frac{\cos\gamma - \sqrt{(\cos^2\gamma - \cos^2\phi)}}{\cos\gamma + \sqrt{(\cos^2\gamma - \cos^2\phi)}} = \cdot 62\,; \quad \therefore\ r' = 299 \times \cdot 62 = 185\cdot 4,$$

and average intensity of stress = 92·7 lbs. per square foot,

$$R' = \frac{r'}{2} \times D\ \text{sq. ft.} = 278\ \text{lbs.}, \quad H = R'\cos\gamma = 231\cdot 6\ \text{lbs.},$$

$$M = H \times \frac{D}{3} = 231\cdot 6\ \text{ft.-lbs.}$$

88. A cutting having 3 feet retaining walls is made on ground sloping at 20° to the horizon. Weight of earth is 120 lbs. per cubic foot, and its angle of repose 30°. Find the horizontal thrust and the overturning moment—1st, when cutting runs horizontal; 2nd, when cutting runs up steepest declivity.

Data : $D = 3$ feet, $\gamma = 20^\circ$, $w = 120$ lbs., $\phi = 30^\circ$.

(1*st*) $r = wD\cos\gamma = 338$ lbs. per square foot

= stress on sloping layer at depth D, being vertical,

$$\frac{r'}{r} = \frac{\cos\gamma - \sqrt{(\cos^2\gamma - \cos^2\phi)}}{\cos\gamma + \sqrt{(\cos^2\gamma - \cos^2\phi)}} = \frac{\cdot 576}{1\cdot 304} = \cdot 442\,;$$

$\therefore$ $r' = 338 \times \cdot 442 = 149\cdot 4$ lbs. per square foot

= conjugate stress on vertical face of wall, being in sloping layer inclined at γ,

$r'\cos\gamma = 140\cdot 4$ lbs. per square foot

= horizontal thrust on wall, at foot of wall.

Average do. = 70·2.

Total do. = average intensity × area = 70·2 × 3 = 210·6 lbs. per lineal foot of wall.

Moment $= 210\cdot 6 \times \frac{D}{3} = 210\cdot 6$ ft.-lbs.

(2*nd*)

$$\frac{q}{r} = \frac{1 - \sin\phi}{\cos\gamma + \sqrt{(\cos^2\gamma - \cos^2\phi)}} = \frac{\cdot 5}{1\cdot 304} = \cdot 383\,;$$

$\therefore$ $q = 338 \times \cdot 383 = 129\cdot 3$ lbs. per square foot

= least principal stress in section on greatest declivity

= also third principal stress which is horizontal on face of vertical wall.

Average do. = 65 lbs. per square foot.

Total do. = 65 × area = 65 × 3 = 195 lbs. per lineal foot of wall.

Moment $= 195\ \text{lbs.} \times \frac{D}{3} = 195$ ft.-lbs.

CHAPTER V.

THE SCIENTIFIC DESIGN OF MASONRY RETAINING WALLS.

In treating this subject analytically, we will consider retaining walls as being built of blocks which touch each other at the joints, and which can exert pressure and friction, but not tension. Some cements are so strong that the whole structure may be considered as one piece, in which case arise questions of strength. In what follows we do not take account of this action of the cement, but consider the joints as being able to resist pressure only. The two conditions which must be fulfilled for a joint of this kind are: (1) The resultant pressure on the joint should fall well within that joint; and (2) the line of action of this pressure should not be inclined to the normal to the joint at an angle exceeding the angle of repose for masonry. When these two conditions are fulfilled, the joint is said to have stability of position and stability of friction.

In order to find the direction and amount of earth pressure on the wall, Rankine's method of the ellipse of stress is employed; and from the results obtained for earths whose natural slopes are $\phi = 30°$ and $\phi = 45°$, and whose free surfaces are horizontal and inclined at the natural slope ϕ, the thicknesses of walls of depth 20 feet are calculated.

The angle of repose for earth is its natural slope, and is the greatest inclination to the horizon at which its free surface will permanently remain; and we assume for earth what is true for a granular mass, that "It is necessary for stability that the direction of the pressure between the portions into which a mass of earth may be divided by any plane, should not at any point make with the normal to that plane an angle exceeding the angle of repose."

Proposition.—For every state of stress parallel to one plane, there are two planes perpendicular to each other, on each of which the stress is wholly normal. These two directions are called principal axes of stress; and the stresses, which are conjugate, are called principal stresses. A pair of stresses, each acting on a plane parallel to the direction of the other, are said to be conjugate.

Rectangular Wall.—$A\ C\ F\ K$, in fig. 65a, represents the wall in cross section; depth, $d = 20$ feet; length, $l = 1$ foot; it supports a bank of earth whose upper surface is horizontal, and whose natural slope is 30°.

Let w = weight of masonry = 140 lbs. per cubic foot,

w' = weight of earth = 120 lbs. per cubic foot,

ϕ = angle of repose of earth = 30°,

t = thickness of wall at base in feet.

In this case, r and r' are principal stresses, and for distinction may be replaced by p and q (p. 74, or fig. 49);

$$\frac{q}{p} = \frac{1 - \sin \phi}{1 + \sin \phi} = \frac{1}{3}. \qquad (1)$$

That is, the horizontal pressure at any point of $A\ C$, the back of the wall, is one-third of the vertical pressure due to that depth of earth. At C, the base of the wall, the vertical and horizontal pressures of earth are therefore

$$p = 20 \times 120 = 2400 \text{ lbs. per square foot,}$$

$$q = \frac{p}{3} = 800 \text{ lbs. per square foot.}$$

This amount q is represented by $C\ T$. By drawing $A\ T$, a triangle is formed; and the horizontal earth pressure at any point of $A\ C$ is given by the line drawn from the point to meet $A\ T$, and parallel to $C\ T$. The area of $A\ C\ T$ represents the total overturning force on $A\ C$ due to the earth pressure, and it may be taken as acting through the centre of gravity of $A\ C\ T$; thus

$$Q = \frac{800 \times 20}{2} = 8000 \text{ lbs.}$$

acting at E, $6\frac{2}{3}$ feet above the base C.

The ellipse of stress for a point at the average depth of 10 feet is drawn; OX, OY (fig. 65b) are the principal axes of stress; the semi-diameters represent p and q, now of half the values above; CC' represents a portion of the plane AC (fig. 65*a*), on which the intensity of pressure is required. ON is drawn at right angles to CC', and along it, OM is taken equal in length to $\frac{p+q}{2} = 800$ lbs.; $MR = \frac{p-q}{2} = 400$ lbs. is drawn so that $RMN = 2\theta$ where the angle $XON = \theta$; in this case $\theta = 90°$, and MR lies in MO; the point R thus found lies on the ellipse of stress, and the line RO represents the intensity and direction of pressure on CC' at the average depth of 10 feet.

To find the thickness of wall required for stability of position, let G (fig. 65*a*) be the centre of gravity of wall, and from G draw downwards a vertical line; produce the line of action of Q through E, and let H be the point of intersection; from H draw $HV = Q = 8000$ lbs., and $HD = W =$ weight of wall (not yet determined); complete the parallelogram of forces and draw the diagonal HZ, producing it, if necessary, to cut the base in L; draw LU at right angles to HV. The point L just found must lie within the base FC; and in order that the bed-joints near C the heel of the wall should not have any tendency to open nor to crush at F the toe, it is necessary to have the point L not further from the centre of FC than about $\frac{3}{10}$ths of that base; that is, the distance from centre of base of wall to the centre of pressure should not exceed $\cdot 3t$. Taking moments round L, we have

$$Q \times \overline{LU} = W \times \overline{LI}, \qquad (2)$$

$$w'd\left(\frac{1-\sin\phi}{1+\sin\phi}\right) \times \frac{d}{2} \times \frac{d}{3} = d \times t \times 140 \times \cdot 3t.$$

t is nearly 8 feet.

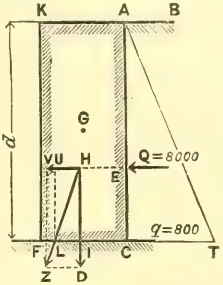


Fig. 65*a*.

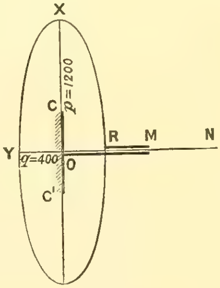


Fig. 65*b*.

For stability of friction, considering the masonry built in horizontal courses, the angle LHI should not exceed the angle of repose for masonry—38°. Now,

$$\tan LHI = \frac{Q}{W} = \frac{8000}{22400} = \cdot 357;$$

$< LHI$ is 20° nearly, a quantity well within the assigned limit for friction.

Trapezoidal Wall (fig. 66*a*).—Front and back of wall inclined at 80° to horizon; upper surface of earth horizontal. The ellipse of stress (fig. 66*b*) for a point 10 feet deep is drawn in position, and is similar to that shown in fig. 65*b*; CC' is now inclined at 80° to the horizon. When the triangle OMR is constructed as described for the previous case, $OR = r = 448{\cdot}6$ lbs. per square foot, and $\gamma = ROM = 17° \, 52'$. The stress on AC (fig. 66*a*) is represented by the triangle ACT; it is zero at A, and increases to $CT = 897{\cdot}2$ lbs. per square foot at base of wall; $AC = 20{\cdot}3$ feet, and the total pressure on AC is $R = 9106$ lbs.

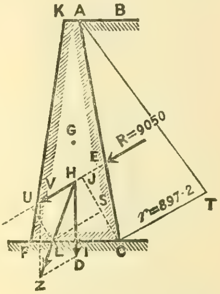


Fig. 66*a*.

Let LU and CJ be perpendiculars on the line of action of R, and LS be perpendicular to CJ; then taking moments round L, as before, we have

$$R \times \overline{LU} = W \times \overline{LI}, \qquad (2)$$

observing that

$$LU = CJ - CS,$$

$$R \times (CE \cos 17° \, 52' - {\cdot}8t \sin 27° \, 52')$$
$$= (t - 3{\cdot}52) \times 20 \times 140 \times {\cdot}3t,$$

from which

$$t = 8{\cdot}2 \text{ ft.} = \text{thickness at base of wall,}$$
$$t - 3{\cdot}52 \times 2 = 1{\cdot}2 \text{ ft.} = \text{thickness at top of wall.}$$

For this case the angle $LHI = 25°$, a quantity less than the angle of repose for masonry.

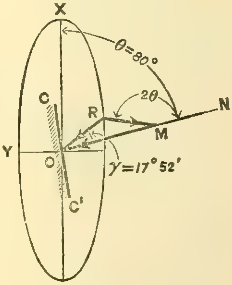


Fig. 66*b*.

Surcharged Rectangular Wall.—The earth is surcharged at its natural slope $\phi = 30°$ (fig. 67*a*), and the conjugate pressures are equal (*cor.*, p. 79). The ellipse of stress (fig. 67*b*) is drawn for a point 10 feet deep, and it may be noted that the major axis of the ellipse is midway between the directions of r and r'; that is, the major diameter is inclined to the vertical at the angle

$$45° - \frac{\phi}{2} = 30°.$$

The intensity of the earth pressure on a horizontal surface at a depth d, due to the weight of the column above it, is wd; on a plane inclined to the horizon at the angle ϕ, the intensity is diminished to $wd \cos \phi$; and thus the intensity of the two equal conjugate pressures r and r', for a point 10 feet deep, is 1039·2 lbs. per square foot.

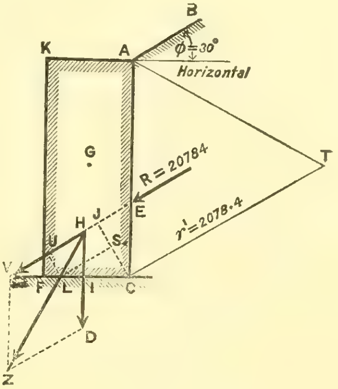


Fig. 67a.

In drawing the triangle OMR, proceed as in first case:

$$\angle XON = \theta = 60^\circ;$$

$$\angle RMN = 2\theta = 120^\circ;$$

$$\angle ROM = \gamma = 30^\circ;$$

$$OR = 1039{\cdot}2;$$

$$OM = \frac{p+q}{2} = 1200;$$

$$MR = \frac{p-q}{2} = 600$$

as found by calculation or graphic construction; from which we have $p = 1800$, $q = 600$ lbs. per square foot as the greatest and least conjugate stresses at the point O, that is, at a point 10 feet deep. The triangle ACT (fig. 67a) represents the pressure on AC as before; $CT = 2078{\cdot}4$ lbs. per square foot at base, and the total earth pressure on AC is $R = 20784$ lbs. Taking moments round L,

$$R \times \overline{LU} = W \times \overline{LI}, \qquad (2)$$

and, as before,

$$LU = CJ - CS = CE \cos 30^\circ - {\cdot}8t \sin 30^\circ,$$

$$W = 140td, \quad LI = {\cdot}3t,$$

$$20784\,(5{\cdot}77 - {\cdot}4t) = 840t^2, \quad t = 8 \text{ ft.}$$

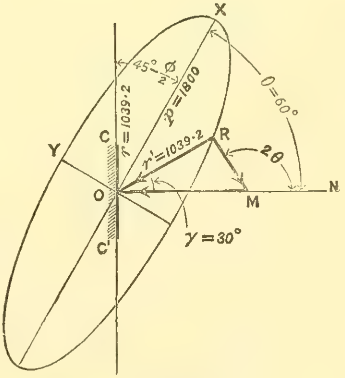


Fig. 67b.

The angle $LHI = 29°$, a quantity only a few degrees within the assigned limit for friction.

Surcharged Trapezoidal Wall.—In fig. 68*a* the front and back of wall are inclined at 80° to the horizon; the earth is surcharged at its natural slope $\phi = 30°$. The ellipse of stress (fig. 68*b*) for a point 10 feet deep is drawn in position, and is similar to that shown in fig. 67*b*; CC' is now inclined at 80° to the horizon. When the triangle OMR is constructed as described for fig. 65*b*, $OM = 1200$, $MR = 600$, $RO = 1245$ lbs., $<XON = \theta = 50°$, $RMN = 2\theta = 100°$, $ROM = \gamma = 28° \, 20'$.

In fig. 68*a*, the earth pressure on AC is represented as before by the triangle ACT; $CT = 2490$ lbs. per square foot at base of wall, and the total earth pressure $R = 25286$ lbs. inclined at 38° 20′ to the horizon.

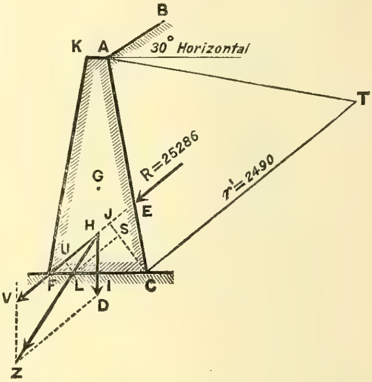


Fig. 68*a*.

Taking moments round L, we have

$$R \times \overline{LU} = W \times \overline{LI}, \quad (2)$$

$$LU = CJ - CS = 6{\cdot}77 \cos 28° \, 20' - {\cdot}8t \sin 38° \, 20',$$

$$W = (t - 3{\cdot}52) \times 20 \times 140,$$

$$t = 8{\cdot}9 \text{ ft.} = \text{thickness at base},$$

$$t - 3{\cdot}52 \times 2 = 1{\cdot}8 \text{ ft.} = \text{thickness at top of wall.}$$

The angle $LHI = 33°$, a quantity exactly equal to the angle of repose for masonry; the courses of masonry should therefore have their bed-joints dipping from front to back of wall at an angle of say 10° to the horizon.

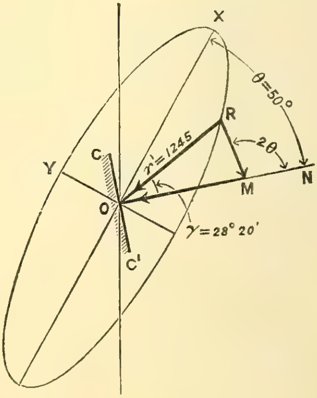


Fig. 68*b*.

Trapezoidal Wall.—In fig. 69 the back of the wall is vertical, and the face batters; upper surface of earth is horizontal. The wall here represented is that shown in fig. 65*a* with the wedge, whose cross-section is $KK'F$ (fig. 69), removed. The centre of gravity of the triangle $KK'F$ is vertically above L, and the moment of stability of the wall is not altered if this wedge be removed. To find the position of K in $K'A$, where K' is vertically above F, take $K'K = 3FL = \cdot 6t$.

The thickness of the wall will therefore be

$$t = 8 \text{ ft. at base,} \quad t - \cdot 6t = 3\cdot 2 \text{ ft. at top.}$$

The angle $LHI = 27°$, a few degrees within the limit for friction.

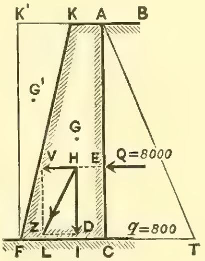


Fig. 69.

Battering Wall of Uniform Thickness.—In fig. 70, the face and back of wall incline backwards, making an angle $a = 10°$ with the vertical; earth surface is horizontal. If we suppose AC the back of the wall to be made up of a number of rectangular steps, vertical and horizontal, the horizontal earth pressure on a vertical face at a depth d will be, as before,

$$w'.d.\frac{1-\sin\phi}{1+\sin\phi};$$

this horizontal pressure will tend to cause the earth to spread upwards, and the vertical earth pressure on a horizontal face looking downwards at depth d will be

$$w'.d.\frac{1-\sin\phi}{1+\sin\phi}.\frac{1-\sin\phi}{1+\sin\phi} = w'.d.\left(\frac{1-\sin\phi}{1+\sin\phi}\right)^2.$$

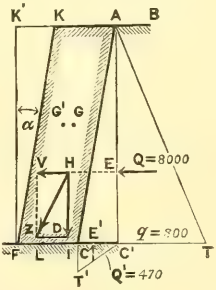


Fig. 70.

The straight line AC may now be considered as the limit of these steps, and the horizontal and vertical pressures on AC will be represented by the triangles $AC'T$ and $C'CT'$; the horizontal pressure is zero at A, and increases to $C'T = 120 \times 20 \times \frac{1}{3} = 800$ lbs. per square foot at the base; the vertical pressure is zero at A, and increases to $CT' = 120 \times 20 \times \frac{1}{9} = 267$ lbs. per square foot at the base. The total horizontal pressure on AC is $Q = 8000$ lbs., as for fig. 65a; the total vertical pressure on AC is

$$Q' = \tfrac{1}{2}\overline{CC'} \times \overline{CT'} = \tfrac{1}{2} \times 3\cdot 52 \times 267 = 470 \text{ lbs.},$$

and the points of application are E and E'.

Taking moments round L,

$$W \times \text{leverage} = Q \times \text{leverage} + Q' \times \text{leverage}$$

$$w\,.\,d\,.\,t\left(\cdot 3t + \frac{d}{2}\tan a\right) = w'd\,\frac{1 - \sin\phi}{1 + \sin\phi}\,\frac{d}{2} \times \frac{d}{3}$$

$$+ w'.d.\left(\frac{1 - \sin\phi}{1 + \sin\phi}\right)^2 \frac{d\,.\tan a}{2}\,(\cdot 8t + \tfrac{1}{3}d\tan a),$$

taking $w = 140$, and $w' = 120$ lbs. per cubic foot, $\phi = 30°$, and $d = 20$ feet,

$$t = 8\sqrt{1 + 4\tan^2 a} - 15{\cdot}4\tan a. \tag{3}$$

The vertical through G, the centre of gravity of wall, must fall within the base; and if it be not allowed to deviate further from the centre of the base than $\frac{3}{10}$ths of the breadth of base, we obtain

$$\tan a = \frac{\cdot 3t}{\frac{1}{2}d} = \cdot 03t, \quad \text{when} \quad d = 20 \text{ feet};$$

putting this value in equation (3), we get

$$t = 8(1 + 4 \times \cdot 0009t^2)^{\frac{1}{2}} - 15{\cdot}4 \times \cdot 03t,$$

$$t = 5{\cdot}78 \text{ ft.} = FC, \text{ the thickness of wall.}$$

The angle a is 10° nearly; that is to say, the back of the wall should not be inclined to the vertical at an angle greater than 10°.

In fig. 70, G' is the centre of vertical forces, viz., the downward weight of the wall, and the upward pressure of earth; HD is the vertical drawn through G', and is equal to $W - Q'$ or 15713 lbs.

The angle $LHI = 27°$, a few degrees less than the limit for friction.

Wall with Vertical Face and Stepped Back.—In fig. 71, the steps are taken at vertical intervals of 5 feet; the upper surface of earth is horizontal. The base of the wall FC supports the masonry $A'CFK$, and the earth $A'AC$ vertically above that base; and the triangle ACT represents the horizontal earth pressure on the back of the wall; G is the centre of gravity, and W is the weight of masonry *and* earth vertically above FC.

Taking moments round L as before,

$$Q \times \overline{LU} = W \times \overline{LI}. \tag{2}$$

To find the thickness of upper 5 feet of wall, proceed as for fig. 65*a*; and obtain $t_5 = 2$ feet.

For t_{10}, thickness at 10 feet deep, we have $Q_{10} = 2000$ lbs. acting at $\frac{10}{3}$ foot above this assumed level; and taking moments round the point corresponding to L

$$Q_{10} \times \tfrac{10}{3} = 10 \times t_5 \times 140 \times (\cdot 5t_5 - \cdot 2t_{10})$$
$$+ 5(t_{10} - t_5)\,(140 + 120)\,(\cdot 2t_{10} + \cdot 5t_5)$$

from which $t_{10} = 4\cdot 1$ ft.; similarly $t_{15} = 6\cdot 3$ ft.; and $t_{20} = 8\cdot 4$ ft., the thickness at base.

The angle $LHI = 20°$, a quantity well under the limit for friction.

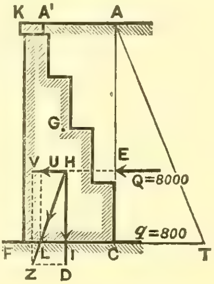


Fig. 71.

Wall with Battering Face and Stepped Back.—In fig. 72 the steps are taken 2 feet wide, and at vertical intervals of 5 feet; the batter of face is 1 in 12, and the earth is surcharged at its natural slope $\phi = 30°$. The ellipse of stress (fig. 67*b*) applies to this case; the depth AC is increased to 23·5 feet by the earth slope; the pressure at base, $CT = 2440$ lbs. per square foot; the total earth pressure on AC is $R = 28600$ lbs. acting at E. On account of the battering face, the point F projects 1·7 feet beyond the vertical through K; and on account of the steps of the back of wall, the base projects 6 feet beyond the vertical through A'; the thickness of wall at base may be represented thus

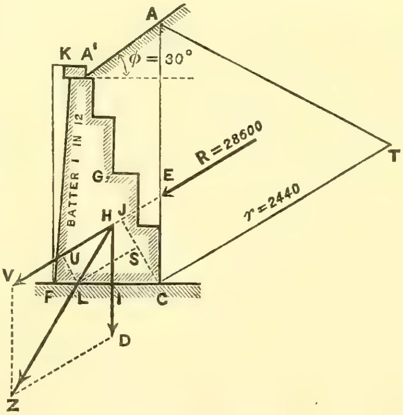


Fig. 72.

$$t = 1\cdot 7 + t' + 6 = t' + 7\cdot 7 \text{ ft.}$$

In order to find the value of t', take moments round F, thus:—

Weight of masonry $= 2800t' + 10780$.

Moment of masonry round $F = 1400t'^2 + 13160t' + 36500$.

Weight of earth vertically over $FC = 8400$.

Moment of earth round $F = 8400 + 44800$.

Moment of earth and masonry round $L = 840t'^2 + 13436t' + 51950$.

Equating this to

$$R \times \overline{LU} = 28600\ \{6{\cdot}77 - {\cdot}4\ (t' + 7{\cdot}7)\},$$

we get $t' = 2$ feet, $t = 9{\cdot}7$ feet, thickness of wall at base.

The angle $LHI = 33°$, a quantity exactly equal to the angle of repose for masonry; the courses of masonry should have their bed-joints dipping from front to back of wall at an angle of, say, 10° to the horizon, or at right angles to FK the battering face.

Tabulated Dimensions.

Since the stability of a wall is proportional to $d \,.\, t^2$, where d = depth and t = thickness, and since the overturning pressure of earth is proportional to d^3, it follows that the thickness of a wall should be proportional to its depth. Having calculated the thickness for a given depth $d = 20$, it is easy to fix on the thickness required for any other depth. If the depth of wall be taken as 100, the accompanying table gives the other dimensions of walls, as deduced from the results given in this Chapter, when the earth has for its angle of repose $\phi = 30°$, or $\phi = 45°$, or slopes of 1 vertical in 1·73 horizontal, and 1 in 1.

For comparison, corresponding values are given for water whose heaviness is 62·5 lbs. per cubic foot.

It will be observed that the thickness of a wall for resisting water pressure is much greater than for the varieties of earth considered; this, of course, is caused by ϕ becoming zero in the case of water. It is, therefore, of the utmost importance that the earth behind a retaining wall should be kept dry; and for this purpose weeping holes through the walls are formed near the level of the original surface of ground, and a dry stone backing from 12 inches to 18 inches thick is laid behind the wall for conveying the water easily to the weeping holes.

SPREAD AND DEPTH OF FOUNDATION.

Because of the obliquity of the downward thrust on the foundation, the concrete or masonry forming the substantial part of it must spread out in front of the wall in steps as it goes deeper. The bottom of the trench may have to be consolidated by driving *packing piles*, or *bearing piles* may have to be driven to a firm stratum, and surmounted by a staging generally dipping back as the friction is then also precarious. In old consolidated earth two conditions serve to determine the *spread* and *depth* of the trench (see first wall, fig. 75). The total vertical thrust of the bed of the trench on the base of the concrete must equal the weight of the wall and the concrete. Also the centre of this upward thrust must be vertically below (F on fig. 75, but L on figs. 72, 71) the assumed centre of stress at the lowest bed-joint of the wall. But besides these two conditions which concern the equilibrium of the wall we have the limiting conditions concerning the equilibrium of the earth surrounding the concrete, namely, the upward thrust of the unit-cube under the toe of the concrete is at most *nine* times a column of earth the depth of the bed of trench below the surface of the earth in front of wall, which is then on the point of *heaving-up*. And for the unit-cube at heel of trench, it is as a practical lower limit, *one-third* of the column of earth behind the wall.

The two equations obtained can best be solved by trial and error. Taking the height of the wall as sensibly $3t$, where t is its thickness, we will try the *spread* 45 per cent. of the thickness, and the depth 20 per cent. of the depth, or 60 per cent. of the thickness.

Weight of wall and concrete is $wt^2(3 + 1{\cdot}45 \times {\cdot}6) = 3{\cdot}87wt^2$. Upward stress at toe of trench is $9 \times {\cdot}6w't = 4{\cdot}32wt$; while at the heel it is $\frac{1}{3}(3 + {\cdot}6)w't = {\cdot}96wt$. Multiplying their average value by $1{\cdot}45t$, we get the total upward stress $= 3{\cdot}83wt^2$ which satisfies the first condition. If the centre of this upward stress be distant y_a and y_b from the heel and toe of bed of trench respectively, then for the centre of gravity of the trapezium of stress drawn below first wall, fig. 75, we have, as explained at fig. 191,

$$y_a : y_b :: (4{\cdot}32 + \tfrac{1}{2} \times {\cdot}96) : (\tfrac{1}{2} \times 4{\cdot}32 + {\cdot}96), \text{ and } y_a + y_b = 1{\cdot}45t,$$

so that $y_a = {\cdot}88t$ or $\frac{7}{8}t$ nearly, which satisfies the second condition, as F on fig. 75 is at most $\frac{7}{8}t$ from back of wall.

TABLE OF THICKNESSES REQUIRED FOR MASONRY RETAINING WALLS OF MODERATE HEIGHTS NOT EXCEEDING ABOUT SIXTY FEET.

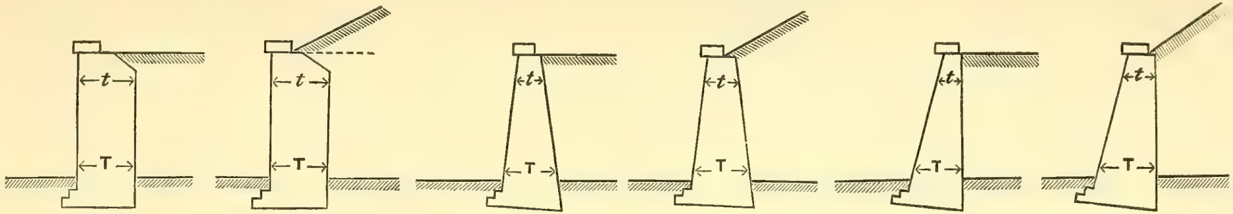
PROFILE.	Earth—Natural slope, $\phi = 30°$.				Earth—Natural slope, $\phi = 45°$.				Water.	
	Earth level on top.		Earth sloping at $\phi = 30°$.		Earth level on top.		Earth sloping at $\phi = 45°$.			
	T	t	T	t	T	t	T	t	T	t
A	40	40	45	45	30	30	35	35	50	50
B	40	2*	45	2*	30	2*	35	2*	55	2*
C	40	16	45	18	30	12	35	14	50	20
D	30	30	35	35	25	25	30	30	45	45
E	45	2*	50	2*	35	2*	40	2*	65	2*
F	45	2*	50	2*	35	2*	40	2*	65	2*

Height of wall 100, measured from actual surface of ground.

T = thickness at base of wall. t = thickness at top of wall. 2* = 2 ft., an assumed minimum thickness of wall.

Centre of stress a fifth of T in from face of wall.

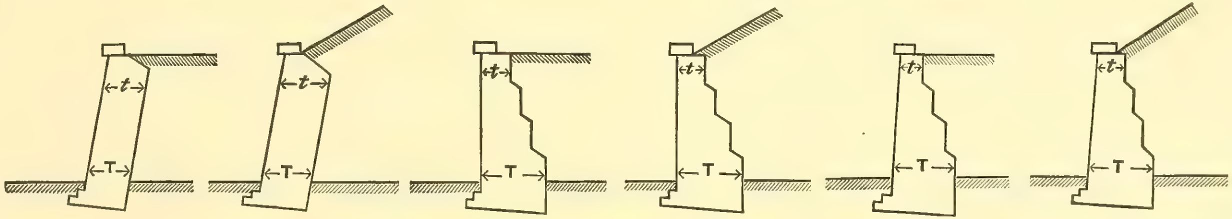
PROFILES OF MASONRY RETAINING WALLS.



A—Of uniform thickness.

B—With front and back at the same inclination to vertical.

C—With battering face and vertical back.

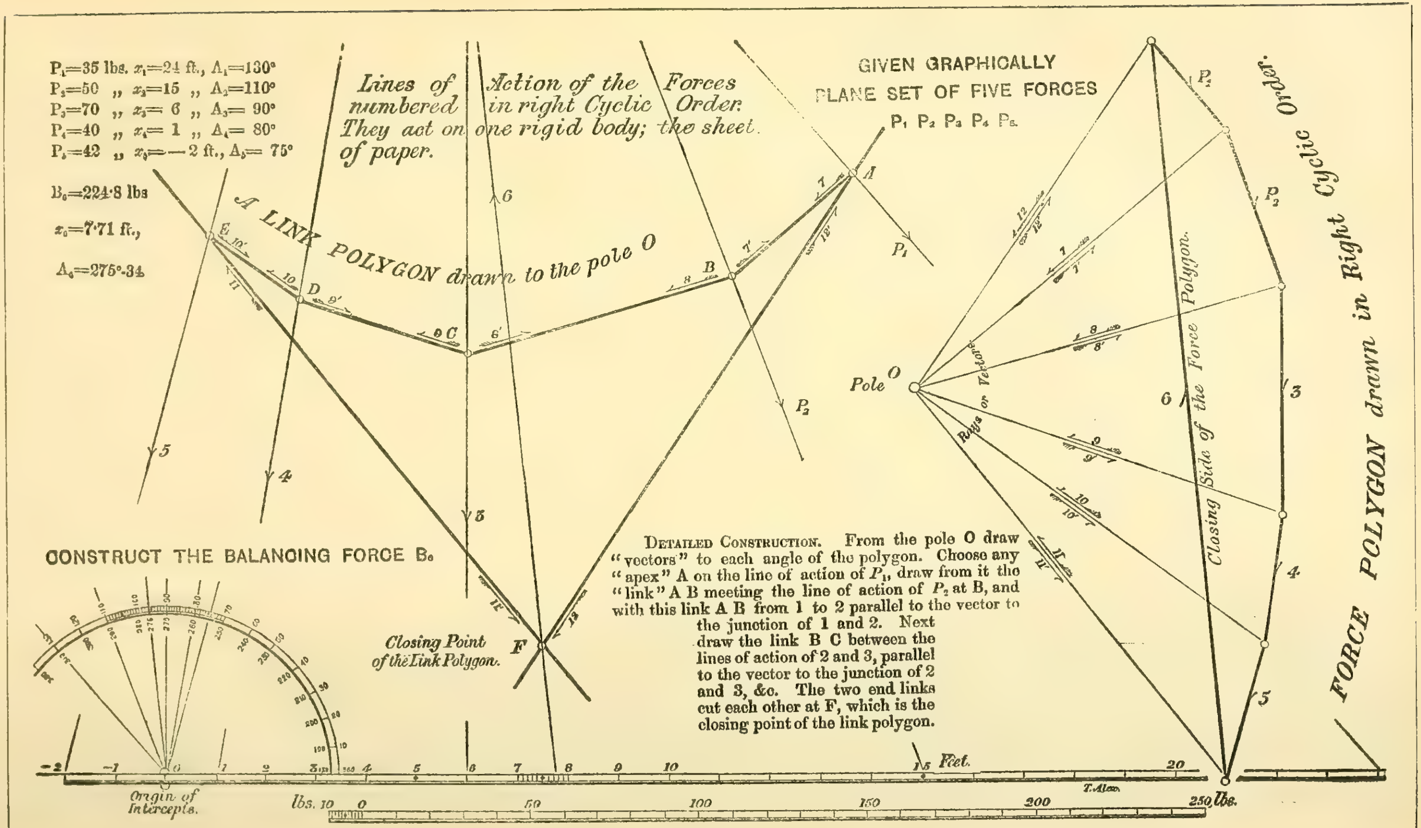


D—Leaning back, with face and back parallel and battering at 10°.

E—With vertical face and stepped back.

F—With face battering at 1 in 12, and stepped back.

Fig. 73.



$P_1=35$ lbs. $x_1=2\frac{1}{4}$ ft., $A_1=130°$
$P_2=50$ „ $x_2=15$ „ $A_2=110°$
$P_3=70$ „ $x_3=6$ „ $A_3=90°$
$P_4=40$ „ $x_4=1$ „ $A_4=80°$
$P_5=42$ „ $x_5=-2$ ft., $A_5=75°$
$B_0=224{\cdot}8$ lbs
$x_0=7{\cdot}71$ ft.,
$A_0=275°{\cdot}34$
Lines of numbered Action of the Forces in right Cyclic Order. They act on one rigid body; the sheet of paper.
GIVEN GRAPHICALLY
PLANE SET OF FIVE FORCES
P_1 P_2 P_3 P_4 P_5
A LINK POLYGON drawn to the pole O
A
B
C
D
E
F
P_1
P_2
Pole O
Rays or Vectors
Closing Side of the Force Polygon.
FORCE POLYGON drawn in Right Cyclic Order.
CONSTRUCT THE BALANCING FORCE B_0
Closing Point of the Link Polygon.
Origin of Intercepts.
Feet.
lbs.
T. Alex.

Detailed Construction. From the pole O draw "vectors" to each angle of the polygon. Choose any "apex" A on the line of action of P_1, draw from it the "link" A B meeting the line of action of P_2 at B, and with this link A B from 1 to 2 parallel to the vector to the junction of 1 and 2. Next draw the link B C between the lines of action of 2 and 3, parallel to the vector to the junction of 2 and 3, &c. The two end links cut each other at F, which is the closing point of the link polygon.

Fig. 74.

GRAPHICAL SOLUTION.

Graphic statics is a study by itself, and in this book we only intend to use the very first elements.

In graphic statics a force is represented by a pair of lines; one a short thick finite line giving the magnitude of the force upon a scale of forces; the ends of this thick line are marked by the centres of two little rings; in a hand-made drawing the centres are pricked on the paper. The other is a long thin line giving the actual position of the force on the plane of the paper relative to some structure or solid body upon which the force acts, and relative to other forces also acting on it. It will be seen then that a second scale of feet is necessary to measure the distances along the solid body, and among the forces. These force lines are furnished with a barb or arrow-head to indicate the sense of the force in that line; near the barb stands a numeral which is really the suffix of a letter such as P_3; the force is called 3 when simply being referred to, but when speaking about it as a quantity it is P_3. Often the force line is so long that it reaches to the edge of the paper, and although only a portion may be ultimately inked in, still a little left at each margin may be indicated. The line is fine and long for the practical purpose of setting the rollers or T square accurately parallel to it. A plane set of forces all acting on one body, which may at first be assumed to be the sheet of paper, are numbered in *any* order; but for a successful issue they must be numbered in *cyclic order*. The thick lines drawn parallel to the thin, each to each, form a polygon round which the corresponding numerals, printed heavier or larger, run consecutively; for some purposes it is convenient to put half barbs on the thick lines. This is the "Polygon of Forces." When the forces are all vertical the sides would lap on each other, but this is avoided by slightly displacing some, and drawing a polygon which often looks like a gridiron pendulum; the "eyes," however, should all be in one vertical line. In this case, the downward forces being all vertical, the polygon is often called "the load line"; while the closing upward sides constitute the reactions or supports.

Given, graphically, fig. 74, a plane set of forces to construct the balancing force or resultant. On the upper left-hand corner is shown an analytical definition of five plane forces in terms of intercepts and angles and lbs., agreeable to which the force polygon and lines of action are drawn to a pair of scales.

CONSTRUCTION*.—Draw a line closing the "force polygon," and scale off its length in lbs. on the force scale for the magnitude of B_6. Reckon its "sense" in the same order round the force polygon as the other forces. Choose any pole O (not on the closing side of the force polygon), and, to that pole, construct a "link polygon" $A\ B\ C\ D\ E\ F$ among the lines of action. Through F, the closing point of the link polygon, draw the line of action of B_6 parallel to the closing side of the force polygon. When scaled off, the values of B_6, x_6, and A_6 should be nearly the same as the values given on fig. 74 which were calculated by trigonometry.

PROOF.—Consider the set of six forces, 1, 2, 3, 4, 5, 6; they are a balanced set of forces. For, add a pair of balanced forces 7 and 7′ equal and opposite to each other, having the link $A\ B$ for their common line of action, and having the vector, which comes to the junction of 1 and 2, for their common magnitude. Add another pair of balanced forces 8 and 8′, with $B\ C$ for their common line of action, and the vector, coming to the junction of 2 and 3, for their common magnitude. Add in the same way the pairs of forces 9 and 9′, 10 and 10′, 11 and 11′, 12 and 12′. We have now altogether a set of eighteen forces 1, 7, 12′; 2, 8, 7′; 3, 9, 8′; 4, 10, 9′; 5, 10′, 11; 6, 12, 11′, which are exactly an equivalent set to the set of six forces with which we began, since all the forces we added balanced in pairs. Now, of the set of eighteen, the first group of three, 1, 7, 12′, act at the same point A and have magnitudes proportional to the homologous sides of a triangle; they are, therefore, a balanced set of three; similarly the second group of three, 2, 8, 7′, act at one point *B* and have magnitudes proportional to the three homologous sides of the triangle. In the same way each group of three is balanced; hence the set of eighteen is balanced; hence the original set of six is balanced.

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* The student should make the construction two or three times on a scroll copy pricked through, with different coloured pencils, taking different positions of the pole and starting the link polygon from different points A on P_1. He will satisfy himself that he gets the same final result. He should note that some positions of the pole and initial point A throw the construction off the paper; others cause the link polygon to loop awkwardly upon itself, and that the construction fails altogether, for the pole on the closing side of the force polygon, simply because the two end links are then parallel and do not meet to define F. He will observe that the position of the pole chosen on the accompanying solution gives what is called *good definition*, that is, the points A, B, &c., are defined by drawing links cutting the lines of action nearly at right angles, and that the two end links define F by cutting each other at an angle greater than 60°. He will thus learn that a good choice of the pole is of *great practical importance*, although theoretically it may be chosen anywhere. He should recognise further that the force polygon might have been drawn in any order, but because it is of the *greatest practical importance* that it should be drawn in *cyclic order*; for reasons which will develope themselves later on, the idea of cyclic order is brought prominently before his notice even in this first exercise.

Cor.—The graphical conditions of equilibrium of a plane set of forces are two in number. The force polygon must close. A link polygon must close.

The proof given here is very important, for, in the first place, conceive the rigid body to be the paper, say, a sheet of brass, and let the six balanced forces be attached to it at the points *A*, *B*, *C*, *D*, *E*, and *F*. Suppose the brass all cut away except the narrow strips under the thick lines *A B C D E F*, which may further be supposed to be pin-jointed at these points. These strips of brass actually apply to the pins, the pairs of forces given by the vectors, viz. 7, 8, 9, 10, 11, and 12, and we have designed an articulated structure in equilibrium under the given load system.

This is a balanced polygonal lineal frame or rib, and if each strip of brass be sectioned for the load on it, we have then an actual balanced frame. A model of this frame sits on a horizontal table in the Engineering Laboratory of Trinity College. It is in *unstable equilibrium*, for a sharp blow on the table causes it to distort.

The pair of diagrams, figs. 75 and 76, show designs for retaining walls. The load on the back of the wall is constructed by the method of the ellipse of stress and its auxiliary figure. This load is reduced so as to be expressed in terms of the weight w of a cubic foot of masonry.

Triangular or rectangular blocks of masonry are added one after another, and the partial resultant constructed graphically till the last resultant, that with the greatest number of barbs, at last passes through a centre of stress deemed to be sufficiently far in from the face of the wall.

The data and construction are sufficiently given on the face of the diagrams, which, however, are small, having been reduced half the lineal sizes of a set of graphical exercises, now in the press, so as to lie on the page.

It will be seen that the slopes of the earth, both actual and limiting, are given 3 to 1 and 2 to 1 which are nearly the same as $\gamma = 20°$, and $\phi = 30°$.

The steps of the calculations corresponding to those graphical solutions are as follows. The fourth is only approximate so that one auxiliary figure may serve for two walls.

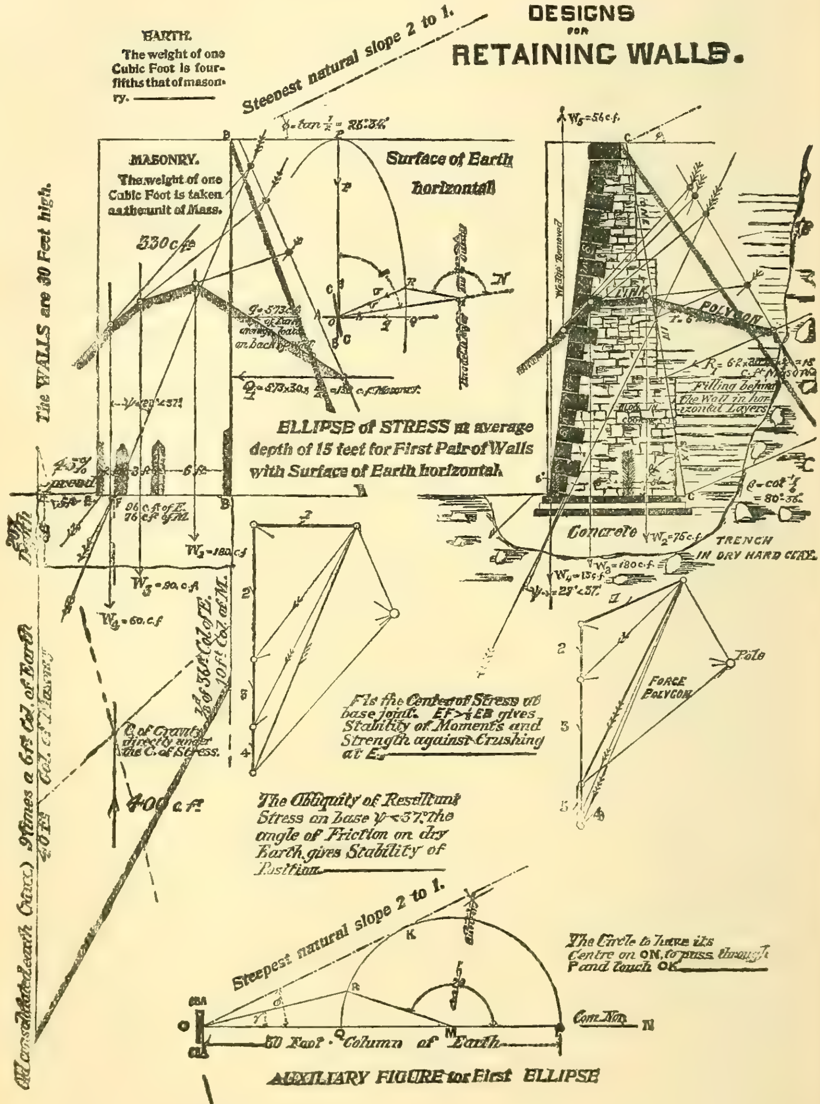

DESIGNS
FOR
RETAINING WALLS.
EARTH.
The weight of one Cubic Foot is four-fifths that of masonry.
Steepest natural slope 2 to 1.
φ = tan ½ = 26°34′
MASONRY.
The weight of one Cubic Foot is taken as the unit of Mass.
Surface of Earth horizontal
330 c.ft
The WALLS are 30 Feet high.
q = 5.73 c.ft. of Earth average load on back of wall
Q/2 = 5.73 × 30 × 4/5 = 138 c.f. Masonry.
ψ = 23°< 37°
45% spread
3 ft
6 ft
F
B
E
96 c.ft of E.
76 c.ft of M.
W₂ = 180 c.f
W₃ = 90 c.f
W₄ = 60 c.f
ELLIPSE of STRESS at average depth of 15 feet for First Pair of Walls with Surface of Earth horizontal.
A consolidated earth (9 times a 6 ft Col. of Earth)
40 ft Col. of Masonry
⅓ of 36 ft Col of E.
10 ft Col. of M.
C. of Gravity directly under the C. of Stress.
400 c.ft
F is the Center of Stress at base joint. EF > ⅓ EB gives Stability of Moments and Strength against Crushing at E.
The Obliquity of Resultant Stress on base ψ < 37° the angle of Friction on dry Earth gives Stability of Position.
W₅ = 56 c.f.
Wedge Removed
POLYGON
R₁ = 6.2 × 30 × 4/5 = 15 c.ft Masonry
Filling behind the Wall in horizontal Layers
θ = cot⁻¹ ⅙ = 80°36′
Concrete
W₂ = 75 c.f.
TRENCH IN DRY HARD CLAY.
W₃ = 180 c.f.
W₄ = 15 c.f.
ψ = 23° < 37°
Pole
FORCE POLYGON
1
2
3
4
5
Steepest natural slope 2 to 1.
K
R
O
Q
M
N
60 Feet Column of Earth
Con. Nor.
The Circle to have its Centre on ON, to pass through P and touch OK
AUXILIARY FIGURE for First ELLIPSE

Fig. 75.

EXAMPLES.

89. First wall on fig. 75, the corresponding analytical solution is

$$\phi = \tan^{-1}\tfrac{1}{2} = 26^\circ\ 34\,; \qquad k = \frac{1-\sin\phi}{1+\sin\phi} = {\cdot}382.$$

$$p = 15w'\,; \qquad q = kp = 5{\cdot}73w'.$$

$$H = Q = q \times 30 = 171{\cdot}9w' = 137{\cdot}5w,$$ or 138 times the weight of a cubic foot of masonry.

$$M = 10H = 1375w\,; \qquad \mathrm{M} = 20wt \times \tfrac{3}{8}t = 11{\cdot}25wt^2.$$

Equating, $t^2 = 122$, and $t = 11$ feet nearly.

90. Second wall on fig. 75. Approximate practical solution. In the last suppose an upward force S to act through the centre of gravity of the wedge removed from the back of the wall equal to the excess weight of masonry over earth.

$$S = \tfrac{1}{2} \times 30 \times 5 \times \frac{w}{5} = 15w,$$

and its lever about the centre of stress is $(\tfrac{7}{8}t - \tfrac{5}{3}) = 8$ nearly as t is almost 11. Correcting the equation of moments

$$11{\cdot}25wt^2 = 1375w + 15w \times 8, \quad \text{or} \quad t^2 = 133 \text{ and } t = 11{\cdot}5 \text{ feet.}$$

91. Second wall on fig. 75, solution exactly corresponding to the graphic solution,

$$OM = \frac{p+q}{2} = \frac{w'}{2}(15 + 5{\cdot}73) = 10{\cdot}365w'.$$

$$MR = \frac{p-q}{2} = \frac{w'}{2}(15 - 5{\cdot}73) = 4{\cdot}635w'.$$

$$OM^2 = 107{\cdot}5, \quad MR^2 = 21{\cdot}5\,; \text{ dropping } w'.$$

$$OR^2 = OM^2 + MR^2 + 2OM \,.\, MR \cos 2\theta.$$

Now $\theta = \beta = \cot^{-1}1\tfrac{1}{6} = 80^\circ\ 33'$, so that $2\theta = 161^\circ\ 06'$, the sup. of which is $18^\circ\ 54'$.

$$OR^2 = 129 - 96{\cdot}07 \times {\cdot}946 = 38{\cdot}12.$$

$$r = OR = 6{\cdot}174w',$$

and CC the back of the wall is $30 \operatorname{cosec}\beta = 30{\cdot}4$ square feet.

$$R = 30{\cdot}4r = 187{\cdot}7w' = 150w.$$

$$\frac{\sin\gamma}{\sin 2\theta} = \frac{MR}{OR}\,; \quad \therefore\ \sin\gamma = \frac{4{\cdot}635}{6{\cdot}174} \times {\cdot}324 = {\cdot}245, \quad \gamma = 14^\circ\ 11'.$$

To γ add $9^\circ\ 27'$, the complement of θ, and we obtain $23^\circ\ 38'$ as the inclination of R to the horizon.

Resolving R into horizontal and vertical complements

$$H = R \,.\, \cos 23^\circ\ 38' = 150w \times {\cdot}916 = 137{\cdot}4w.$$

$$V = R \,.\, \sin 23^\circ\ 38' = 150w \times {\cdot}401 = 60{\cdot}15w.$$

The equation of moments about F, the centre of stress, is

$$1374 = 30\,(t-5)\,(\tfrac{3}{8}t - \tfrac{5}{2}) + 75\,(\tfrac{7}{8}t - \tfrac{10}{2}) + 60\,(\tfrac{7}{8}t - \tfrac{5}{3}),$$

$$t^2 - 1{\cdot}16t = 120, \quad \text{or} \quad t = 11{\cdot}5 \text{ feet.}$$

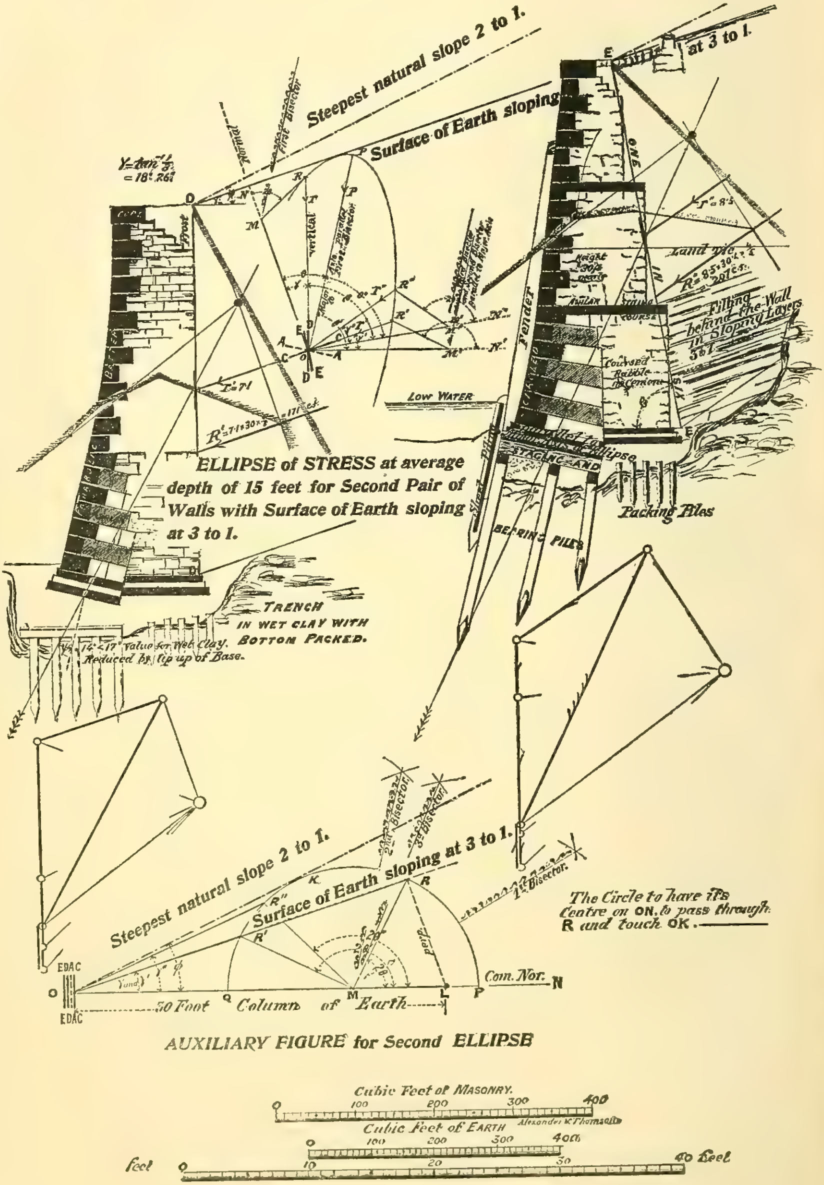

Steepest natural slope 2 to 1.
Surface of Earth sloping at 3 to 1.
First Bisector
Normal
Y = tan⁻¹ ⅓ = 18°.26'
O
Frost
Batter
Cope
vertical
parallel to First Bisector
P
T
R
M
N
E
D
C
A
R'
R''
M'
N'
T' = 7.1
R' = 7.1 × 30 × ⅓ = 71 c.ft.
T'' = 8.5
Land Tie
R'' = 8.5 × 30' × ⅓ = 287 c.ft.
Fender
Height 30 ft
Ashlar
Coursed Rubble in Cement
Filling behind the Wall in Sloping Layers 3 to 1
LOW WATER
Staging and
Bearing Piles
Packing Piles
Sheet Piles
ELLIPSE of STRESS at average depth of 15 feet for Second Pair of Walls with Surface of Earth sloping at 3 to 1.
TRENCH IN WET CLAY WITH BOTTOM PACKED.
14° < 17° Value for Wet Clay. Reduced by tip up of Base.
2nd Bisector
3rd Bisector
1st Bisector
K
L
Q
Com. Nor.
EDAC
30 Foot Column of Earth
The Circle to have its Centre on ON, to pass through R and touch OK.
AUXILIARY FIGURE for Second ELLIPSE
Cubic Feet of MASONRY. 0 100 200 300 400
Cubic Feet of EARTH 0 100 200 300 400
feet 0 10 20 30 40 feet
Alexander K Thomson

Fig. 76.

92. The first wall on fig. 76 has the additional datum $\gamma = \tan^{-1}\tfrac{1}{3} = 18^\circ\ 26'$.

$$r = 15w' \cos\gamma = 15w' \times {\cdot}9487 = 14{\cdot}23w'.$$

$$k' = \frac{\cos\gamma - \surd(\cos^2\gamma - \cos^2\phi)}{\cos\gamma + \surd(\cos^2\gamma - \cos^2\phi)} = \frac{{\cdot}9487 - \surd({\cdot}9000 - 8000)}{{\cdot}9487 + \surd({\cdot}9000 - 8000)} = \frac{9487 - 3162}{9487 + 3162} = {\cdot}5.$$

$$r' = k'r = 7{\cdot}115w', \quad \text{also} \quad \gamma' = \gamma = 18^\circ\ 26'.$$

$$R' = 30r' = 214w' = 171w; \quad H' = R' \cos\gamma' = 162w.$$

$$V' = R' \sin\gamma' = 42w. \quad \text{Also} \quad W = 30wt.$$

Equation of moments about F, the centre of stress, at $\tfrac{1}{8}$th of t in from the face of the wall.

$$10H' = W \times \tfrac{3}{8}t + V' \times \tfrac{7}{8}t,$$

$$1620 = 11{\cdot}25t^2 + 36{\cdot}8t, \quad \text{and} \quad t = 10{\cdot}5 \text{ feet.}$$

93. Approximately for second wall on fig. 76, by removing wedge at back.

$$1620 = 11{\cdot}25t^2 + 36{\cdot}8t - 15 \times 8, \quad \text{and} \quad t = 11 \text{ feet.}$$

94. Detailed calculations for second wall on fig. 76 corresponding to the graphical solution shown on it.

$$\frac{p}{r} = \frac{1 + \sin\phi}{\cos\gamma + \surd(\cos^2\gamma - \cos^2\phi)} = \frac{1 + {\cdot}4472}{{\cdot}9487 + {\cdot}3162} = 1{\cdot}144.$$

$$p = 1{\cdot}144r = 1{\cdot}144 \times 14{\cdot}23w' = 16{\cdot}27w'.$$

$$q = kp = {\cdot}382 \times 16{\cdot}17w' = 6{\cdot}21w'.$$

$$OM = \tfrac{1}{2}(p + q) = 11{\cdot}24w'; \quad MR = \tfrac{1}{2}(p - q) = 5{\cdot}03w'.$$

$$\frac{\sin 2\theta}{\sin\gamma} = \frac{OR}{MR} = \frac{r}{\frac{1}{2}(p - q)} = \frac{14{\cdot}23}{5{\cdot}03}.$$

$$\sin 2\theta = \frac{1423}{503} \times {\cdot}3162 = {\cdot}8956; \quad 2\theta = 63^\circ\ 34'; \quad \theta = 31^\circ\ 47'.$$

Now the angle between the normals ON' and ON'' can be expressed in two ways.

$$\theta + \theta'' = \gamma + \beta, \quad \text{or} \quad \theta'' = 18^\circ\ 26' + 80^\circ\ 33' - 31^\circ\ 47' = 67^\circ\ 12'.$$

$2\theta'' = 134^\circ\ 24'$ the sup. of which is $45^\circ\ 36'$.

$$(OR'')^2 = OM^2 + MR^2 + 2OM\,.\,MR \cos 2\theta''.$$

$$\left(\frac{r''}{w'}\right)^2 = 11{\cdot}24^2 + 5{\cdot}03^2 - 2 \times 11{\cdot}24 \times 5{\cdot}03 \times {\cdot}6997 = 72{\cdot}53.$$

$$r'' = 8{\cdot}516w'; \quad R'' = r'' \times 30{\cdot}4 = 259w' = 207{\cdot}2w.$$

$$\frac{\sin\gamma''}{\sin 2\theta''} = \frac{MR''}{OR''} = \frac{5{\cdot}03}{8{\cdot}516}; \quad \therefore \quad \sin\gamma'' = \frac{5030}{8516} \times {\cdot}7145 = {\cdot}422, \text{ and } \gamma'' = 24^\circ\ 58'.$$

Obliquity of OR'' to the vertical is

$$\beta - \gamma'' = 80^\circ\ 33' - 24^\circ\ 58' = 55^\circ\ 35'.$$

$$H'' = R'' \sin 55^\circ\ 35' = 171w, \quad \text{and} \quad V'' = R'' \cos 55^\circ\ 35' = 117w.$$

$$10H'' = 30w(t - 5)\left(\tfrac{3}{8}t - \tfrac{5}{2}\right) + 75w\left(\tfrac{7}{8}t - \tfrac{10}{3}\right) + V''\left(\tfrac{7}{8}t - \tfrac{5}{3}\right),$$

or

$$t^2 + 3{\cdot}27t = 158{\cdot}2, \quad \text{and} \quad t = 11{\cdot}05 \text{ feet.}$$

CHAPTER VI.

TRANSVERSE STRESS.

In the preceding chapters we have considered the internal stress at any point within a solid, and have shown that it can be expressed by means of three principal stresses. We began with one principal stress, the other two being zero; this was illustrated by pieces strained under one direct simple stress, such as tie rods and struts; and at each point in these pieces the strain was similar in every respect. We next considered two principal stresses, the third being zero or identical with one of those two; this was illustrated by small rectangular prisms of earth under foundations, or loaded with the weight of superincumbent earth, the prism being strained by two (or three) direct simple stresses upon its pairs of opposite faces. There we saw that the strain at all points, in certain parallel planes, was similar in every respect; varying, however, as we passed from points in one to points in another of those parallel planes. It was pointed out that earth might have the stress in one horizontal direction artificially increased by a direct external stress, in which case there would be three principal stresses at each point, the intensities of which might be different at different points.

In all such examples, the internal stresses were due to strain produced in the simplest manner possible, viz., by *direct* external stresses; and in many the stresses at internal points were given, without specifying what the solid was, or in what manner it was strained. These exercises served to illustrate methods, but it will afterwards appear that the data specifying the stress at such points were obtained by supposing that the body was strained by external stresses, definite though by no means either simple or direct.

We now come to consider the stresses at points within solids, due to strains produced in the next simplest manner, viz., by external stresses which are all *parallel.* Pieces under such stresses are called *beams,* and the stress is called *transverse stress.* The case in which both ends of the beam are simply supported will be primarily considered. For simplicity, the external stresses, as shown on the diagrams, are all vertical; they consist of the two upward thrusts concentrated at the extremities, and the loads concentrated on intermediate portions and acting downwards. These external stresses are uniform in the direction normal to the paper; and whatever be the breadth of the beam, they may be replaced by forces all in one plane, the plane of the paper.

On fig. 77, $A\,A'\,B'\,B$ is the longitudinal section of a beam of length $2c$, depth h, and breadth b, and OX is any line chosen as axis. W_1 is a force in the plane of the paper, replacing a stress spread uniformly over the breadth of the beam, as shown on the cross-section below it. Similarly P and Q are forces at the extremities and in the plane of the paper. In order to have these forces specified, it is necessary to know their amounts, and the distances measured from some origin O, say at one end of the beam, to the points where their lines of action cross OX. Such distances are called the abscissæ of the places of application of the loads. Thus P acts at O, W_1 at x_1, and Q at $2c$. C is the centre of span; its abscissa is c.

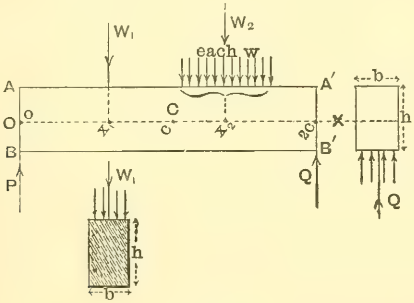


Fig. 77.

The varieties of load to be considered are:—

1°. Loads concentrated at one or more points of the span as W_1.

2°. Loads *uniformly* spread over the whole or parts of the span, as w lbs. per running foot. Such a load is represented on fig. 77 by a set of arrows, each equal to w, and consequently they are one foot apart; to save trouble, it is more convenient, as on fig. 78, to represent such a load by means of a parallelogram surrounding all the arrows.

3°. Combinations of such loads.

The loads concentrated at points might be the ends of cross beams resting on such points, or the wheels of carriages, &c. The weight of the beam itself is often to be considered as a load spread uniformly over the span.

To find the relations among the external forces, we consider the equilibrium of the beam as a whole. The beam is to be considered as perfectly rigid and indefinitely strong. In order to find the supporting forces P and Q, we require to know the amounts and positions of the loads, and the length of the beam.

Since the forces are all parallel and in one plane, there are two conditions of equilibrium:—

I. The algebraic sum of the forces is zero.

II. The algebraic sum of the moments of the forces about any point is zero.

From the first condition we have

$$P + Q = W_1 + W_2 + W_3 + \&c. = \Sigma(W) \qquad (1)$$

where $\Sigma(W)$ represents the sum of all the quantities

$$W_1,\ W_2,\ W_3,\ \&c.$$

If we take moments about O, then P has no moment, and Q tends to turn the beam in one direction about O, while the loads all tend to turn it in the other direction. By the second condition the sum of these moments is zero, and we may, if we choose, put the moment of Q equal to the sum of the moments of the loads, thus

Q × leverage = sum of the products got by multiplying each load by its leverage,

or $$Q \,.\, 2c = W_1x_1 + W_2x_2 + W_3x_3 + \&c. = \Sigma(Wx);$$

hence $$Q = \frac{\Sigma(Wx)}{2c}, \qquad (2)$$

where $\Sigma(Wx)$ represents the sum of all the quantities

$$W_1x_1,\ W_2x_2,\ \&c.$$

P may be found in a similar manner by taking moments about the other end; or it may be found at once, since we know $P + Q$ by equation (1).

An uniform load, such as w lbs. per running foot, spread over a portion of span, is to be treated as one force equal to the amount, and concentrated at the middle of that portion.

EXAMPLES.

95. The span of a beam is 20 feet, and there is a load of 80 tons at 5 feet from the left end. Find the supporting forces.

$$2c = 20\,; \quad W_1 = 80, \quad \text{and} \quad x_1 = 5\,; \quad Q\,.\,2c = W_1 x_1.$$

$$Q = \frac{W_1 x_1}{2c} = \frac{400}{20} = 20 \text{ tons}, \quad P + Q = 80 \text{ tons}, \quad P = 60 \text{ tons}.$$

Otherwise,

$$Q = \frac{\text{load}}{\text{span}} \times \text{segment remote from } Q = \frac{80}{20} \times 5 = 20 \text{ tons}\,;$$

and
$$P = \frac{\text{load}}{\text{span}} \times \text{segment remote from } P = \frac{80}{20} \times 15 = 60 \text{ tons}.$$

96. A beam of span 24 feet supports loads of 20, 30, and 40 tons concentrated, in order, at points which divide its length into four equal parts. Find the supporting forces.

$$W_1 = 20\,;\ W_2 = 30\,;\ W_3 = 40\,;\ x_1 = 6\,;\ x_2 = 12\,;\ x_3 = 18\,;\ 2c = 24.$$

$$Q = \frac{\Sigma(Wx)}{2c} = \frac{20 \times 6 + 30 \times 12 + 40 \times 18}{24} = 50 \text{ tons}.$$

$$P = \Sigma(W) - Q = (20 + 30 + 40) - 50 = 40 \text{ tons}.$$

97. A beam 30 feet span supports three wheels of a locomotive which transmit each 6, 14, and 8 tons; the distances measured from the left end of the beam to the wheels are 8, 18, and 24 feet respectively. Find the supporting forces.

$$W_1 = 6\,; \quad W_2 = 14\,; \quad W_3 = 8\,; \quad \Sigma(W) = 28 \text{ tons}\,;$$

$$x_1 = 8\,; \quad x_2 = 18\,; \quad x_3 = 24\,; \quad 2c = 30 \text{ feet}.$$

Ans. $P = 11{\cdot}6$ tons; $Q = 16{\cdot}4$ tons.

98. The span of a beam is 60 feet; an uniform load of 2000 lbs. per running foot is spread over the portion of the span beginning at 40 and ending at 50 feet from the left end. Find the supporting forces.

See fig. 77, and suppose the spread load alone on the beam.

Replacing $ww\ldots$ by

$$W_2 = w(50 - 40) = 2000 \times 10 = 20000 \text{ lbs.},$$

concentrated at

$$x_2 = \tfrac{1}{2}(40 + 50) = 45 \text{ feet}, \quad P = 5000, \quad \text{and} \quad Q = 15000 \text{ lbs.}$$

99. A beam 60 feet span, and weighing 100 tons, supports an uniform load of 2 tons per running foot, which extends from the left end of the span to a point 20 feet therefrom. Find the supporting forces.

$w = 2$; $W_1 = w \times 20 = 40$ tons; $x_1 = 10$, the middle point of uniform load.

Weight of beam, $W_2 = 100$ tons; $x_2 = c = 30$.

Ans. $P = 83\tfrac{1}{3}$ tons; $Q = 56\tfrac{2}{3}$ tons.

100. A beam 60 feet span, and weighing 100 tons, supports a locomotive as in example 97, and an uniform load of 2 tons per running foot which extends from the middle to the right end of the span. Find the supporting forces.

$$Q \times 60 = 6 \times 8 + 14 \times 18 + 8 \times 24 + 60 \times 45 + 100 \times 30.$$

Ans. $Q = 103{\cdot}2$, and $P = 84{\cdot}8$ tons.

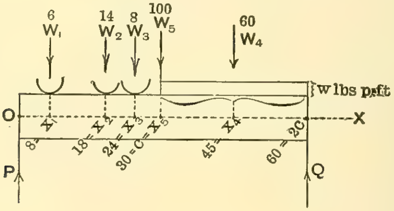


Fig. 78.

101. A beam 36 feet span weighs one ton per lineal foot. The first half is loaded uniformly with 2 tons, and the second half with 3 tons per running foot. Find P and Q.

Ans. $P = 58{\cdot}5$, and $Q = 67{\cdot}5$ tons.

102. A beam 42 feet span supports five wheels of a heavy locomotive. The fore wheel is one foot from the left end, and the distances between the wheels, in order, are 5, 8, 10, and 7 feet, and the loads transmitted, in order, are 5, 5, 11, 12, and 9 tons. Find the supporting forces. See fig. 79.

$$W_1 = 5\,; \quad W_2 = 5\,; \quad W_3 = 11\,; \quad W_4 = 12\,; \quad W_5 = 9\,; \quad \Sigma(W) = 42.$$

$$x_1 = 1\,; \quad x_2 = 6\,; \quad x_3 = 14\,; \quad x_4 = 24\,; \quad x_5 = 31\,; \quad 2c = 42.$$

$$\therefore \quad Q \times 42 = 5 \times 1 + 5 \times 6 + 11 \times 14 + 12 \times 24 + 9 \times 31.$$

Ans. $P = 24$, and $Q = 18$ tons.

For a system of loads such as W_1, W_2, &c., there is a point at which, if they were all concentrated, the supports would share the load as they do for the actual distribution at different points. This point is called the *centre of gravity* of the load system; its position will be marked G, and its abscissa OG will be denoted by $\bar{x}$. Hence, supposing the total force $\Sigma(W)$ concentrated at G, we have

$Q \,.\, 2c = \Sigma(W) \,.\, \bar{x}$ for the single force $\Sigma(W)$.

$Q \,.\, 2c = \Sigma(Wx)$ for the actual distribution; see equation 2 on page 110.

Hence $$\Sigma(W) \,.\, \bar{x} = \Sigma(Wx), \quad \text{or} \quad \bar{x} = \frac{\Sigma(Wx)}{\Sigma(W)},$$

which gives the position of G.

Having calculated $\bar{x}$, we can now find P and Q as for the single load $\Sigma(W)$ at $\bar{x}$ from the left end; see example 95, second method.

$$\text{The supporting force at either end} = \frac{\text{total load}}{\text{span}} \times \text{remote segment},$$

or $$P = \frac{\Sigma(W)}{2c}\,(2c - \bar{x})\,; \quad Q = \frac{\Sigma(W)}{2c} \,.\, \bar{x}.$$

103. Solve exercise 102 by the method just described.

We have

$$\bar{x} = \frac{\Sigma(Wx)}{\Sigma(W)} = \frac{5 \times 1 + 5 \times 6 + 11 \times 14 + 12 \times 24 + 9 \times 31}{5 + 5 + 11 + 12 + 9} = 18 \text{ feet,}$$

and the other segment $(2c - \bar{x}) = 24$ feet.

$$\therefore \quad P = \frac{\text{total load}}{\text{span}} \times \text{remote segment} = \frac{42}{42} \times 24 = 24 \text{ tons,}$$

and

$$Q = \frac{\text{total load}}{\text{span}} \times \text{remote segment} = \frac{42}{42} \times 18 = 18 \text{ tons.}$$

The position of G relative to the loads W_1, W_2, . . . can be found, although the span of the beam and the position of the loads upon it be unknown, provided the amounts of the loads and their distances apart be given. On fig. 79, let S be the point where the first load W_1 is situated, and let a_1 (= 0), a_2, a_3, $\bar{a}$, a_4, a_5 be the distances from S to W_1, W_2, W_3, G, W_4, W_5, respectively. Taking moments about S, the moment of $\Sigma(W)$ acting at $\bar{a}$ will equal the sum of the moments of W_1, W_2 . . . acting at a_1, a_2 . . . .

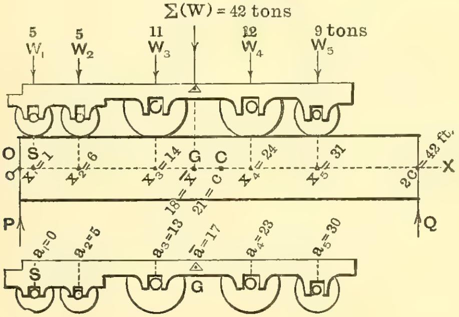


Fig. 79.

That is

$$\Sigma(W) \times \bar{a} = \Sigma(Wa); \quad \therefore \quad \bar{a} = \frac{\Sigma(Wa)}{\Sigma(W)}$$

gives position of G measured from S the left end of the load.

104. In example 103, find G the position of the centre of gravity of the load measured from the fore wheel.

$$W_1 = 5, \quad W_2 = 5, \quad W_3 = 11, \quad W_4 = 12, \quad W_5 = 9, \quad \Sigma(W) = 42.$$

$$a_1 = 0, \quad a_2 = 5, \quad a_3 = 13, \quad a_4 = 23, \quad a_5 = 30.$$

Then

$$\bar{a} = \frac{\Sigma(Wa)}{\Sigma(W)} = \frac{0 + 5 \times 5 + 11 \times 13 + 12 \times 23 + 9 \times 30}{5 + 5 + 11 + 12 + 9} = 17 \text{ feet.}$$

See G for this locomotive, fixed on fig. 153, by graphic construction of fig. 74.

105. In example 97, find $\bar{a}$ the distance of G from the fore wheel.

$$W_1 = 6, \quad W_2 = 14, \quad W_3 = 8. \qquad a_1 = 0, \quad a_2 = 10, \quad a_3 = 16;$$

$$\therefore \quad \bar{a} = 9{\cdot}57 \text{ feet.}$$

It is convenient to calculate $\bar{a}$, if it be required to find values of P and Q, as in examples 97 and 102, corresponding to the given load system shifted into some new position upon the beam; thus

106. In example 102, find P and Q when the locomotive shifts till its fore wheel is 6 feet from the left end.

$\bar{a} = 17$: hence adding six feet we have $\bar{x} = 23$, so that P and Q will now be the same as for a single load $\Sigma(W) = 42$ tons concentrated at G, a point dividing the span into the segments 23 and 19.

$$\therefore \quad Q = \frac{\text{load}}{\text{span}} \times \text{remote segment} = \frac{42}{42} \times 23 = 23 \text{ tons.}$$

Similarly

$$P = \frac{42}{42} \times 19 = 19 \text{ tons.}$$

In like manner P and Q may be found with great convenience for other positions of the locomotive, all of whose wheels *must*, however, be on the beam; because, if one wheel goes off, the beam is under a different load system altogether.

107. Find P and Q in example 97 when the locomotive is shifted so that its fore wheel is 10 feet from the left end of the beam.

Here

$$\bar{a} = 9{\cdot}57 \text{ feet,} \quad \text{and} \quad \Sigma(W) = 28.$$

Ans. $P = 9{\cdot}73, \quad Q = 18{\cdot}27$ tons.

Neutral Plane and Neutral Axis.

The phenomena which accompany transverse stress are :—

Every horizontal straight line parallel to the axis of the beam becomes a curve, one line on the diagram showing the curved condition of all lines lying in the same horizontal layer.

All points in the beam, except those over the supports, arrive at a lower level.

The consequence is, that some horizontal layers are shorter and others are longer than they were before the stress was applied. The $\left\{\begin{matrix}\text{top}\\ \text{bottom}\end{matrix}\right\}$ layer is that which is most $\left\{\begin{matrix}\text{compressed}\\ \text{extended}\end{matrix}\right\}$, and one nearer the $\left\{\begin{matrix}\text{top}\\ \text{bottom}\end{matrix}\right\}$ is more $\left\{\begin{matrix}\text{compressed}\\ \text{extended}\end{matrix}\right\}$ than one not so near. Since this condition of being extended diminishes gradually as you pass upwards from layer to layer, and passes into a condition of being compressed, there must be one intermediate layer which is neither extended nor compressed. This layer is indefinitely thin, is in fact a plane, and is called the *neutral plane* of the beam, and the line which is its trace upon the diagram is called the *neutral axis of* the beam.

On fig. 80, the straight line OX, the neutral axis, while the beam is unstrained, is chosen as an axis of reference. Let S be any point on the neutral axis after the beam has been strained, s its distance from O measured along the curve; let S' be the point on OX directly above S, and x its distance from O measured along OX. It is to be observed that the curvature, although exaggerated on the diagrams, is really in practice so slight that x and s will be sensibly equal to each other, and x may be put for the amount of either unless where it is absolutely necessary to distinguish between them. $S'S$ is called the *deflection* of the point S; the greatest value of this is called *the deflection* of the beam, and when the beam is symmetrically loaded, it occurs at the centre of span.

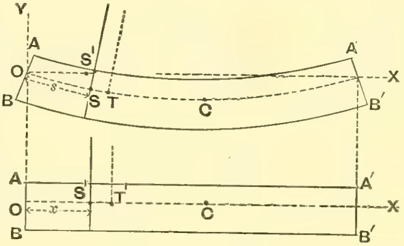


Fig. 80.

Let T be another point on the curve. Draw SH and TH normals to the curve at S and T meeting each other at H; then H will be the centre of a circle which will coincide with the arc ST. Draw also SK a tangent at S (fig. 81), meeting any horizontal line KL at K. Then

ds = arc ST is the small difference between the values of s for the two points T and S.

$dx = S'\,T'$ is the small difference of the values of x, the abscissæ of T and S.

$ds = dx$ so far as value is concerned.

$\rho = SH$ is called the *radius of curvature* at S;

$\frac{1}{\rho}$, its reciprocal, is called the *curvature* at S;

and H is called the *centre of curvature* at S

i = the angle SKL is called the *slope* at S; its greatest value is at one end, and is called *the slope* of the beam.

di = the difference of the slopes at T and S

$$= \frac{\text{arc } ST}{SH} = \frac{ds}{\rho} = \frac{dx}{\rho} \text{ sensibly.}$$

These angles are in circular measure.

Let y_a be the height of the top layer AA above, y_b the depth of the bottom layer BB below the neutral axis, and let $\{\mp\}\ y$ be the distance of any layer $CC \begin{Bmatrix}\text{above}\\ \text{below}\end{Bmatrix}$ the neutral axis.

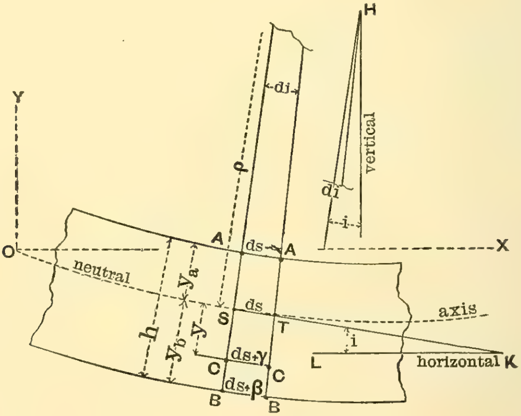


Fig. 81.

The portions of these layers intercepted between the two radii HS and HT before being strained were all equal to ds in length. Let $(ds - \alpha)$, $(ds + \beta)$, and $(ds \pm \gamma)$ be their lengths respectively when strained; then by similar triangles

$$\frac{\text{arc } AA}{AH} = \frac{\text{arc } ST}{SH}, \quad \text{or} \quad \frac{ds - \alpha}{\rho - y_a} = \frac{ds}{\rho};$$

$$\therefore \quad ds - \alpha = ds - y_a \cdot \frac{ds}{\rho}, \quad \text{or} \quad \alpha = y_a \frac{ds}{\rho}; \quad \therefore \quad \frac{\alpha}{ds} = \frac{y_a}{\rho}.$$

Again $\dfrac{\text{arc } BB}{BH} = \dfrac{\text{arc } ST}{SH}$, or $\dfrac{ds + \beta}{\rho + y_b} = \dfrac{ds}{\rho}$;

$$\therefore \quad ds + \beta = ds + y_b \cdot \frac{ds}{\rho}, \quad \text{or} \quad \beta = y_b \frac{ds}{\rho}; \quad \therefore \quad \frac{\beta}{ds} = \frac{y_b}{\rho}.$$

Again $\dfrac{\text{arc } CC}{CH} = \dfrac{\text{arc } ST}{SH}$, or $\dfrac{ds \pm \gamma}{\rho \pm y} = \dfrac{ds}{\rho}$;

$$\therefore \quad ds \pm \gamma = ds \pm y \cdot \frac{ds}{\rho}, \quad \text{or} \quad \gamma = y \frac{ds}{\rho}; \quad \therefore \quad \frac{\gamma}{ds} = \frac{y}{\rho}.$$

Now the intensity of the longitudinal strain on the layer AA at the point A is

$$\frac{\text{augmentation of arc } AA}{\text{original length of arc } AA} = \frac{-\alpha}{ds}. \qquad \text{(See p. 4.)}$$

Similarly the intensity of the longitudinal strain on the layer BB at the point B is $\dfrac{\beta}{ds}$, and that on CC at any point C is $\dfrac{\pm\gamma}{ds}$. Hence, from the above equations, we have

$$\frac{\alpha}{ds} : \frac{\beta}{ds} : \frac{\gamma}{ds} : : y_a : y_b : y, \qquad \text{or in words—}$$

The intensity of the longitudinal strain on each layer at the point where it crosses a section AB, is proportional to the distance of the layer from the neutral axis.

ELEMENTS OF THE STRESS AT AN INTERNAL POINT OF A BEAM.

To specify the stress at a point within a beam, it is necessary and sufficient to find the intensity and obliquity of the stress at that point upon any two planes through it; for convenience we take two planes at right angles to the plane of the paper: see page 35. In fig. 82, OX and OY are rectangular axes; OX coincides with the neutral axis of the beam, and the origin O is at the middle of its length. Let distances measured along OX to the $\left\{\begin{matrix}\text{left}\\ \text{right}\end{matrix}\right\}$, along OY $\left\{\begin{matrix}\text{downwards}\\ \text{upwards}\end{matrix}\right\}$, be $\left\{\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right\}$. Then c and $-c$ will be the abscissæ of the two ends of the beam; x_1, x_2, &c., the abscissæ of the weights. Let H be any point in the beam, its coordinates being x and y; that is, on the diagram, x is the distance of H to the right or left of O, and y its distance above or below the neutral axis. Of the planes at right angles to the paper and passing through H, choose two, viz., AB and CD, vertical and horizontal. According to custom, CD may be called the *plan* through H, and AB, the *cross-section*, or shortly the *section*; further, it is called the section at x, meaning that the abscissa of every point in the section is x. H may be *any* of the points on the cross-section at the distance y from the neutral plane; and as all these points are exactly under the same stress, it is unnecessary to say which of them H is; or, in other words, it is not necessary to give the third or Z coordinate of H required to specify its distance from the plane of the paper. The trace of the neutral plane upon the cross-section is a horizontal straight line, dividing the cross-section into an upper and under portion, and this line is called the *neutral axis* of the cross-section. In order to specify the stress at H, we find the intensities and obliquities of the stresses at that point upon the two rectangular planes through it. The stresses upon these two planes are due to the strain upon the beam; that is, to the fact of its being bent at the section AB, leaving out of account the particular forces which actually cause the beam to be bent. We may

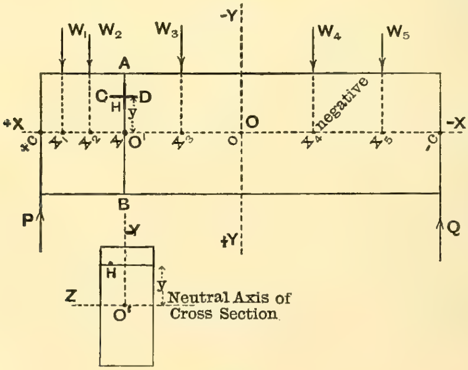


Fig. 82.

suppose that the beam is bent by these particular forces, and surrounded by an envelope of some rigid material, and then that these particular forces are removed.

Upon this consideration it is evident that the stress on *CD* will have no *normal component.* Certainly, if one of the weights of the actual load happened to be at *A*, then a normal stress on *CD* would be directly transmitted to it; such a stress, however, being accidental is left out of the present investigation. Having remarked this about the plane *CD*, we leave its further consideration for some time, and give our attention to the section *AB.* At the point *H* on the section *AB*, we see there will be a normal component stress; and we know, further, that it is a *thrust*, since the horizontal fibre through *H* is compressed. If, however, *H* be on the neutral axis of the cross-section, there is no normal component stress, since a horizontal fibre through such a point is unstrained. On the other hand, if *H* be below the neutral axis, that is, if its ordinate y be positive, there is a normal component stress; and we know, further, that it would be a *tension*, since the horizontal fibre through such a point is stretched. Generally, then, at a point *H* on a section *AB*, there will be a normal component stress of opposite signs for points situated on opposite sides of the neutral axis of the cross-section; and generally, also, there will be a tangential component stress acting in the same direction for all points on the section, as will afterwards be seen.

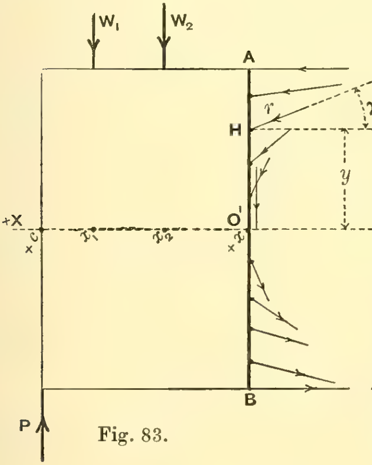


Fig. 83.

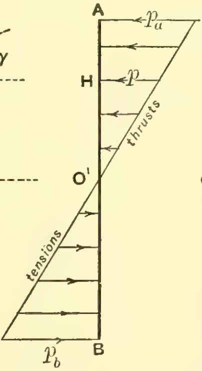


Fig. 84.

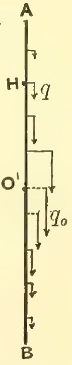


Fig. 85.

To ascertain the stresses at points on the section AB; suppose the beam cut into two portions at that section, and consider the equilibrium of one, say the left portion, as a rigid body (fig. 83). The forces acting on it are P, W_1, W_2, shown on the figure by strong arrows, together with the stress opon the cut surface, shown by fine arrows. Let P be greater than the sum of W_1 and W_2; then the vertical components of the fine arrows must act downwards to conspire with W_1 and W_2 in balancing P. For some positions of the section, $W_1 + W_2 +$ &c., the sum of the loads on the portion of the beam to the left of the section AB, may exceed P; and on such sections all the vertical components of the fine arrows must act upwards; while for other positions of the section, the sum of W_1, W_2, &c., may equal P, and on these sections the fine arrows will be normal to AB. Let r be the intensity of the stress on the section at the point H, and θ its obliquity. Resolve the fine arrows into vertical and horizontal components—that is, resolve the stress at each point into a tangential and normal component stress. On figs. 84 and 85, p and q are the components of r shown separately, one set on each diagram; p_a and p_b are the values of p at the highest and lowest points of the cross-section, and the value of p at the neutral axis is zero; q_0 is the value of q at the neutral axis, and, as will afterwards appear, the value of q at the highest and lowest points is zero.

The equilibrium of these forces gives the three following conditions:—

I. The algebraic sum of the arrows p is zero.

II. The algebraic sum of the external forces (strong arrows) together with the arrows q is zero.

III. The algebraic sum of the moments of all the forces about O' is zero.

Condition I. is equivalent to—The sum of the arrows p which are thrusts acting on the portion of the section above the neutral axis, equals the sum of the arrows p which are tensions acting on the portion of the section below the neutral axis. Hence the resultant of all the arrows p is a pair of equal and opposite forces not in one straight line; or, in other words, a couple in the plane of the paper. Since this couple, by condition III., balances the sum of the moments of the external forces about O', therefore the moment of the couple is equal to that sum, and acts in the opposite direction.

DEFINITIONS.—F_x, the algebraic sum of the external forces acting on a portion of a beam included between one end and the cross-section at x, and comprising the supporting force (if any) at that end and the loads on that portion, is called the *shearing force* at that cross-section.

M_x, the moment of these external forces about any point on the cross-section at x, is called the *bending moment* at that cross-section.

F_x, the amount of the tangential component stress on the cross-section at x, is called the *resistance to shearing* of that cross-section.

M_x, the couple which is the moment of the total stress on the cross-section at x about any point of the cross-section, or what is the same thing, the moment of the normal component stress on the cross-section about any point in the plane of the paper, is called the *moment of resistance to bending* of that cross-section.

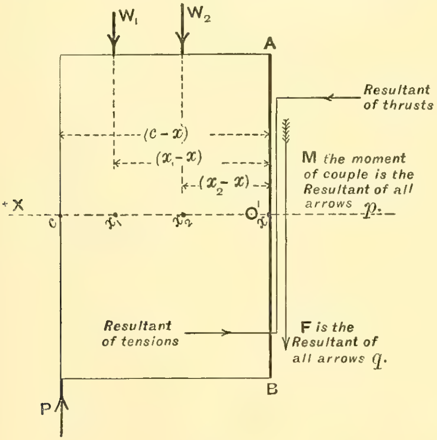


Fig. 86.

On fig. 86, these four quantities are—

$$F_x = P - W_1 - W_2.$$

$$M_x = P(c-x) - W_1(x_1-x) - W_2(x_2-x).$$

F_x = Resultant of arrows q.

M_x = Resultant of the pair of equal and opposite forces, one of which is the resultant of all the thrusts p, the other of all the tensions p.

The conditions of equilibrium for any portion of the beam comprised between one end and a cross-section at x are—

$$F_x = \mathsf{F}_x. \tag{1}$$

$$M_x = \mathsf{M}_x. \tag{2}$$

Since F_x and M_x are calculated from external forces alone, they are independent of the size or form of the cross-section, and depend only upon the amount and distribution of the load; and, when calculated at a sufficient number of cross-sections, they form the data from which to design the beam. On the other hand, F_x and M_x depend only upon the size and form of the cross-section at x, and upon the material of which the beam is to be made. Having fixed upon a material for which we know the working strengths to resist tension, thrust, and shearing, the two equations above enable us to design the different dimensions of the cross-section at x in any required form. The form to be adopted in any particular case depends in some degree upon the shapes in which the material is naturally obtained, or in which it can be manufactured cheaply; and when it can with equal facility be manufactured in several forms, that one is to be preferred which takes greatest advantage of any difference in the resistance of the material to the various *kinds* of stress; since, by doing so, we require the least quantity of material. The form of cross-section chosen must be suitable for the particular nature of the load, and locality of the beam. Having thus designed the cross-sections at a sufficient number of places, we are said to have designed the beam so as to have sufficient *strength* to resist the given loads.

Besides the above qualities of suitability, cheapness, and economy of material, a beam must also have sufficient *stiffness*. We will, further on, derive equations to enable us to find the form of economy mentioned above, and also to select the ratio of the dimensions of the cross-section to fulfil the condition of stiffness.

The cross-section at x either being given or having been designed, equations (1) and (2) enable us to calculate the elements of the stress at any internal point H.

THE CANTILEVER.

A beam may be supported by *one* prop placed exactly below the centre of gravity of the load, as shown on fig. 87. It is evident from the definitions, that, in this case, the shearing force and bending moment are maxima at the point of support; because the external forces upon one of the portions into which the section through this point divides the beam, say upon the left portion, are the loads on that portion and they all act in one direction. The bending moment at that section is the *arithmetical* sum of each of these loads into its leverage; and for any other section to the left, the number of loads to be reckoned in calculating the bending moment may be fewer, and at the same time all the leverages are shorter. In considering the left portion alone, it is unnecessary to draw the other; and instead, the left portion may be supposed, as on the figure, to terminate at its right extremity in a vertical plane; this plane is supposed to give the necessary resistance to balance the shearing force and bending moment at the point of support.

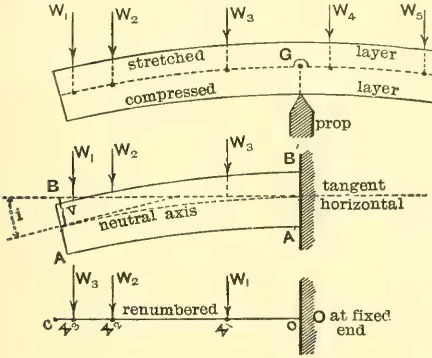


Fig. 87.

Such a piece is called a *Cantilever*; as shown on diagram, the right is its *fixed* and the left its *free* end; its length is *c*. When the cantilever is strained, that is bent, the tangent to the axis at the fixed end remains horizontal; the upper layer is stretched while the lower is compressed, and the lettering will be accordingly. The deflection *v* of the cantilever is the difference of level of the two ends, and its slope *i* is the inclination of the tangent to the axis at its free end with the horizontal.

We shall now proceed to calculate the bending moments on, and construct bending moment diagrams for, beams and cantilevers under various loadings; in such a diagram, the horizontal and vertical dimensions are respectively the span and bending moments at the different points of the span drawn to scale. Each of these diagrams requires two scales—a scale say of feet for horizontal, and a scale such as foot-tons for vertical, measurements; there may be required, also, a scale, say of tons for loads, if the loads are drawn to scale. Such diagrams must be drawn upon a large scale if intended to be used as graphical solutions, in which case only the construction requires to be known. The diagrams of this treatise are too small for such purposes, but they serve to show clearly the constructions, and are principally useful as maps upon which to note the analytical results. In every case, these diagrams are constructed so that one of their boundaries is straight, is in fact the span. This is a matter of importance, as the diagram so constructed assumes a suitable form for a practical purpose which will be afterwards pointed out.

One boundary of such diagrams is generally a parabola in certain simple positions, and we will now give a short chapter on the equations to, and the construction of, this curve.

CHAPTER VII.

THE PARABOLA.

On fig. 88, X and Y are two rectangular axes passing through A any point of reference. Let distances measured from A to the $\left\{\begin{matrix}\text{left}\\\text{right}\end{matrix}\right\}$, and from the axis of X $\left\{\begin{matrix}\text{downwards}\\\text{upwards}\end{matrix}\right\}$, be $\left\{\begin{matrix}\text{positive}\\\text{negative}\end{matrix}\right\}$.

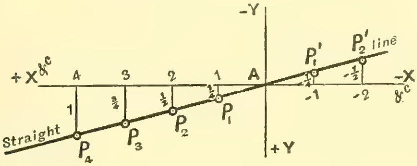


Fig. 88.

The points P_1, P_2, &c., have their ordinates $1P_1$, $2P_2$, &c., proportional to their abscissæ $A1$, $A2$, &c.; that is, if X and Y be the coordinates of any point P, then

$$Y = mX,$$

where m is any number, whole or fractional, and the locus of P is a straight line passing through A. In the figure $m = \frac{1}{4}$, and for any point P, X, and Y are both positive or both negative. The coordinates of points P_1, P_2, &c., are marked; *e.g* for P_3, $X = 3$, and $Y = \frac{3}{4}$.

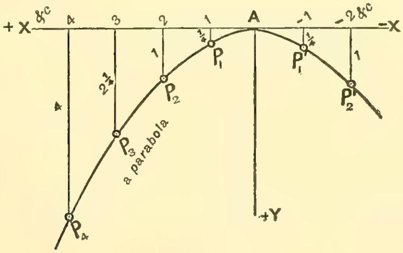


Fig. 89.

On fig. 89, P_1, P_2, &c., are points corresponding to the points marked similarly on fig. 88, but in this case the points P_1, P_2, &c., have their ordinates $1P_1$, $2P_2$, &c., proportional to the squares of their abscissæ; that is

$$Y = mX^2.$$

This is the *principal equation* to the parabola, and the quantity m is the *modulus*.

The locus of this equation, P, is the parabola: and it is altogether on one side of the axis of X, since, although X the abscissa of any point P is positive or negative, the quantity X^2 is always positive.

In the figure $m = \frac{1}{4}$, and for any point P, while X may be positive or negative, Y is always positive. The coordinates of the points P_1, P_2, &c., are marked; *e.g.* for P_3, $X = 3$, and $Y = 2\frac{1}{4}$. The point A is called the vertex, and the line AY the axis of the parabola; the curve is symmetrical about this axis. The points P_1, P_2, &c., thus found are points on the curve; and if a sufficient number of such points be found and a fair curve drawn through them, the curve will be sensibly a parabola.

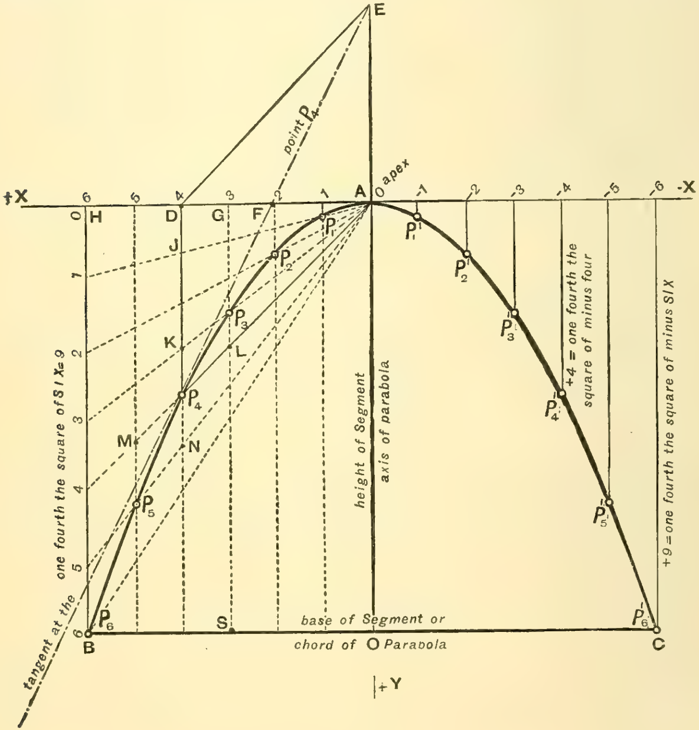


Fig. 90.

Whenever we have an equation of the above form, we conclude that the locus is a parabola whose vertex is at the origin, and whose axis lies along the axis of Y.

PARABOLIC SEGMENT.

Any line as BC, fig. 90, meeting the curve in two points, and drawn parallel to the axis of X, may be called a *right chord*; and the figure enclosed between this chord and the curve may be called a *parabolic right segment*. This chord is the *base* of the segment, and AO the distance of the vertex A from the base is the *height* of the segment.

The following is a convenient construction for drawing a parabolic right segment of a given height and on a given base BC, fig. 90. Plot A at the required height above O the middle of the base, and complete the rectangle $BOAH$. Let it be required to construct accurately twelve points at equal horizontal intervals. Divide AH and HB each into six equal intervals, and number them as on the figure. From each number on AH draw vertical lines, and from the point A draw a ray to each number on HB. Then P_1, the point of intersection of the vertical through 1 and the ray $A1$, is a point on the parabola; so also is P_2, the point of intersection of the vertical through 2 and the ray $A2$; &c.

It is evident that, for instance, DP_4 is proportional to the square of AD; for, calling $1P_1$ unity, then $DJ = 4$, and $DP_4 = 4DJ = 16 = 4^2$, while the abscissæ of the points P_1 and P_4 are proportional to 1 and 4; similarly with the other points. Hence *one fourth* is the common ratio of the depth of each point below AH to the square of its distance laterally from AO. We will return to the subject of drawing a parabolic segment, and will now show how to draw a

TANGENT AT ANY POINT OF A PARABOLA,

say P_4; on P_4A and P_4D construct a parallelogram and draw its diagonal P_4E: this line will be the tangent required. P_4K is the same fraction of P_4D that P_4L is of P_4A, and LP_3 is parallel to P_4K. If KP_3 were parallel to P_4L, then would KL be a parallelogram similar to DA, and so P_3 would be on the diagonal P_4E. But since KP_3 is not parallel to P_4L, but converges towards it, the point P_3 where KP_3 meets LG lies to the right of the line P_4E. Produce EP_4 through P_4. P_4N is the same fraction of P_4D that P_4M is of P_4A; if NP_5 were parallel to P_4M, then P_5 would be on the diagonal EP_4 produced; but, since NP_5 diverges from P_4M, the point P_5 where NP_5 meets MP_5 lies to the right of the line P_4E produced through P_4. In the same manner every point on the curve, either above or below P_4, lies to the right of P_4E; that is, P_4E is a tangent at the point P_4. Now the diagonal P_4E bisects the other diagonal AD in F, and the most convenient way of drawing a tangent at any point P_4 is to project P_4 on the horizontal through the apex A; bisect AD in F, and draw P_4F; this is the tangent required.

To plot the locus of an equation of the form $Y = mX^2$; in other words, to draw the parabola whose modulus is m. For instance, let $m = \frac{1}{4}$. Draw any line BC parallel to the axis of X, fig. 90, as base; lay off OA equal to $\frac{1}{4}OB^2$; then draw the segment by plotting accurately a number of points P_1, P_2, &c., by the construction already given, and draw a fair curve through them. Usually we fix only a few points on the curve accurately, and from these the rest of the curve is sketched in. By making the number of these points sufficiently great, we can draw the curve as accurately as we please.

A parabolic segment might be constructed on, and cut out of, a piece of cardboard, and used exactly like a set square. The parabola could then be quickly drawn on our diagrams thus:—If the apex A be given, place the parallel rollers to the axis of X, shift the rollers, slide the cardboard segment along them till the apex is at A, and draw the curve. Again, suppose we are given (fig. 90) the axis OE and a point B on the curve, and that we wish to draw the curve; place the rollers at right angles to the given axis OE, slide the cardboard segment with its apex on this axis till the curved edge passes through B, and then draw the curve. Instead of cardboard we may use a parabolic segment cut out of a slip of pear-tree or brass, and with one such segment we can, if we choose, draw all parabolas. Thus, suppose our pear-tree segment to have the modulus $\frac{1}{4}$, and that we require to draw a segment whose modulus is 1. We may choose as suitable for horizontals, a scale of 10 parts to an inch; lay off the base with this scale and draw the curve with the pear-tree segment; if we now measure the verticals on the scale, we find for every point $Y = \frac{1}{4}X^2$. If we draw a *new scale* of 40 parts to an inch for vertical measurements, then for every point on the curve we will have, as required,

$$Y = 1 \times X^2 = X^2.$$

In like manner the curve drawn by this pear-tree segment may represent *any* parabola, if the verticals be measured on a suitable scale; this is similar to the common practice of exaggerating the vertical scale for sections. All the diagrams which immediately follow may be very conveniently drawn with *one such segment*, since it is much easier to draw an additional scale than to construct a new parabola; for example, suppose the segment, fig. 90, so drawn, and that we wish it to represent the parabola $Y = \frac{1}{3}X^2$.

Using a horizontal scale of four parts to an inch, we have by measurement $OB = 6$, and OA or $HB = 9$; that is, $HB = \frac{1}{4}OB^2$; but we wish HB to measure $\frac{1}{3}OB^2$, that is 12. It is only necessary then to divide HB into 12 equal parts, and lay off a scale of such parts to be used for verticals; this scale is evidently that of 3 inches divided into 16 equal parts.

EQUATIONS TO THE PARABOLA.

In figs. 91 and 92, let O the middle of the span be the origin of rectangular coordinates, the span BC being taken as the axis of X; let distances measured from O to the $\left\{\begin{matrix}\text{left}\\ \text{right}\end{matrix}\right\}$, and from the axis of X $\left\{\begin{matrix}\text{upwards}\\ \text{downwards}\end{matrix}\right\}$, be $\left\{\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right\}$. In fig. 91, the axis of the curve passes through O, whereas in fig. 92 it passes to one side. Let x, y be the coordinates of any point P; and for the apex let $x = K$, and $y = H$.

In fig. 91, $K = 0$, since K is on the axis of Y; and in order to find the equation to the curve with the origin at O, we have

$$Y = -mX^2 \qquad \text{(origin at } A\text{)}.$$

Instead of X and Y, we substitute their values, thus

$$y - H = -mx^2; \qquad y = H - mx^2.$$

In what immediately follows, let C_0, C_1, and C_2 be *constant* quantities; then, if we have an equation of the form $y = C_0 + C_2x^2$, consisting of a term not containing x, and a term in x^2, we conclude that the locus is a parabola, with its axis vertical, its apex on the axis of Y and at the distance C_0 from the origin; and that the modulus of the parabola is C_2, the coefficient of x^2, so that the principal equation is $Y = C_2X^2$.

For example, suppose we wish to plot a number of points above the span BC, such that the coordinates of each point may fulfil the equation

$$y = \tfrac{1}{4}(36 - x^2) = 9 - \tfrac{1}{4}x^2;$$

we conclude that all the points are on a parabola whose axis is the vertical through O; that the apex A is at the height $H = 9$, the term not containing x; that the modulus is $-\frac{1}{4}$, the coefficient of x^2; and therefore that the principal equation is

$$Y = -\tfrac{1}{4}X^2; \quad \therefore \; H = 9 = \tfrac{1}{4}OD^2, \quad \text{or} \quad OD = 6.$$

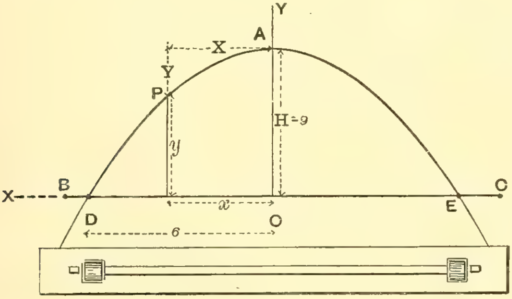


Fig. 91.

To draw the locus, lay off $OA = 9$, $OD = OE = 6$, and construct points on the curve, as already shown on fig. 90; or more quickly, by means of the pear-tree segment we may draw a curve through the points D and E just found, and construct a scale for verticals upon which OA measures 9.

Again, on fig. 92, the equation to the curve with the origin at O is derived from the principal equation by substituting $x - K$ and $y - H$ for X and Y, thus

$$y - H = -m(x - K)^2,$$

$$\text{or} \qquad y = H - mx^2 + 2mKx - mK^2, \quad \text{or} \quad y = (H - mK^2) + 2mKx - mx^2,$$

which is an equation of the form $y = C_0 + C_1x + C_2x^2$, consisting of a term not containing x, a term in x, and a term in x^2; and we conclude that the locus is a parabola, with its axis vertical. In this equation, if C_1C_2 is $\left\{\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right\}$, the apex is to the $\left\{\begin{matrix}\text{right}\\ \text{left}\end{matrix}\right\}$ of the axis of Y; if $C_0 - \dfrac{C_1^2}{4C_2}$ is $\left\{\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right\}$, the apex is $\left\{\begin{matrix}\text{above}\\ \text{under}\end{matrix}\right\}$ the axis of X.

To find the value of K and of H, arrange the equation thus:—

$$y = m(2K - x)x + H - mK^2.$$

Now, when $y = H$, its value is a maximum, and the corresponding value of x is K; so that to ascertain K, we have only to find that value of x which makes

$$y = m(2K - x)x + H - mK^2 = \text{maximum}.$$

This is a maximum when $(2K - x)x$ is a maximum, since the rest of the expression is constant for all values of x.

Suppose that $2K$ is the length of a line, which is divided at a point into two segments; let x be the length of one of them; then $2K - x$ is the length of the other, and $(2K - x)x$ is the rectangle contained by the two. We know by Euclid that this rectangle is greatest when the segments are equal, each being half of the line $2K$. So that, when $x = K$, $(2K - x)x$ is a maximum, $m(2K - x)x + H - mK^2$ is also a maximum, and $y = H$.

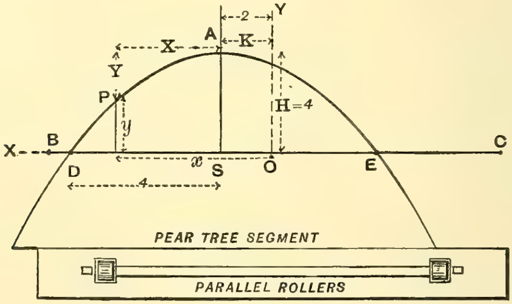


Fig. 92.

For example, suppose we wish to plot a number of points whose coordinates x, y fulfil the equation

$$y = \tfrac{1}{4}(4 - x)x + 3 = 3 + x - \tfrac{1}{4}x^2.$$

From the form of this equation, we conclude that all the points are on a parabola, whose axis is vertical and to the left of O; that the apex A is above BC; that the modulus is $-\tfrac{1}{4}$, the coefficient of x^2, and therefore that the principal equation is

$$Y = -\tfrac{1}{4}X^2.$$

The distance K at which the axis lies to the left of O is found thus:—the value of x which makes

$$y = \tfrac{1}{4}(4 - x)x + 3 = \text{a maximum},$$

is that in which $\quad 4 - x = x, \quad$ or $\quad x = 2; \quad$ that is, $\quad K = 2.$

Again, to find the height of the segment; when

$$x = K, \quad y = H, \quad \text{and} \quad H = \tfrac{1}{4}(4 - 2)\,2 + 3 = 4.$$

Substituting in the principal equation, we have

$$4 = H = \tfrac{1}{4}SD^2; \quad \text{therefore} \quad \text{half-base} = SD = \pm 4.$$

To draw the locus:—lay off $OS = 2$, and draw a vertical through S; from S lay off $SD = SE = 4$, and $SA = 4$; and construct points on the curve, as in fig. 90; or more quickly, by means of the pear-tree segment, we may draw a curve through the points D and E just found, and construct a scale for verticals upon which SA measures 4.

Theorem A.—Fig. 93. The quadrant of a parabola and a right-angled triangle stand on a common horizontal base, with the right angle of each at one end (the right end in the figure); let a be the height of the apex of the parabola, and b the height of the vertex of the triangle, above that end. If, at each point of the base, the ordinate of the parabola be added to that of the hypotenuse of the triangle, and a new curve be plotted; it will be the *same* parabola with its axis still vertical, and having its apex shifted beyond that end (the right in the figure) of the base above which the apex was, and the curve will still pass through the other end of the base. In like manner, if the ordinate of the parabola be deducted from that of the hypotenuse, a similar result is obtained; the apex of the new figure, however, is shifted to the other side. The horizontal distance through which the apex shifts is the same fraction of the base, that the height of the vertex of the triangle is of twice the height of the apex of the parabola; or if α be the lateral distance through which the apex shifts, then

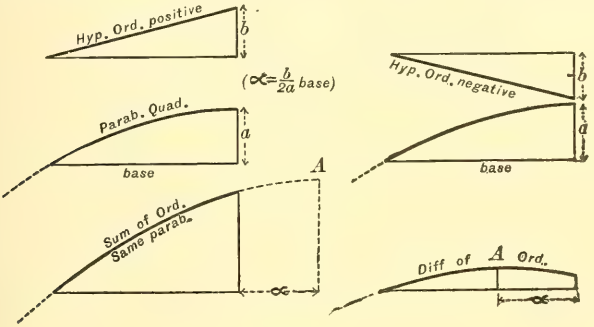


Fig. 93.

$$\alpha = \frac{b}{2a} \times \text{base of the quadrant.}$$

The addition does not affect the coefficient of x^2, so that the modulus is unchanged. In short, the algebraic addition of the ordinates of the straight slope makes the parabolic base segment shift on the rollers from the position on fig. 91 to that on fig. 92. Compare the text at fig. 116, p. 166.

Theorem B.—If to the ordinates of a parabola, axis vertical and apex to left or right of origin, the ordinates of another parabola, with axis vertical and passing through the origin, be added; then the new locus is a parabola, its modulus is the sum of their moduli, and its axis is vertical; its apex lies on the same side of the origin as the apex of the first parabola; and the abscissa of its apex is the same fraction of the abscissa of the first parabola's apex, that the modulus of the first parabola is of the sum of the moduli. Fig. 94. See also figs. 91 and 92.

For the parabolas whose apexes are A_1 and A_0, respectively,

$$y'_1 = y_0 + y_1 = H_0 - m_0x^2 + z\,(H_1 - mK_1^2) + 2mK_1x - mx^2,$$

or
$$y'_1 = (H_0 + H_1 - mK_1^2) + (m_0 + m)\left(\frac{2mK_1}{m_0 + m} - x\right)x, \tag{1}$$

which is a parabola of modulus $(m_0 + m)$, its axis is vertical, its apex lies on the same side of origin as the apex of the first parabola; and if K'_1 is the value of x which makes $\left(\frac{2mK_1}{m_0 + m} - x\right)x$ greatest, then

$$K'_1 = \frac{m}{m_0 + m}K_1 \tag{2}$$

Q. E. D.

Theorem C.—If to the ordinates of a locus consisting of two parabolas, with axes vertical, with a common modulus m, and intersecting at F, there be added the ordinates of a parabola, axis vertical, apex above origin, and modulus m_0; then the new locus consists of a pair of parabolas with axes vertical, having the common modulus $(m_0 + m)$, and intersecting at F' on the same vertical as F. Fig. 94.

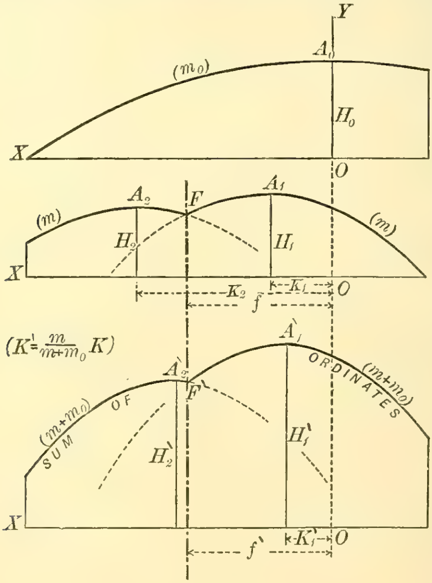


Fig. 94.

For the parabolas whose apexes are A_1 and A_2, respectively,

$$y_1 = (H_1 - mK_1^2) + 2mK_1x - mx^2; \quad y_2 = (H_2 - mK_2^2) + 2mK_2x - mx^2.$$

If f be the abscissa of the point of intersection, then where $y_1 = y_2$, $x = f$; subtracting, we have

$$0 = (H_1 - H_2, - m(K_1^2 - K_2^2) + 2m(K_1 - K_2)f, \tag{3}$$

which gives the value of f.

Again, by the previous theorem, equation (1),

$$y'_1 = (H_0 + H_1 - mK_1^2) + 2mK_1x - (m_0 + m)x^2;$$

similarly

$$y'_2 = (H_0 + H_2 - mK_2^2) + 2mK_2x - (m_0 + m)x^2.$$

Let f' be the abscissa of the point of intersection, then where $y'_1 = y'_2$, $x = f'$; subtracting, we have

$$0 = (H_1 - H_2) - m(K_1^2 - K_2^2) + 2m(K_1 - K_2)f', \tag{4}$$

hence

$$f' = f. \tag{5}$$

Q. E. D.

Theorem D.—If two parabolas with axes vertical and a common modulus m intersect at a point, then the horizontal projection of any double chord drawn through that point is constant and is equal to the double horizontal chord drawn through it. And conversely, if a line whose horizontal projection is equal to the double horizontal chord through the point of intersection be placed with one extremity on each parabola, either the line or the line produced passes through the point of intersection. (Fig. 95.)

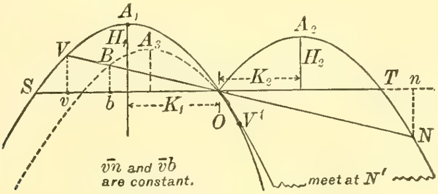


Fig. 95.

The equation to the parabola (fig. 92) is

$$y = (H - mK^2) + 2mKx - mx^2;$$

choosing the origin at O the point of intersection, then

$$H_1 = mK_1^2, \quad \text{and} \quad H_2 = mK_2^2; \quad \text{hence} \quad (H_1 - mK_1^2) = (H_2 - mK_2^2) = 0;$$

and considering K_1 and K_2 positive, we have

$$y_1 = 2mK_1x - mx^2, \tag{6}$$

$$y_2 = -2mK_2x - mx^2. \tag{7}$$

The equation to any straight line through O may be written

$$y = \mu x; \tag{8}$$

equating (6) and (8), we find for V the point of intersection $x = 2K_1 - \dfrac{\mu}{m}$; similarly for N, $x = -2K_2 - \dfrac{\mu}{m}$.

Hence

$$\overline{vn} = 2(K_1 + K_2) = ST. \tag{9}$$

Further, if VN be such that $\overline{vn} = 2(K_1 + K_2)$, and V moves on parabola No. 1, while N moves on parabola No. 2, it is evident that $\overline{VN}$ always passes through O. When V arrives at O, then $\overline{VN}$ is a tangent at O to parabola No. 1; and if V moves to V', then $N\,V'$ produced passes through O. *Q. E. D.*

Theorem E.—While the double chord VON turns, if for each position there be taken on it a point B whose horizontal projection b divides $\overline{vn}$ in a constant ratio; then this point traces out a parabola whose axis is vertical, which has the same modulus m as the parabolas Nos. 1 and 2, and which passes through the point of intersection O. (Fig. 95.)

x, y being the coordinates of V; let X, Y be the coordinates of B; and $\overline{vb} = C$ a constant; then

$$x = X + C, \quad \text{and} \quad Y = \frac{y}{x}X.$$

Now $$y = 2mK_1x - mx^2, \quad \text{or} \quad \frac{y}{x} = m(2K_1 - x);$$

therefore $$Y = m(2K_1 - C - X)X, \tag{10}$$

the locus of B; and it is a parabola with axis vertical, of modulus m, and passing through O. The abscissa of the apex A_3 is equal to $K_1 - \frac{1}{2}C$. *Q. E. D.*

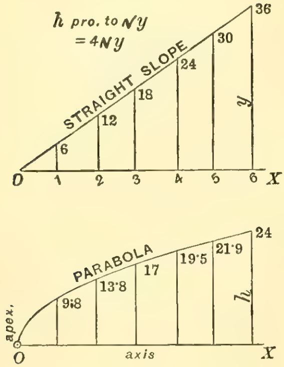


Fig. 96.

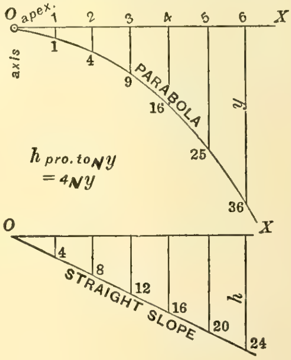


Fig. 97.

Degradation of a Locus.

Definition.—If a locus be drawn whose ordinates are proportional to the square roots of those of a given locus, the new locus is the given locus *degraded.*

Theorem F.—Fig. 96. If a locus which is a straight slope be degraded, the new locus is a parabola with its axis horizontal, and its apex at the point where the loci cross the horizontal base.

For the straight slope, $y = ax$, if we take the origin at the point where it cuts the base; if h be the corresponding ordinate of the new locus,

$$h^2 \propto y \quad \text{or} \quad ax; \quad \text{that is,} \quad x \propto h^2,$$

a parabola with axis parallel to OX, and apex at the origin.

Theorem G.—Fig. 97. If a locus, which is a parabola with axis vertical and apex on the base, be degraded, the new locus is a straight slope crossing the base at the apex.

For a parabola with origin at apex, $y = mx^2$; but $h^2 \propto y$; therefore

$$h \propto \sqrt{m}x,$$

a straight line passing through the origin.

Theorem H.—Fig. 98. If a locus which is a parabolic segment be degraded, the new locus is an ellipse with its centre at the middle of the base, the base being a major or minor diameter.

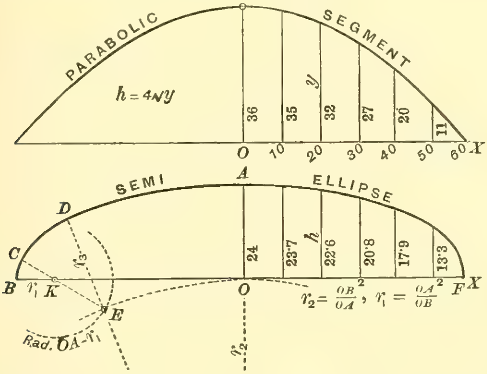


Fig. 98.

Taking the origin at the centre of base, we have for the equation to a parabola, $y = m(c^2 - x^2)$, where $2c$ is the base. Let $h' = \mu h$, where μ is a quantity such that $h'^2 = y$; then since $h^2 \propto y$, we have

$$\frac{h'^2}{m} = c^2 - x^2, \quad \text{or} \quad \frac{h'^2}{m} + x^2 = c^2; \quad \therefore \quad \frac{\mu^2 h^2}{mc^2} + \frac{x^2}{c^2} = 1;$$

the central equation to an ellipse whose semidiameters are c and $\frac{c\sqrt{m}}{\mu}$. The ratio of the semidiameters depends on m.

On fig. 98, the ordinates of the degraded figure have been multiplied by 4; thus OA is 24, or four times the square root of 36. So that BAF is the degraded figure if the vertical scale be four times as coarse as the horizontal. If the multiplier were 10, then BAF would be a semicircle.

Hence a figure consisting entirely of arcs of the same parabolic segment can, when degraded, be represented by a series of arcs of circles, provided a suitable vertical scale be constructed.

The false semi-ellipse, fig. 98, is readily struck from five centres. The first at a distance $r_2 = 60^2 \div 24 = 150$ below A. With this radius AD is swept out, and BC with $r_1 = 24^2 \div 60 = 10$ nearly. While DC is drawn from the centre E defined as shown on the diagram.

CHAPTER VIII.

BENDING MOMENTS, AND SHEARING FORCES FOR FIXED LOADS.

DEFINITION.—*The Bending Moment at any cross-section of a beam, or,* as we may more conveniently say, *at any point of the span is—The sum of the moments about that point of all the external forces, acting upon the portion of the span on either side of the point.*

For convenience in the case of beams supported at both ends, we calculate this bending moment from the forces acting upon the portion to the left of the point. These forces comprise (*vide* fig. 82, p. 118, and fig. 86, p. 121) the supporting force P acting upwards at the left end, and the loads acting downwards between that end and the point. Taking the centre of span as origin, the abscissa of P is c; and, if x be the abscissa of the point about which moments are taken, then $(c - x)$ is the leverage of P, and P tends to break the beam at the point by bending the left portion *upwards* with a moment $P(c - x)$; the abscissa of W_1 is x_1, its leverage is $(x_1 - x)$, and it tends to break the beam at the point by bending it *downwards* with a moment $W_1(x_1 - x)$; all the other loads to the left of the point have an effect on the beam similar to that of W_1; and since, in this case, the left portion of the beam is bent upwards at the point, the moment $P(c - x)$ exceeds the sum of all these moments $W_1(x_1 - x) + W_2(x_2 - x)$, &c.; and if M_x represent the bending moment at any point x, then

$$M_x = P(c - x) - W_1(x_1 - x) - W_2(x_2 - x), \text{ \&c.,}$$

all the loads on the portion of the beam to the left of the point being taken into account.

A BENDING MOMENT DIAGRAM is a figure having a horizontal straight line for its base, equal in length to the span on a scale for horizontals which should accompany the diagram. Above this base is an outline or *locus* consisting of a curve, a polygon with straight sides, or a polygon with curved sides, and such that the height of any point on the outline gives the bending moment at the point of the span over which it stands, measured on a scale (say) of ft.-lbs. or of ft.-tons for verticals which also should accompany the drawing. It is evident that this outline always meets the horizontal base at both ends, since the bending moment at each end is zero. It will be seen that x and M_x are respectively the abscissa and ordinate of a point on this locus or outline; an equation between those two quantities, such that, when you substitute into it any value for x, it gives you the corresponding value of M_x, is called the *equation to the bending moment.*

An approximate method of drawing a bending moment diagram is, to calculate the bending moments at a number of points of the span, say at equal short intervals; plot these to scale, and then join the tops of the verticals with straight lines, or draw through these points a fair curve. Such a diagram will give the bending moments accurately at the points which were plotted, and approximately at intermediate points; and its principal use is to mark the calculated results thereon. On the other hand, if upon investigation we find the locus to be of a form which we can draw readily, then, drawing the diagram first, we may afterwards, by measurement from it, find the value of the bending moment at any point of the span. Such a method of proceeding is called a *graphical solution.*

The MAXIMUM BENDING MOMENT is that value of the bending moment, than which no other value is greater; if this value be at one particular point of the span, that point is called the *point of maximum bending moment;* sometimes this value extends between two points of the span, in which case any point intermediate may be so called. The determination of the maximum bending moment, and of the point at which it occurs, is of great importance. A graphical method is peculiarly successful in giving these, as we know or readily find the *highest* point on the diagram; the height or ordinate of this point is, of course, the maximum bending moment, and its abscissa is the point at which it occurs. In all the cases which follow, that point on the diagram is either the angular point or side of a polygon, or the apex of a parabolic right segment.

DEFINITION.—*The Shearing Force at any cross-section of a beam*, or, as we may more conveniently say, *at any point of the span is—The algebraic sum of the external forces acting upon the portion of the beam to either side of the point.*

We will denote this shearing force by F_x, and calculate its amount systematically from the forces acting on the portion of the beam to the left of the point; thus for a beam

$$F_x = P - W_1 - W_2 - \&\text{c.}, \tag{1}$$

the external forces on the left portion.

Similarly, for a cantilever

$$F_x = - W_1 - W_2 - \&\text{c.} \tag{2}$$

On the beam then, F_x is $\left\{\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right\}$ according as P is $\left\{\begin{matrix}\text{greater}\\ \text{less}\end{matrix}\right\}$ than the sum of the weights to the left of the section at x. The beam may be divided at a certain point into two segments, such that for every point in the $\left\{\begin{matrix}\text{left}\\ \text{right}\end{matrix}\right\}$ segment, the shearing force is $\left\{\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right\}$; at this point the shearing force changes sign. The shearing force on either segment may be considered as positive; for convenience we have chosen as just stated. Having fixed the sign thus, on a cantilever as shown on the diagrams, that is with its fixed end to the right, the shearing force is everywhere negative; and in considerations regarding beams *and* cantilevers, it is necessary to take this into account; when, however, the cantilever alone is considered, it will be better to reckon the shearing forces as positive.

It may be observed that the symbolical expression for F_x does not depend for its form upon the position of the origin; in this respect it is unlike M_x.

A Shearing Force Diagram is a figure having a horizontal base representing the span on a convenient scale, and an outline or locus. For a cantilever this locus lies wholly under the base; for a beam, it lies above the base for the segment to the left, under the base for the segment to the right, and crosses the base at an intermediate point. The *height* of any point on the locus, measured on a scale for forces, say tons or lbs., gives the shearing force at the point of the span over which it stands. Both these scales should accompany the drawing.

Maximum Shearing Force.—On a cantilever it is evident that the greatest value is at the fixed end. On a beam there are two values each greater than the others near it, one at the left end and positive, one at the right and negative; the value of the greater of these is the greatest for the whole span.

This locus, in the diagrams for all the cases of fixed loads which we consider, consists of straight lines. For a portion of the span between any two adjacent loads, the straight line is evidently horizontal; for portions uniformly loaded it slopes at a rate given by the number which indicates the intensity of the load; thus if w lbs. per foot be the intensity of the uniform load, then w vertical to one horizontal is the slope. Where a weight is concentrated at a point, the line is vertical; that is, at such a point the locus makes a sudden change of level, the change being equal to the weight.

On a cantilever, the shearing force at the fixed end is equal to the load, and at the free end it is zero; we can readily draw the straight lines as above described to suit the nature of the load, and so complete the diagram.

On a beam the shearing force at the left end is P, at the right end it is $-Q$, and at some intermediate point the locus intersects the base and changes sign; the manner of fixing the position of this point will be explained immediately. The whole locus is then readily completed by drawing the lines as above to suit the nature of the load.

Theorem.—The Shearing Force at any point of a beam or cantilever is the rate of variation of the bending moment at that point; and on the beam the shearing force changes sign at the point of maximum bending moment.

As before,

$$F_x = P - \Sigma(W),$$

where $\Sigma(W)$ means the sum of the loads to the left of x; and

$$M_x = P(c - x) - \Sigma(W.x),$$

where $\Sigma(W.x)$ means the sum of the products got by multiplying each load to the left of x by its leverage about x. Hence if we suppose F_x to be positive, and take the bending moment at any interval d further to the right, the second bending moment will exceed the first by $F_x.d$, if there be no load on the portion d; and by $F_x.d$, minus the load over the portion d into its leverage about x, if there be a load on the portion d.

Since the leverage of the load which is on the portion d is less than d, we can by taking d small enough make this product as small as we please; that is, $F_x.d$ is the change of the bending moment in passing from x through a small interval d. Now the rate of change of M_x means the change in passing from x through an unit interval, say one foot, if the change continued uniform throughout that interval, and at the same rate as at x; in other words, the change in M_x for d reckoned equal to unity, without taking into consideration any additional loads which may be over that interval; hence the

$$\text{rate of change of } M_x = F_x;$$

also, as we pass from the left to the right end of the span, if F_x be $\begin{Bmatrix}\text{positive}\\\text{negative}\end{Bmatrix}$, M_x is $\begin{Bmatrix}\text{increasing}\\\text{decreasing}\end{Bmatrix}$, and at the point where F_x changes sign M_x is a maximum.

For fixed loads the shearing force diagram greatly assists the construction of the bending moment diagram, besides readily locating the maximum bending moment. It is convenient to consider and draw these two diagrams conjointly.

Beam with Unequal Weights at Irregular Intervals.

Let a beam 42 feet span (fig. 99) support weights, viz., $W_1 = 5$, $W_2 = 5$, $W_3 = 11$, $W_4 = 12$, and $W_5 = 9$ tons at points whose abscissæ, reckoned from O the centre, are $x_1 = 20$, $x_2 = 15$, $x_3 = 7$, $x_4 = -3$, and $x_5 = -10$ feet. This is the problem of example 103, fig. 79, and we have $P = 24$ tons, at $c = 21$ feet, and $Q = 18$ tons, at $-c = -21$ feet.

To calculate the bending moment at any point x, $x = -3$ for instance, we may take the forces upon the right portion, and

$$M_{-3} = Q \times 18 - W_5 \times 7 = 18 \times 18 - 9 \times 7 = 261 \text{ ft.-tons.}$$

We will now calculate the bending moments at the points where the weights stand, in each case systematically from the forces upon the left portion.

Ft.-tons

$M_{x_1} = P(c - x_1)$;

$M_{20} = 24 \times 1 =$ 24

$M_{x_2} = P(c - x_2) - W_1(x_1 - x_2)$;

$M_{15} = 24 \times 6 - 5 \times 5 =$ 119

$M_{x_3} = P(c - x_3) - W_1(x_1 - x_3) - W_2(x_2 - x_3)$;

$M_7 = 24 \times 14 - 5 \times 13 - 5 \times 8 =$ 231

$M_{x_4} = P(c - x_4) - W_1(x_1 - x_4) - W_2(x_2 - x_4) - W_3(x_3 - x_4)$;

$M_{-3} = 24 \times 24 - 5 \times 23 - 5 \times 18 - 11 \times 10 =$ . . . 261

$M_{x_5} = P(c - x_5) - W_1(x_1 - x_5) - W_2(x_2 - x_5) -$ &c.;

$M_{-10} = 24 \times 31 - 5 \times 30 - 5 \times 25 - 11 \times 17 - 12 \times 7 = 198$

In the interval between W_3 and W_4, consider any two sections A and B, the latter being situated further to the right; let the distance between them be d. The leverage of P is greater at B than at A by the quantity d, and the *upward* bending moment of P is greater at B than at A by the quantity $P \, . \, d$; again, the leverages of W_1, W_2, and W_3, respectively, are greater at B than at A by the quantity d, and the *downward* bending moment due to W_1, W_2, and W_3 is greater at B than at A by the quantity $(W_1 + W_2 + W_3)\,d$. Hence M_B is to be derived from M_A by adding $P \, . \, d$ and subtracting $(W_1 + W_2 + W_3)\,d$; in our example $P > (W_1 + W_2 + W_3)$, so that the quantity to be added *exceeds* the quantity to be subtracted; that is

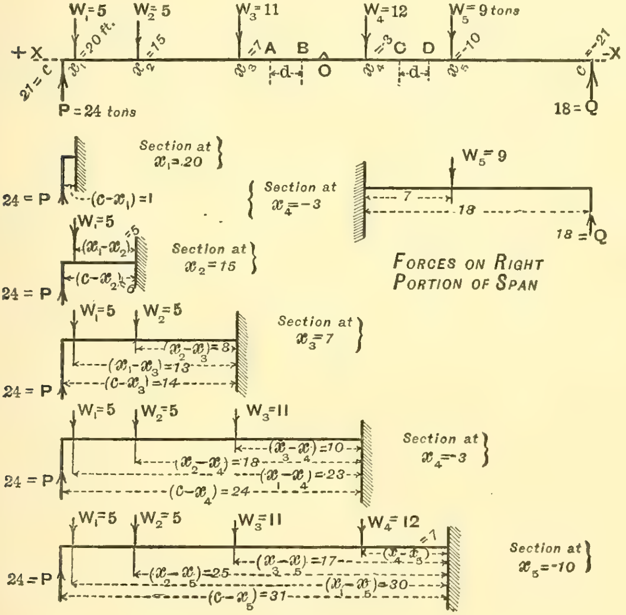


Fig. 99.

$$M_B > M_A, \quad \text{by} \quad (P - W_1 - W_2 - W_3)\,d,$$

a quantity proportional to d.

Again, for any two points C and D in the interval between W_4 and W_5, M_D is to be derived from M_C by adding $P \, . \, d$ and subtracting $(W_1 + W_2 + W_3 + W_4)d$; in this case, however, $P < (W_1 + W_2 + W_3 + W_4)$, so that the quantity to be added is *smaller* than the quantity to be subtracted; that is

$$M_D < M_C, \quad \text{by} \quad (W_1 + W_2 + W_3 + W_4 - P)d,$$

a quantity proportional to d.

In both cases, as we pass towards the right from one point to another in the interval between two weights, the change in the bending moment is uniform, and is proportional to the horizontal distance passed over—uniformly increasing or uniformly decreasing according as the supporting force P is greater or less than the sum of all the weights to the left of the points being considered; and if for some interval, P is equal to the sum of the weights to the left, the bending moment at points in that interval is constant.

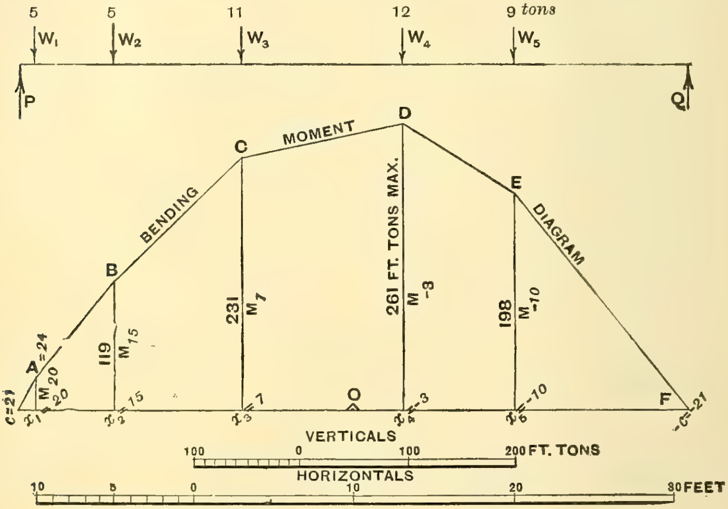


Fig. 100.

Bending Moment Diagram.—With a scale of feet for horizontals, lay off the span, fig. 100, and plot the positions of the loads; at each of these points erect a vertical equal to the bending moment thereat upon a suitable scale of, say, ft.-tons for verticals; join the tops of these ordinates by straight lines, and join each end of span to the top of the nearest ordinate. The lines just drawn give ordinates which vary uniformly in each interval, so that at any point whatever, the ordinate gives the bending moment at that point in ft.-tons when measured on the vertical scale. The bending moment at each load having been calculated analytically, a diagram thus constructed is a *graphical solution* for every other point.

Maximum Bending Moment.—In our example, since $P > W_1$, the line AB slopes *up* towards the right, that is, the bending moment increases in this interval; BC also slopes *up* towards the right since $P > (W_1 + W_2)$, and CD slopes *up* towards the right since $P > (W_1 + W_2 + W_3)$; DE slopes *down* towards the right since $P < (W_1 + W_2 + W_3 + W_4)$, and, lastly, EF slopes *down* towards the right since $P < (W_1 + W_2 + W_3 + W_4 + W_5)$. That is, beginning at the left end and passing towards the right, we find that the sides of the polygon slope $\left\{\begin{matrix}\text{up}\\ \text{down}\end{matrix}\right\}$ towards the right, while P is $\left\{\begin{matrix}\text{greater}\\ \text{less}\end{matrix}\right\}$ than the sum of the weights we have passed. For some loads there is one interval where P is *equal* to the sum of the weights passed, and in such cases the side of the polygon above that interval is horizontal. Evidently the highest ordinate is that of the angle of the polygon made by the last side which slopes up to the right, either with the first side which slopes down, or with the horizontal side if there is one. On fig. 100, that angle is D, and its ordinate is the maximum bending moment; so that, in our example, the maximum bending moment occurs at W_4, that is at the point $x = -3$, and its value is 261 ft.-tons. If one side of the polygon be horizontal, the ordinate to any point of this line is constant, and is a maximum.

The *Point of Maximum Bending Moment* is found thus—From P subtract the quantities W_1, W_2, W_3, &c., in succession until the remainder becomes zero, or first negative; when the remainder becomes zero; the maximum bending moment occurs at the weight last subtracted, at the weight next in order, and at every point between them: when the remainder becomes negative for the first time, the maximum occurs at the weight last subtracted, and at that point only. In our example, from $P = 24$, subtract $W_1 = 5$, $W_2 = 5$, $W_3 = 11$, and the remainder is $+ 3$; subtract $W_4 = 12$, and the remainder is negative for the first time; at this weight, that is, at $x_4 = -3$, the maximum bending moment 261 ft.-tons occurs.

Graphical Solution for Bending Moment Diagram.—The following purely graphical solution for the same problem requires no analysis, but must be drawn with accurate instruments and upon a large scale. Draw the vertical lines

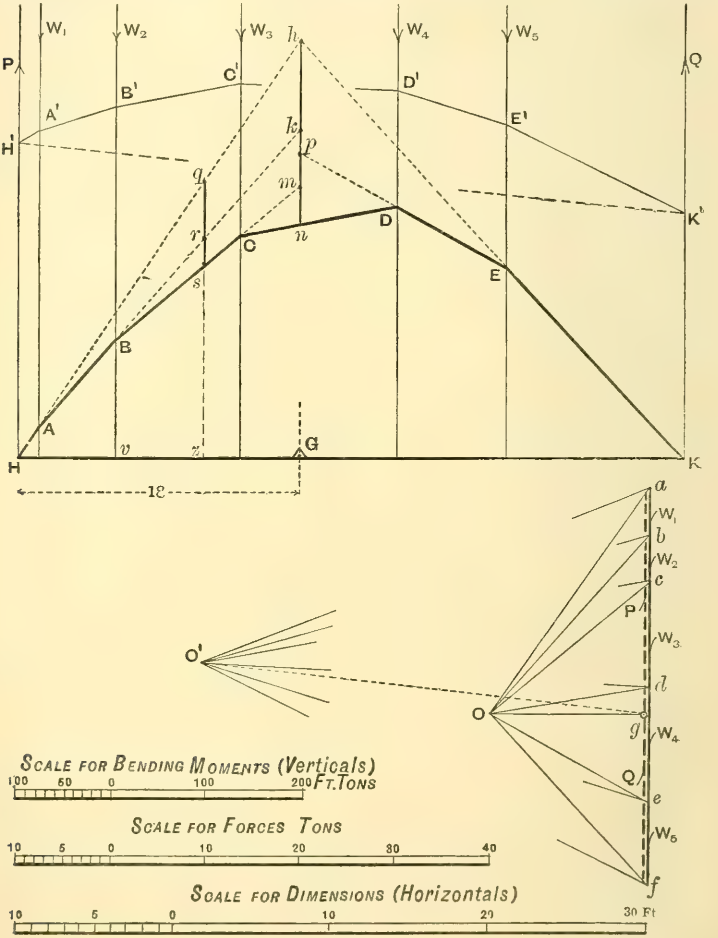


Fig. 101.

P, W_1, W_2, W_3, W_4, W_5, and *Q*, fig. 101, at the given horizontal distances apart upon a scale for dimensions; draw *ab*, *bc*, *cd*, *dc*, *ef* equal respectively to the forces W_1, W_2, W_3, W_4, W_5 upon a scale for forces, that is equal to 5, 5, 11, 12, and 9 tons in our example. Choose any pole O', and join it to a, b, c, d, e, and f. From H' any point on P draw $H'A$ parallel to $O'a$, and meeting W_1 at A'; draw $A'B'$, $B'C'$, $C'D'$, $D'E'$, and $E'K'$ parallel respectively to $O'b$, $O'c$, $O'd$, $O'e$, and $O'f$. Join $H'K'$; and draw $O'g$ parallel to $H'K'$, meeting af in g. Then, upon the scale for forces, $fg = Q$, and $ga = P$. In the example, $fg = 18$ tons, and $ga = 24$ tons; so far this is a graphical solution for finding the supporting forces. The polygon $H'A'B'C'D'E'K'$ is a bending moment diagram; but as it is inconvenient to have $H'K'$ sloping, we draw another polygon thus:—Choose a new pole O on the horizontal line passing through g, and such that the distance Og is some convenient integral number upon the scale for dimensions, on the diagram $Og = 10$; draw Oa, Ob, Oc, Od, Oe, Of, and Og. From H any point on P, draw HA parallel to Oa, and meeting W_1 at A; similarly, draw AB, BC, CD, DE, and EK parallel respectively to Ob, Oc, Od, Oe, and Of.

We have a new polygon, $HABCDEK$, which is the bending moment diagram, and HK is horizontal as desired. The vertical ordinates give the bending moments at each point of the span, upon a scale for verticals obtained by subdividing the divisions on the scale for forces by the number which Og measures on the scale for dimensions; thus on the figure this new scale is made by subdividing each division on the scale for forces into 10 equal parts, because we made $Og = 10$ on the scale for dimensions. We can now find the bending moment at z, any point of the span, by measuring the ordinate zs upon the proper scale; for instance, the ordinate at D measures 261, the bending moment in foot-tons at that point.

Produce the two sides HA and KE to meet at h; then G the point below h is the centre of gravity of the loads W_1, W_2, W_3, W_4, and W_5; on fig. 101, $HG = 18$ on the scale for dimensions: see example 103, fig. 79, p. 113.

Proof.—Draw the vertical hn, and produce all the sides of the polygon to meet it; at z, any point of span, draw the vertical zq, and let the sides of the polygon that lie to the left, or those sides produced, meet it in the points r and s; then sr is called the *intercept* on the vertical through z, made by the two sides of the polygon which meet at the angle B. Other intercepts on this vertical are zq and rq, made by the pairs of sides from the angles H and A respectively. Again, on the vertical through h we have the intercepts Gh, kh, mk, and nm made respectively by the pairs of sides from the angles H, A, B, and C; and also the intercepts Gh, ph, and np made respectively by the pairs of sides from the angles K, E, and D.

The triangle *Bsr* on the base *sr*, and of height *zv*, is similar to the triangle *Ocb* on the base *cb*, and of height *gO*, because their sides are respectively parallel; the bases of these triangles measured on one scale will be in the same proportion to each other as the heights measured on any other scale; hence

$$\overline{sr} : \overline{cb} \begin{smallmatrix}\text{on scale}\\\text{of forces}\end{smallmatrix} :: \overline{zv} : \overline{gO} \begin{smallmatrix}\text{on scale of}\\\text{dimensions}\end{smallmatrix};$$

or

$$\overline{sr} \begin{smallmatrix}\text{on scale}\\\text{of forces}\end{smallmatrix} = \frac{\overline{cb} \begin{smallmatrix}\text{on scale}\\\text{of forces}\end{smallmatrix} \times \overline{zv} \begin{smallmatrix}\text{on scale of}\\\text{dimensions}\end{smallmatrix}}{\overline{gO} \begin{smallmatrix}\text{on scale of}\\\text{dimensions}\end{smallmatrix}}$$

$$= \frac{W_2 \text{ tons} \times \overline{zv} \text{ feet}}{10} = \frac{\text{moment of } W_2 \text{ about } z}{10}.$$

But $\overline{sr}$ measures 10 times as much upon the scale for verticals as it does upon the scale for forces, since we made the scale for verticals by subdividing each division on the scale for forces into ten parts; therefore

$$\overline{sr} \text{ on vertical scale} = \text{moment of } W_2 \text{ about } z.$$

That is, measuring on the vertical scale, the intercept on the vertical through any point *z*, made by the pair of sides from any angle of the polygon, equals the moment about *z* of *W*, the force at that angle.

To show that if $P = \overline{ga}$, and $Q = \overline{fg}$, they will balance $W_1 + W_2 + W_3 + W_4 + W_5$. It is evident that $P + Q = \Sigma W$; and if you produce *Hh* to meet the vertical through *K*, the intercept by the pair of sides from *H* equals the moment of *ga*, that is of *P*, about *K*; the five intercepts made on the vertical through *K* by the pairs of sides from *A*, *B*, *C*, *D*, and *E*, are the moments about *K* of the weights W_1, W_2, W_3, W_4, and W_5 respectively, and it is evident that the first of these intercepts is identically equal to the sum of the other five. Hence the moment of *P* equals the sum of the moments of W_1, W_2, W_3, W_4, and W_5 about *K*; that is, *ga* exactly represents the value of *P*.

To show that *G* is the centre of gravity of the weights W_1, W_2, W_3, W_4, and W_5. The sum of the moments about *G* of all the weights to the left of *G*, that is of W_1, W_2, and W_3, is $hk + km + mn = hn$; again, the sum of the moments about *G* of all the weights to the right of *G*, that is of W_4 and W_5, is $np + ph = hn$; hence the sum of the moments about *G* of all the weights to the left equals the sum of the moments about *G* of all the weights to the right; and since these moments tend to turn the beam in opposite directions, the one sum destroys the other, or the sum of the moments of all the weights W about G is zero; that is, G is the centre of gravity of the weights.

To show that $\overline{zs}$, the ordinate of the polygon, measured on the vertical scale equals the bending moment at z,

$$\begin{aligned} \overline{zs} &= \overline{zq} - \overline{qr} - \overline{rs}, \text{ all measured on the scale for verticals} \\ &= \text{Mom. of } P - \text{Mom. of } W_1 - \text{Mom. of } W_2, \text{ all about } z \\ &= \text{Bending Moment at } z. \end{aligned}$$

This is Culman's Theorem. It will be seen that this is a special case of fig. 74, p. 100. Also *HABCDEK* is a balanced frame, the vectors from O giving the thrust on the bars, &c.

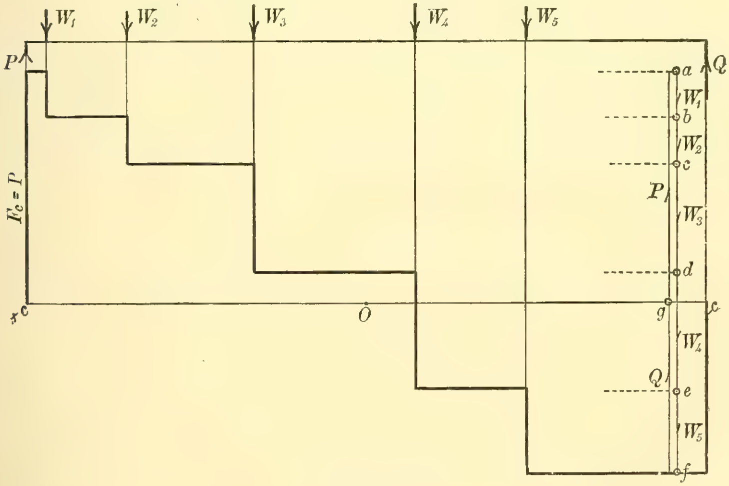


Fig. 102.

In all cases of fixed loads the maximum bending moment occurs where the locus of the shearing force diagram crosses its base.

Graphical Solution for Shearing Force Diagram.—Fig. 102. P and Q are determined graphically by finding the point g, as on fig. 101, or by moments. Through g, fig. 102, draw a horizontal line and produce the lines of action of P, W_1, W_2, ... Q to cut it; join these lines of action in pairs by horizontals through *a*, *b*, *c*, *d*, *e*, *f*, respectively. The scale used for forces is also the scale for shearing forces.

The construction is cyclical; through each of the points *a*, *b*, *c*, *d*, *e*, *f*, *g*, a horizontal line being drawn to join the lines of action of the forces in pairs, viz.:—from *a* joining P and W_1, from *b* joining W_1 and W_2, from *c* joining W_2 and W_3, from *d* joining W_3 and W_4, from *e* joining W_4 and W_5, from *f* joining W_5 and Q, and from *g* joining Q and P.

CANTILEVER WITH UNEQUAL WEIGHTS AT INTERVALS.

A cantilever (fig. 103) 12 feet long supports three weights, $W_1 = 8$, $W_2 = 6$, and $W_3 = 4$ tons at points whose abscissæ reckoned from the fixed end are $x_1 = 12$, $x_2 = 10$, and $x_3 = 4$ feet. This loading is shown also on fig. 87, p. 123, but on that figure the cantilever projects beyond W_1, and so c is greater than x_1; it is evident, however, that the part which so projects is not strained, so that although a cantilever does so project, yet, for purposes of calculation, its length may be considered as the distance from the fixed end to the most remote load; this is shown on fig. 103, where $c = x_1$.

We will now calculate the bending moments at the fixed end, and at points where the weights stand, systematically from the forces upon the left-hand portion.

Ft.-tons

$M_{x_1 \text{ or } c}$ = ... 0

$M_{x_2} = W_1(x_1 - x_2)$; $M_{10} = 8 \times 2$ = ... 16

$M_{x_3} = W_1(x_1 - x_3) + W_2(x_2 - x_3)$; $M_4 = 8 \times 8 + 6 \times 6 = 100$

$M_0 = W_1x_1 + W_2x_2 + W_3x_3$; $= 8 \times 12 + 6 \times 10 + 4 \times 4 = 172$

The Bending Moment at any point is the sum of the products got by multiplying each weight to the left of that point by its distance therefrom; the above is an arithmetical sum, since all the weights tend to bend the left portion downwards. Having found the bending moment at one point, we may derive the bending moment at another point nearer the fixed end and such that no weight intervenes, by adding the product of the sum of all the weights to the left into the distance between the two points, since no *new* weights have to be considered, and all the leverages have increased by the distance between the two points. Hence the bending moments increase uniformly in each interval as you move towards the fixed end.

Bending Moment Diagram.—With a scale of feet for horizontals, lay off the length (fig. 103) and plot the positions of the loads; at each of these points, and at the fixed end, draw a vertical ordinate *downwards*, equal to the bending moment thereat, upon a suitable scale of foot-tons for verticals; join the ends of the ordinates by straight lines. The ordinates are drawn *downwards* to signify that the moments on a cantilever are of a different sign, as compared to those on a beam. In either case the moments are all of one sign, which will always be reckoned as positive.

Maximum Bending Moment.—It is evident that the maximum bending moment is at the fixed end, and is

$$M_0 = \Sigma(Wx).$$

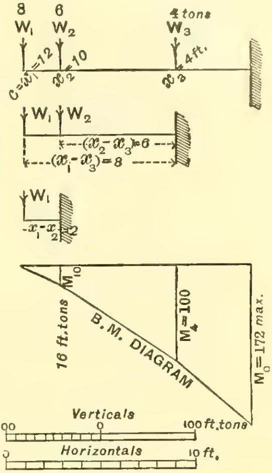


Fig. 103.

Graphical Solution for Shearing Force Diagram.—Fig. 104. From *a, b, c, d*, joints of load-line, draw horizontals joining the lines of action of the forces in pairs, the vertical through the fixed end being reckoned as a line of action. Thus draw horizontals, from *a* joining the vertical through the fixed end and W_1, from *b* joining W_1 and W_2, from *c* joining W_2 and W_3, and from *d* joining W_3 and the vertical through the fixed end. The scale for forces is also the scale for shearing forces.

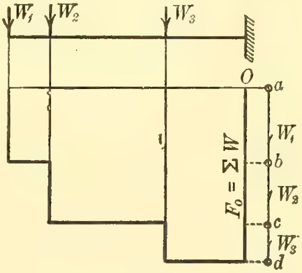


Fig. 104.

Graphical Solution for Bending Moment Diagram.—Draw a vertical line through *K*, the fixed end, and draw the vertical lines W_1, W_2, W_3 at the given horizontal distances apart upon a scale for dimensions (fig. 105). Draw *ab, bc, cd*, equal respectively to the forces W_1, W_2, W_3 upon a scale for forces; that is, equal to 8, 6, and 4 tons in our example. Choose a pole *O* on the horizontal line passing through *a*, and such that the distance *Oa* is some convenient integral number upon the scale for dimensions; on the diagram $Oa = 10$; and draw *Ob, Oc*, and *Od*. From *A* any point on W_1 draw *AB* parallel to Ob, and meeting W_2 at B; similarly, draw BC, CD, and AK parallel, respectively, to Oc, Od, and Oa. Then $ABCDK$ is the bending moment diagram; its vertical ordinates give the bending moment at each point of the span upon a scale for verticals, obtained by subdividing the divisions on the scale for forces by the number which Oa measures on the scale for dimensions; thus, on the figure, this new scale is made by subdividing each division on the scale for forces into 10 equal parts, since $Oa = 10$ on the scale for dimensions.

Proof.—At any point e draw the ordinate eh, and produce those sides of the polygon that lie to the left till they meet it. On the figure, they meet it at e, f, g, h. As in the proof to fig. 101, the intercept on the vertical through e made by the pair of sides from any angle of the polygon, will, when measured upon the scale for verticals, give the moment about e of the force at that angle. Now the ordinate at e is the sum of the intercepts made by the pairs of sides from each angle to the left, and so will give on the vertical scale the sum of the moments of the forces to the left of e; that is, the bending moment at e. On fig. 105, ef the intercept by the pair of sides from A, gives the moment of W_1 about e; fg, the intercept by the pair of sides from B, gives the moment of W_2 about e, and gh gives the moment of W_3 about e. Hence eh gives the sum of these three moments, that is the bending moment at e.

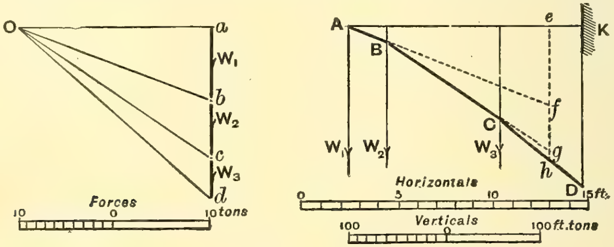


Fig. 105.

SHEARING FORCE DIAGRAM DUE TO ONE LOAD.

For a beam with W at the centre, the shearing force diagram consists of two rectangles of height $\frac{1}{2}W$, one standing above the left half and the other below the right half of span.

For a beam loaded with W at a point dividing the span into any two segments, the shearing force diagram consists of two rectangles, one standing above the left segment, the other below the right segment. The height of each is inversely proportional to the length of the segment on which it stands, and the sum of their heights is W.

In these two cases it is evident, and it can easily be proved for a general case, fig. 102, that the area of the part of the shearing force diagram above the base is equal to the area of the part below the base.

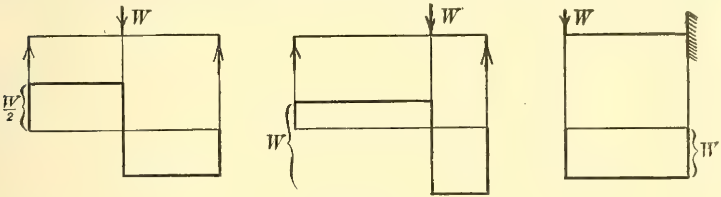


Fig. 106.

For a cantilever with W at its free end, the shearing force diagram is evidently a rectangle of height W, standing below the span.

Beam Loaded at the Centre.

Fig. 107. Let W be the load at the centre. By symmetry $P = Q = \frac{1}{2}W$. For a section distant from the centre x towards the left, consider the forces on the left-hand portion. The only force is P, and its leverage is $(c - x)$: hence

$$M_x = P(c - x) = \frac{W}{2}(c - x),$$

the equation to the bending moment.

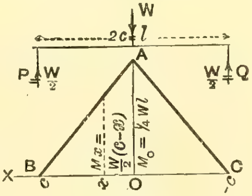


Fig. 107.

The value of M_x is zero at the end, that is where $x = c$; it increases uniformly as x decreases, that is, as you approach the centre, and it is greatest where $x = 0$, that is at the centre; by symmetry for the other half of the span, the value will decrease uniformly till it is again zero at the right-hand end; hence the maximum bending moment

$$M_0 = \frac{W}{2}c = \frac{1}{4}.Wl.$$

In Rankine's "Applied Mechanics," the maximum bending moment in each case is given in the above form, viz., maximum bending moment = constant × total load × span, or $M_0 = m.W.l$.

Throughout a great portion of that work. m stands for this constant, which he calls the *numerical coefficient of the maximum bending moment expressed in terms of the load and span.* The value of m depends upon the *manner* of loading and of support. In the case we have just solved, we specify the mode of support by calling the piece a beam, and the manner of loading when we say that W is at the centre, and we find $m = \frac{1}{4}$.

Bending Moment Diagram.—Upon a convenient scale of feet, lay off BC, fig. 107, equal to the span; construct a triangle with its apex above the centre O. Draw a scale of ft.-lbs. to measure verticals upon, such that OA shall measure upon it one-fourth of the product of the load in lbs. into the span in feet.

Graphical Solution.—The simplest graphical solution is to draw the bending moment diagram as above; one which is purely graphical may be made as in fig. 101, and it will not be necessary to use the first pole O', &c., as we know that $P = Q = \frac{1}{2}W$. Draw ab vertical and equal to the load W; from its middle point draw a horizontal line; choose O at a distance from ab equal to some convenient integral number on the scale for dimensions, and draw Oa and Ob. Then fig. 107 is constructed by drawing BA parallel to Oa, and AC parallel to OB; and a scale for verticals is obtained by subdividing the scale for forces by the number chosen for the distance of O from ab.

Cantilever loaded at the end.—Fig. 108. To find the bending moment at a section distant x from the fixed end K, consider the loads to the left of that section. The only force is W, and its leverage about the section at x is $(c - x)$, and we have

$$M_x = W(c - x),$$

the equation to the bending moment.

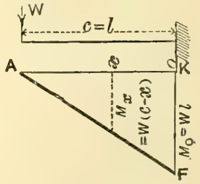


Fig. 108.

The value of M_x is zero when x equals c, that is at the free end; it uniformly increases as x decreases, and is a maximum when $x = 0$, that is at the fixed end; the maximum bending moment is

$$M_0 = Wc = W.l,$$

and the value of the constant is $m = 1$.

Bending Moment Diagram.—Upon a convenient scale of feet, lay off KA (fig. 108) equal to the length; draw below KA the right-angled triangle FAK, with the right angle at the fixed end, and construct a scale of ft.-lbs. for verticals, such that *KF* may measure upon it the product of the load in lbs. into the length in feet.

Graphical Solution.—The simplest graphical solution is to draw the bending moment diagram as above; one which is purely graphical may be made as on fig. 105. Make *ab* equal to *W*, and from *a* draw a horizontal line; choose a point *O* such that *Oa* is a convenient integer on the scale for dimensions, and join *Ob*. Draw *AF* (fig. 108) parallel to *Ob*, and a scale for verticals is obtained by subdividing the scale for forces by the value of *Oa*.

BEAM UNIFORMLY LOADED.

Fig. 109. Let w be the intensity of the uniform load in lbs. per foot of span. This is represented by a load area, consisting of a rectangle of height w feet standing on the span, and weighing one lb. per square foot. The total load W is the area of this rectangle, so that

$$W = 2wc, \quad \text{or} \quad w = \frac{W}{2c}.$$

To find the bending moment at a section distant x from the centre O. Consider the load area standing upon the portion of the span to the left of that section; it consists of a rectangle of length $(c - x)$ feet, its area is $w(c - x)$ square feet, and its weight is $w(c - x)$ lbs. This weight may be considered to be concentrated at the centre of gravity of the area—that is at its middle point; this gives a bending moment about the section equal to that for the actual distribution. We have two forces to the left of the section, $P = wc$ lbs., half the total load acting upwards with a leverage of $(c - x)$ ft., and $w(c - x)$ lbs. acting downwards with a leverage of $\frac{1}{2}(c - x)$ ft.; hence

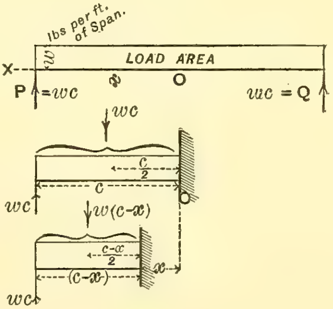


Fig. 109.

$$M_x = P(c - x) - w(c - x) \times \frac{c - x}{2} = wc(c - x) - \frac{w}{2}(c - x)^2$$

$$= \frac{w}{2}(c - x)(2c - c + x) = \frac{w}{2}(c - x).(c + x) = \frac{w}{2}(c^2 - x^2);$$

$$= \frac{W}{4c}(c^2 - x^2), \text{ the equation to the bending moment.}$$

The value of M_x is zero where $x = \pm c$, that is at the two ends; it increases as x decreases numerically from c and from $-c$, that is from both ends towards the centre; and it is greatest where $x = 0$, that is at the centre. By making $x = 0$, we have $M_0 = \frac{1}{4} Wc$; or by considering the section at O, the maximum bending moment is

$$M_0 = P \,.\, c - wc \,.\, \frac{c}{2} = wc \,.\, c - \frac{w}{2} c^2 = \frac{wc^2}{2} = \tfrac{1}{4} Wc, \quad \text{or} \quad \tfrac{1}{8} Wl.$$

In this case, the value of the constant is $m = \frac{1}{8}$.

Bending Moment Diagram.—Examining the equation to the bending moment, we see that the ordinate M_x equals a constant term, minus a term in x^2; hence we know that the locus is a parabola, with its axis vertical and with its apex A (fig. 110) exactly above the origin O at the height $\frac{1}{4} Wc = \frac{1}{8} Wl$; and that the modulus of the parabola is $\frac{W}{4c}$, the coefficient of x^2, so that the principal equation to the parabola, that is, taking A as origin, is

$$Y = \frac{W}{4c} X^2.$$

Graphical Solution.—With a scale of feet for horizontals, lay off the span BC (fig. 110) and draw a vertical OA upwards through O; apply the parallel rollers to BC; place *any* parabolic segment cut upon pear-tree or card-board against the rollers (see fig. 91), with its apex on the vertical through O; shift the rollers till the curved edge passes through B and C, which it will do simultaneously, and draw the curve BAC; construct a scale of ft.-lbs. for verticals, such that

$$OA = \tfrac{1}{8} W \,.\, l.$$

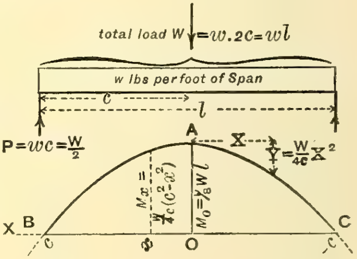


Fig. 110.

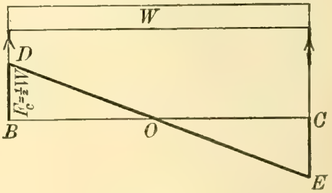


Fig. 111.

Shearing Force Diagram.—Fig. 111. Draw verticals, one upwards from the left end, another downwards from the right end of span, each equal to $\frac{1}{2}W$; join their extremities with a straight line, which will cut the base at O the centre.

The shearing force diagram consists of two right-angled triangles, whose common height is $\frac{1}{2}W$; one is above the left half, the other is below the right half, of span, and the right angle of each is at the end of span.

CANTILEVER UNIFORMLY LOADED.

Fig. 112.—Consider a section at the distance x from the fixed end. The load area standing on the portion to the left of that section is a rectangle of length $(c - x)$ ft. and height w ft.; its area represents a weight of $w(c - x)$ lbs., which may be considered to be concentrated at the centre of gravity of the rectangle; and so, to the left of the section, there is to be taken into account only one force $w(c - x)$ lbs., having a leverage about the section of $\frac{1}{2}(c - x)$ ft.; hence

$$M_x = w(c - x) \cdot \frac{c - x}{2}$$

$$= \frac{w}{2}(c - x)^2 = \frac{W}{2c}(c - x)^2,$$

the equation to the bending moment.

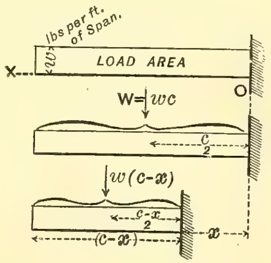


Fig. 112.

The value of M_x is zero where $x = c$, that is at the free end; it increases as x decreases; and it is greatest where $x = 0$, that is at the fixed end. Putting $x = 0$ we have the greatest value, or directly from the figure we find the maximum bending moment

$$M_0 = W \cdot \frac{c}{2} = \tfrac{1}{2}Wl.$$

In this case, the value of the constant is $m = \frac{1}{2}$.

Bending Moment Diagram.—For the sake of comparison with the diagrams for beams, we may consider the bending moments on a cantilever to be negative, when the equation becomes

$$M_x = -\frac{W}{2c}(c - x)^2 = \frac{W}{2c}(2c - x)x - \frac{Wc}{2};$$

the locus is a parabola (see fig. 92) with its axis vertical and to the left of O, and with its apex on the axis of X. The value $x = c$ makes $(2c - x)x$ greatest, and therefore makes $M_c = 0$ a *positive* maximum; so that the apex lies to the left of O at a distance c,—that is, the apex is at the free end. Since the coefficient of x^2 is $\frac{W}{2c}$, the principal equation to the parabola is $Y = \frac{W}{2c}X^2$.

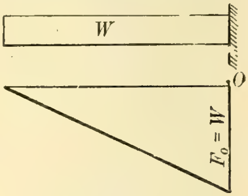


Fig. 113.

Graphical Solution.—With a convenient scale of feet, lay off KA, fig. 114, equal to the length; place the parallel rollers to KA; set *any* parabolic segment against the rollers, with its apex at A; draw the curve AG, till it meets the vertical through K at G. Construct a scale of ft.-lbs. for verticals, such that KG may measure upon it one-half the product of the total load in lbs. into the span in feet.

Shearing Force Diagram.—Fig. 113. Draw a vertical downwards from the fixed end of the base equal to W, and join its extremity to the free end of the base.

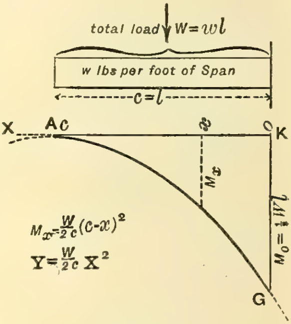


Fig. 114.

The shearing force diagram is a right-angled triangle of height W, standing below the span, and with the right angle at the fixed end.

BEAMS SUBJECTED TO EQUAL LOADS AT EQUAL INTERVALS.

The actual loading on a girder is often of this kind because the dead weight of the platform is transmitted to it by equidistant cross-girders. This again may have to be combined with the uniform weight of the girder itself: see such a combination following. Also the wheels of rolling stock riding over a girder or stringer while at rest are sensibly uniform loads at uniform distances. If the rolling stock move, the problem is changed and will be treated of in a chapter to follow.

On fig. 115. Let W, the total load, be distributed over the span l, in n parts each equal to w, and at equal intervals a apart; then $nw = W$, $(n + 1)a = 2c = l$, $P = \frac{1}{2}nw$. Take S, a point directly under one of the weights, and let $BS = ra$, then r will be a whole number; and if x be the distance of S from O the centre, then $ra = (c - x)$. On the portion of the beam to the left of S, there are in all r forces, viz., $P = \frac{1}{2}nw$ acting upwards with a leverage about S of ra, and $(r - 1)$ forces each equal to w and acting downwards; the nearest to S has a leverage a, the next a leverage $2a$, the next a leverage $3a$, &c., and the last a leverage $(r-1)a$; hence

$$\begin{aligned} M_x &= P\,.\,ra - w\,.\,a - w\,.\,2a - w\,.\,3a\,.\,.\,.\,- w\,.\,(r-1)\,a \\ &= \frac{nw}{2}\,.\,ra - wa\,(1 + 2 + 3\,.\,.\,.\,+ \overline{r-1}) \\ &= \frac{nw}{2}\,.\,ra - wa\,\frac{(r-1)r}{2} = \frac{w}{2}\,.\,ra\,(n+1-r). \end{aligned}$$

But

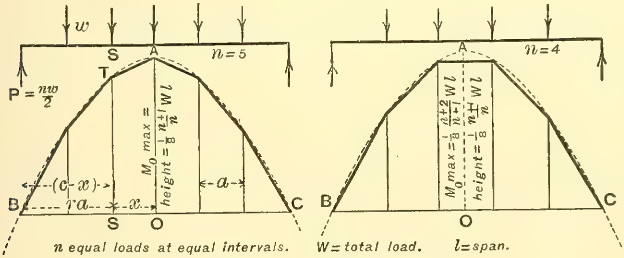
$$w = \frac{W}{n}, \quad ra = c - x, \quad a = \frac{2c}{n+1}, \quad \therefore\ r = \frac{(n+1)(c-x)}{2c};$$

substituting, we have

$$\begin{aligned} M_x &= \frac{W}{2n}(c-x)\left\{n + 1 - \frac{(n+1)(c-x)}{2c}\right\} \\ &= \frac{W}{2n}(c-x)(n+1)\,\frac{c+x}{2c} = \frac{W}{4c}\,\frac{n+1}{n}\,(c^2 - x^2), \end{aligned}$$

which is the equation to the locus of T.

This equation gives the bending moment only at points where weights are, that is for values of x which are multiples of a, but not at intermediate points; it only differs from the equation we had for an uniform load, by the constant factor $\frac{n+1}{n}$. The locus of T, the tops of the ordinates at the points where the weights are situated, is a parabola with its axis vertical, and its apex above O. Putting y instead of M_x for the ordinate to the curve at any point, we have


w
S
A
n=5
T
P = $\frac{nw}{2}$
M_0 max = height = $\frac{1}{8}\,\frac{n+1}{n}\,W\,l$
(c - x)
a
B
ra
x
C
S
O
n=4
A
M_0 max = $\frac{1}{8}\,\frac{n+2}{n+1}\,W\,l$
height = $\frac{1}{8}\,\frac{n+1}{n}\,W\,l$
B
C
O

n equal loads at equal intervals. W = total load. l = span.
If W be the same in both so also is M_0 the max.

Fig. 115.

$$y = \frac{W}{4c}\,\frac{n+1}{n}\,(c^2 - x^2),$$

so that y is a maximum where $x = 0$, and

$$y_0 = \frac{W}{4c}\,.\,\frac{n+1}{n}\,c^2 = \tfrac{1}{8}\,\frac{n+1}{n}\,W\,.\,l$$

a maximum, and the height of the apex A in every case.

On fig. 115, it will be seen that if n be odd, a weight comes exactly at the centre; so that y_0, the ordinate of the parabola, is the maximum bending moment; hence the maximum bending moment

$$M_0 = \tfrac{1}{8}\,\frac{n+1}{n}\,W\,.\,l, \qquad (n \text{ odd}),$$

and the value of the constant is

$$m = \tfrac{1}{8}\,\frac{n+1}{n}.$$

When n is even, the maximum bending moment M_0 is less than y_0, and equals the ordinate of the parabola at the weight on either side of the centre; so that to find the value of M_0, it is only necessary to substitute for x half of an interval, that is $\frac{1}{2}a$, or $\frac{c}{n+1}$; hence the maximum bending moment

$$M_0 = y_{\left(\frac{c}{n+1}\right)} = \frac{W}{4c}\,\frac{n+1}{n}\left\{c^2 - \left(\frac{c}{n+1}\right)^2\right\} = \tfrac{1}{8}\,\frac{n+2}{n+1}\,W\,.\,l, \qquad (n \text{ even}),$$

and the value of the constant is

$$m = \tfrac{1}{8}\,\frac{n+2}{n+1}.$$

The chords of the parabola give the bending moments at points intermediate between the weights, since the bending moment varies uniformly in these intervals.

Cor.—If the load be distributed in equal portions at equal intervals, the maximum bending moment exceeds that for the uniform distribution in the ratio

$$\frac{n+1}{n} \quad \text{or} \quad \frac{n+2}{n+1},$$

according as n, the number of parts into which the load is divided, is odd or even.

Thus suppose M_U is the maximum for uniform distribution; then if the load be concentrated at the centre, that is, if $n = 1$, the maximum bending moment

$$= \frac{+1}{1}\,M_U = 2M_U;$$

see figs. 107 and 110; if concentrated equally at two points dividing the span into three equal intervals, that is if $n = 2$, the maximum bending moment

$$= \frac{2+2}{2+1}\,M_U = \tfrac{4}{3}\,M_U;$$

if concentrated equally at three points, dividing the span into four equal intervals, that is if $n = 3$, the maximum bending moment

$$= \frac{3+1}{3}\,M_U = \tfrac{4}{3}\,M_U; \text{ \&c.}$$

The last two are each $\frac{4}{3}\,M_U$, and are therefore equal to each other; and for the same total load W placed on the span at equal intervals, the maximum bending moment M_0 will be the same whether the load be divided into an *even* number of equal parts, or the *next consecutive odd* number of equal parts.

The Bending Moment Diagram is the polygon formed by the chords of the parabola.

Cor.—If n be great, the polygon nearly coincides with the parabola, $\dfrac{n+1}{n}$ approaches unity, and the parabola is nearly the *same* as that for the load uniformly distributed; that is, if the load be concentrated equally at a great number of points equally apart, as, for instance, when a girder supports, at equal intervals, the ends of cross girders which carry equal loads, then the bending moments will be nearly the same as for the total load uniformly distributed.

Graphical Solution.—With a scale of feet for horizontals, lay off the span BC (fig. 115), and draw a vertical OA upwards through O. Apply the parallel rollers to BC; place *any* parabolic segment cut on pear-tree against the rollers (see fig. 91) with its apex on the vertical through O; shift the rollers till the curved edge passes through B and C, which it will do simultaneously, and draw the dotted curve BAC. Draw up verticals to meet the parabola from the points at which the weights are, and draw the chords of the parabola. Construct a scale of ft.-lbs. for verticals such that $OA = \frac{1}{8}\dfrac{n+1}{n} W \,.\, l$; where W = total load in lbs., l = span in feet, and n = the number of equal parts into which the load is divided.

EXAMPLES.

108. A beam 20 feet span supports a load of 10 tons uniformly distributed. Find the shearing forces at intervals of 5 feet.

> *Ans.* $F_{10} = 5$ tons; $F_5 = 2{\cdot}5$ tons; $F_0 = 0$. On right half of span, the values are the same, but negative.

109. A beam 20 feet span supports a load of 10 tons concentrated at the centre. Find the shearing forces.

> *Ans.* $F_{10\text{ to }0} = 5$ tons, $F_{0\text{ to }-10} = -5$ tons, and the sign changes at the centre.

110. In the example given on page 140 and with the data shown in fig. 99, find the shearing forces.

Ans.

$$\begin{aligned}
F_{21\text{ to }20} &= P &&= 24 \text{ tons.}\\
F_{20\text{ to }15} &= 24 - W_1 &&= 19 \quad ,,\\
F_{15\text{ to }7} &= 19 - W_2 &&= 14 \quad ,,\\
F_{7\text{ to }-3} &= 14 - W_3 &&= 3 \quad ,,\\
F_{-3\text{ to }-10} &= 3 - W_4 &&= -\,9 \quad ,,\\
F_{-10\text{ to }-21} &= -9 - W_5 &&= -\,18 \text{ tons} = (-Q).
\end{aligned}$$

See figs. 100 and 102, and note that the shearing force changes sign, and that the bending moment is a maximum at the point $x = -3$.

111. A beam 24 feet span is loaded with 20, 30, and 40 tons at points dividing it into equal intervals. Find the maximum bending moment and the point where it occurs.

As in example No. 96, we have $P = 40$ tons; deduct 20 and it leaves 20; deduct 30 and the remainder is negative for the first time: hence the maximum occurs under the load 30, that is at 12 feet from the left end, and maximum

$$M_{12} = P \times 12 - W_1 \times 6 = 40 \times 12 - 20 \times 6 = 360 \text{ ft.-tons.}$$

112. In example No. 111, find the bending moments at the other two weights.

$M_6 = P \times 6 = 40 \times 6 =$ . . . . 240 ft.-tons.

$M_{18} = P \times 18 - W_1 \times 12 - W_2 \times 6 = 40 \times 18 - 20 \times 12 - 30 \times 6 =$ 300 ,,

or $M_{18} = Q \times 6 = 50 \times 6 =$ . . . . 300 ,,

113. In example No. 111, find the bending moments at points midway between the weights.

$M_9 =$ an average of M_6 and $M_{12} =$ . . . . 300 ft.-tons.

$M_{15} =$ an average of M_{12} and $M_{18} =$ . . . . 330 ,,

or $M_{15} = P \times 15 - W_1 \times 9 - W_2 \times 3 = 40 \times 15 - 20 \times 9 - 30 \times 3 =$ 330 ,,

114. In example No. 97, find the bending moments at intervals of five feet.

$M_5 = P \times 5 = 11{\cdot}6 \times 5 =$ . . . . 58 ft.-tons.

$M_{10} = P \times 10 - W_1 \times 2 = 11{\cdot}6 \times 10 - 6 \times 2 =$ . . . . . 104 ,,

$M_{15} = P \times 15 - W_1 \times 7 = 11{\cdot}6 \times 15 - 6 \times 7 =$ . . . . . 132 ,,

$M_{20} = P \times 20 - W_1 \times 12 - W_2 \times 2 = 11{\cdot}6 \times 20 - 6 \times 12 - 14 \times 2 =$ 132 ,,

$M_{25} = P \times 25 - W_1 \times 17 - W_2 \times 7 - W_3 \times 1$
$= 11{\cdot}6 \times 25 - 6 \times 17 - 14 \times 7 - 8 \times 1 =$ . . . . . 82 ,,

115. Find the maximum bending moment in example No. 97.

Ans. $M_{18} = 148{\cdot}8$ ft.-tons, maximum.

116. Find the maximum bending moment in example No. 106.

Ans. $M_{19} = 256$ ft.-tons, maximum.

Or, taking the centre as origin, $M_2 = 256$ ft.-tons, maximum, and is at the wheel transmitting $W_3 = 11$ tons.

Thus for the position of the locomotive given in No. 106, that is with its fore wheel 6 feet from the left end, the maximum bending moment 256 ft.-tons occurs at the wheel transmitting $W_3 = 11$ tons; whereas for the position of the locomotive given in figs. 99 and 100, that is with its fore wheel one foot from the left end, the maximum bending moment 261 ft.-tons occurred at the wheel transmitting $W_4 = 12$ tons. In like manner, we may find the maximum bending moment for the locomotive in a variety of positions, in each case observing the wheel under which it occurs. For the two positions which we have investigated, observe that the maximum in the one case is greater than the maximum in the other, and occurs at a different place. It may be that the maximum for some third position is greater than either, and occurs at some other place; so that we do not know that 261 ft.-tons is the greatest possible bending moment that can be produced upon the beam by the locomotive, nor are we even sure that 256 ft.-tons is the greatest bending moment that can possibly occur at the point two feet to the left of the centre of span, for it is possible that some new position of the locomotive may produce a greater bending there. See Chapters XI. and XII.

117. A beam 40 feet span supports four weights $W_1 = 50$, $W_2 = 10$, $W_3 = 20$, and $W_4 = 30$ cwts. at points whose abscissæ, measuring from the centre to left and right, are $x_1 = 10$, $x_2 = 2$, $x_3 = -12$, and $x_4 = -16$ feet. Find the supporting force at the left end, the maximum bending moment, and the place where that maximum occurs.

Ans. $P = 50$ cwts.; and since $P - W_1 = 0$, the maximum bending moment occurs at x_1, at x_2, and at every intermediate point, and $M_{10 \text{ to } 2} = 500$ ft.-cwts., maximum.

118. A beam 50 feet span has weights 5, 8, 9, 12, 9, 8, and 5 cwts. placed at equal intervals of 5 feet, in order, upon the span, and with the load 12 cwts. at the centre. Find the maximum bending moment.

Ans. $M_0 = 500$ ft.-cwts., maximum.

119. A cantilever 12 feet long bears four loads $W_1 = 8$, $W_2 = 6$, $W_3 = 9$, and $W_4 = 12$ tons at distances from the fixed end of $x_1 = 12$, $x_2 = 8$, $x_3 = 6$, and $x_4 = 2$ ft. Find the bending moment at each weight, and also the maximum bending moment.

Ans. $M_{12} = 0$, $M_8 = 32$, $M_6 = 60$, $M_2 = 152$, and $M_0 = 222$ ft.-tons.

120. In the previous example, find the bending moments at points midway between each pair of weights.

Ans. They are averages of those at the weights on each side of such points, or they may be calculated independently. $M_{10} = 16$, $M_7 = 46$, $M_4 = 106$ ft.-tons.

121. A cantilever is loaded with weights of 8, 6, and 4 tons at distances of 12, 10, and 4 feet from the fixed end. Find the bending moment at each weight, and draw a bending moment diagram upon a large scale: see fig 103.

122. Draw also a bending moment diagram by the graphical construction (fig. 105) to large scales. From either diagram, by measurement, find the bending moments at intervals of two feet.

Ans. $M_0 = 172$, $M_2 = 136$, $M_4 = 100$, $M_6 = 72$, $M_8 = 44$, $M_{10} = 16$, $M_{12} = 0$ ft.-tons.

123. A cantilever 12 feet long is uniformly loaded with 3 cwts. per foot-run. Find the equation to the bending moment.

Ans. $M_x = \frac{3}{2}(12 - x)^2$ ft.-cwts.

124. In the previous example, find the bending moments at intervals of two feet by substituting for x into the equation.

Ans. $M_0 = \frac{3}{2}(12 - 0)^2 = 216$ ft.-cwts. $M_2 = \frac{3}{2}(12 - 2)^2 = 150$ ft.-cwts. $M_4 = 96$, $M_6 = 54$, $M_8 = 24$, $M_{10} = 6$, $M_{12} = 0$.

125. In example No. 123, find M_4, directly, by taking a section at the point $x = 4$.

Ans. $M_4 = 3\,(12 - 4) \times \dfrac{12 - 4}{2} = 96$ ft.-cwts.

126. A cantilever 20 feet long is uniformly loaded with 2 tons per foot-run. Find the maximum bending moment.

Ans. $M_0 = m \,.\, W \,.\, l = \frac{1}{2} \times 40 \times 20 = 400$ ft.-tons.

127. For the previous example, find the bending moment at the centre.

The bending moment diagram is a parabola, with its apex at the free end; for the middle point, the horizontal ordinate is half that for the fixed end, measuring from the apex; and since the verticals vary as the squares of the horizontals, the bending moments at the centre and at the fixed end are in the proportion of one and four.

Ans. $M_{10} = 100$ ft.-tons.

128. Find the principal equation to the parabola which, in each case, forms the bending moment diagram in examples Nos. 123 and 126.

Ans. $Y = \frac{3}{2} X^2$, and $Y = X^2$.

129. A beam 20 feet span supports a load of 20 tons at its centre. Find the maximum bending moment, and the bending moments at points midway between the centre and each end.

Ans. $M_0 = m \,.\, W \,.\, l = \frac{1}{4} \times 20 \times 20 = 100$ ft.-tons, and $M_5 = M_{-5} = \frac{1}{2} M_0 = 50$ ft.-tons.

130. A beam 20 feet span supports a load of 20 tons uniformly distributed. Find M_0 and M_5.

Ans. $M_0 = m \,.\, W \,.\, l = \frac{1}{8} \times 20 \times 20 = 50$ ft.-tons; and $M_5 = \frac{3}{4} M_0$, as may readily be seen from a bending moment diagram.

131. Find the equation to the bending moment in the previous example, and calculate M_6 from that equation. Also find the principal equation to the parabola which forms the bending moment diagram.

Ans. $M_x = \dfrac{W}{4c}(c^2 - x^2) = \frac{1}{2}(100 - x^2)$; $M_6 = 32$ ft.-tons. $Y = \frac{1}{2} X^2$.

132. In example 130, find M_6, directly, by taking a section at the point $x = 6$.

$$M_6 = P(10 - 6) - 1 \times (10 - 6) \times \frac{10 - 6}{2} = 10 \times 4 - 4 \times 2 = 32 \text{ ft.-tons.}$$

133. A beam 40 feet span supports seven loads, each two tons, and placed symmetrically on the span at intervals of five feet. Calculate the maximum bending moment by substituting in the proper equation, and calculate at each load the height of the parabola which gives the bending moments.

Here

$$W = 14 \text{ tons}, \quad c = 20 \text{ feet}, \quad l = 40 \text{ feet}, \quad \text{and} \quad n = 7.$$

The maximum bending moment is

$$M_0 = y_0 = \frac{1}{8}\,\frac{n+1}{n}\,W \,.\, l = \frac{1}{8} \times \frac{8}{7} \times 14 \times 40 = 80 \text{ ft.-tons.}$$

The equation to the parabola is

$$y = \frac{W}{4c} \,.\, \frac{n+1}{n}\,(c^2 - x^2) = \frac{14}{80} \times \frac{8}{7}\,(400 - x^2) = \frac{1}{5}\,(400 - x^2);$$

therefore, at the weights,

$$M_{5 \text{ or } -5} = \frac{1}{5}\,(400 - 25) = 75 \text{ ft.-tons}; \quad M_{10 \text{ or } -10} = 60, \quad M_{15 \text{ or } -15} = 35 \text{ ft.tons.}$$

134. In the example No. 133, calculate M_0 independently, by taking a section at the centre.

$$M_0 = P \,.\, c - 2 \times 5 - 2 \times 10 - 2 \times 15 = 7 \times 20 - 10 - 20 - 30 = 80 \text{ ft.-tons.}$$

135. A beam 39 feet span supports twelve loads, each 10 cwts., and placed symmetrically on the span at intervals of 3 feet. Find the maximum bending moment, and the height of the parabola which gives the bending moments; also, from the equation to the parabola find the bending moments $M_{1\cdot5}$, $M_{4\cdot5}$, and M_6, all measured from the centre.

Here

$$W = 120 \text{ cwts.,} \quad c = 19{\cdot}5 \text{ feet,} \quad l = 39 \text{ feet,} \quad \text{and} \quad n = 12.$$

Maximum bending moment,

$$M_0 = \tfrac{1}{8} \,.\, \frac{n+2}{n+1} W \,.\, l = 630 \text{ ft.-cwts.}$$

Height of parabola,

$$y_0 = \tfrac{1}{8} \frac{n+1}{n} W \,.\, l = 633\tfrac{3}{4}.$$

Equation to parabola,

$$y = \frac{W}{4c} \,.\, \frac{n+1}{n} (c^2 - x^2) = \tfrac{5}{3} (380\tfrac{1}{4} - x^2);$$

hence

$$M_{1\cdot5} = \tfrac{5}{3} \left\{380\tfrac{1}{4} - (\tfrac{3}{2})^2\right\} = 630 = M_0,$$

$$M_{4\cdot5} = 600, \quad M_{7\cdot5} = 540, \quad \text{and} \quad M_6 = 570,$$

an average of $M_{4\cdot5}$ and $M_{7\cdot5}$, as it is the ordinate of the middle point of the chord joining the tops of $y_{4\cdot5}$ and $y_{7\cdot5}$.

136. A beam 80 feet span supports a load of 100 tons. Find the bending moments at the centre, and at twenty feet from the end of span; first, if the load be uniformly distributed; and, second, if it be distributed in equal amounts at intervals of two feet.

For uniform load,

$$M_0 = \tfrac{1}{8} W \,.\, l = \tfrac{1}{8} \times 100 \times 80 = 1000 \text{ ft.-tons,} \quad \text{and} \quad M_{20} = \tfrac{3}{4} M = 750 \text{ ft.-tons.}$$

For distributed load,

$$n = 39, \quad W = 100, \quad l = 80, \quad \text{and} \quad c = 40;$$

$$M_0 = \tfrac{1}{8} \frac{n+1}{n} W \,.\, l = \tfrac{1}{8} \times \tfrac{40}{39} \times 100 \times 80 = 1026 \text{ ft.-tons.}$$

$$M_{20} = \frac{W}{4c} \; \frac{n+1}{n} (c^2 - 20^2) = 769 \text{ ft.-tons.}$$

In this case the bending moments exceed by $\frac{1}{39}$th, that is by $\frac{1}{n}$th, of themselves, the bending moments for the uniform load.

CHAPTER IX.

BENDING MOMENTS AND SHEARING FORCES FOR COMBINED FIXED LOADS.

Beam uniformly loaded and with a load at its centre.—Fig. 116. Let U be the amount of the uniform load, then the bending moment at x due to it alone is $\frac{U}{4c}(c^2 - x^2)$; let W be the load at the centre, then the bending moment at x due to it alone is $\frac{W}{2}(c - x)$; summing these, we have

$$M_x = \frac{U}{4c}(c^2 - x^2) + \frac{W}{2}(c - x) = \frac{U}{4c}(c - x)\left(c + x + \frac{2Wc}{U}\right),$$

the equation to the bending moment for positive values of x, that is, for the left half of the span. Putting y instead of M_x, we have

$$y = \frac{U}{4c}(c - x)\left(c + x + \frac{2Wc}{U}\right)$$

a curve, the ordinates of which are the bending moments for the left half of span; this curve is a parabola with its axis vertical, and its apex above BC the span. To find the position of the apex A_1, it is only necessary to find that value of x which makes y greatest; now y is greatest when

$$(c - x)\left(c + x + \frac{2Wc}{U}\right)$$

is a maximum; and since the sum of these two factors is constant, their product is greatest when they are equal; putting then

$$c + x + \frac{2Wc}{U} = c - x, \quad \text{we have} \quad x = -\frac{W}{U}c$$

as the value of x which makes y greatest; the negative sign denotes that A_1 lies to the *right* of O, so that OS_1 is to be laid off towards the right and equal to $\frac{W}{U}c$. The height of A_1 is the value of y when we substitute this value for x; that is,

$$S_1A_1 = \frac{U}{4c}\left(c + \frac{W}{U}c\right)\left(c - \frac{W}{U}c + \frac{2Wc}{U}\right)$$
$$= \left(\frac{W+U}{U}\right)^2 \times (\tfrac{1}{8}Ul) = \left(\frac{W+U}{U}\right)^2$$

times the height of A_0, the apex of the parabola for the uniform load alone.

A_2DC is the same parabola with its apex at the symmetrical point A_2, and the portion DC gives the bending moments for the right half of the span. Since the coefficient of x^2 is $\frac{U}{4c}$, the principal equation to the parabolas A_1DB and A_2DC, each referred to its own apex as origin, is

$$Y = \frac{U}{4c}X^2;$$

this is also the principal equation to the parabola BA_0C for the uniform load alone, so that all three parabolas are identical.

We might suppose the diagram for the uniform load alone to consist of two parabolas lying on the top of each other; and that upon the addition of the load W at the centre, they both move upwards, while the one moves towards the right and the other towards the left. This follows at once from the *Theorem A*, fig. 93, p. 131.

The *Bending Moment Diagram* is BDC. It can be shown that the tangent at D to the parabola A_2DC cuts off CE equal to the height of A_0. $ODEC$ is an *approximate* bending moment diagram made with *straight* lines; and it is *safe*, since the ordinate of any point on DE is greater than the ordinate for the corresponding point on DC.

Graphical Solution.—With a scale of feet for horizontals, lay off BC equal to the span, fig. 116. From the centre O lay off OS_1 and OS_2 to the right and left, each equal to $\frac{W}{U}c$, and draw verticals upwards from S_1, O, and S_2. Apply the parallel rollers to the span BC; place *any* parabolic segment against the rollers, as in fig. 92, with its apex on the vertical through S_1; shift the rollers till the curved edge passes through B, the end of the span on the *opposite* side of the centre from S_1, and draw the curve BDA_1. Again, place the segment with its apex on the vertical through S_2; shift the rollers till the curved edge passes through the end C, and draw the curve CDA_2; then BDC is the bending moment diagram. The scale, of say ft.-lbs., for verticals is to be constructed such that OD measures $\frac{1}{8}(U + 2W)l$, where U and W are in lbs. and l is in feet. The same scale may be constructed as follows:—Place the parabolic segment with its apex on the vertical through O; shift the rollers till the curved edge passes through B and C; draw the curve BA_0C, and make

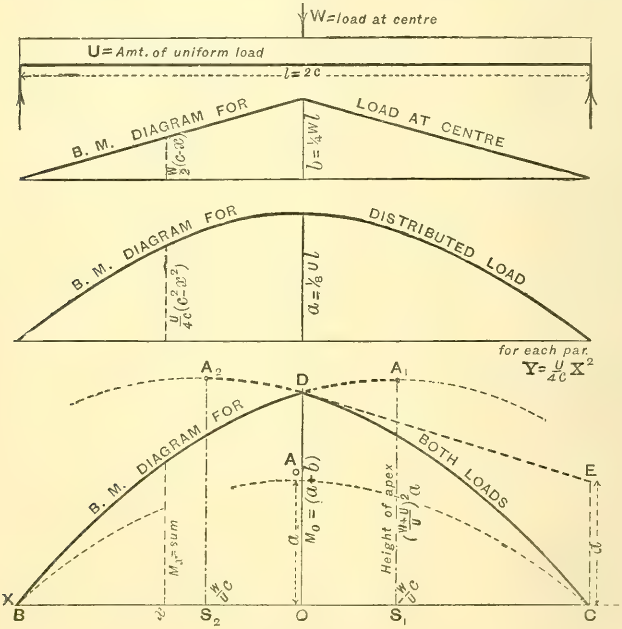


Fig. 116.

a scale of ft.-lbs. for verticals, such that OA_0 measures upon it $\frac{1}{8}Ul$, where U is in lbs. and l is in feet.

Cor.—For the same uniform load, although different loads be put at the centre, A_1DB is always the *same* parabola; as the load at the centre increases, the apex A_1 moves from the centre, and the arc DB is a part of the wing of that parabola further from the apex. Now the wing of a parabola gets flatter as its distance from the apex increases; hence, if U be constant and W be increased, BD becomes flatter and flatter; and if W be very great compared to U, BD is sensibly a straight line.

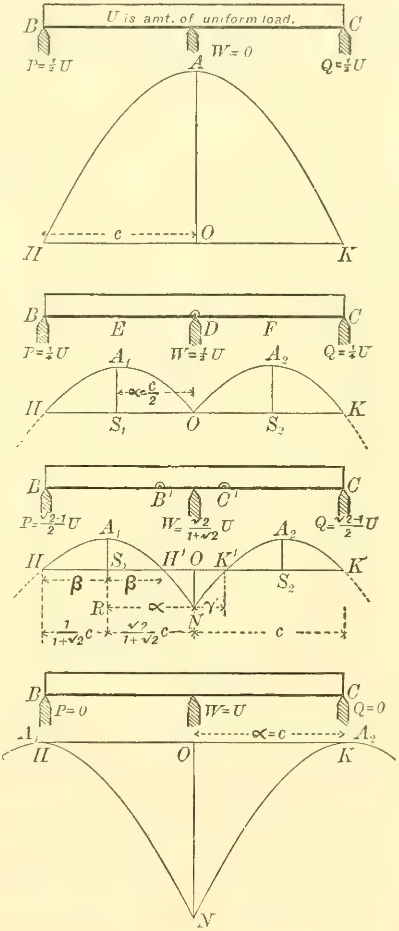


Fig. 117.

Beam uniformly loaded and supported on three props.—Fig. 117. Let BC be a beam with a load U uniformly distributed on it and supported on three props, one at each end and one at the centre; let P, W, and Q be the forces with which they press upwards, so that $P + W + Q = U$ always, and $P = Q$ by symmetry. If $W = 0$, the central prop bears no share, $P = Q = \frac{1}{2}U$, and the bending moment diagram is the parabola HAK, as on fig. 110. If the central prop bears a share, then the beam is loaded with a uniform load U, and a *negative* load W at the centre; and the bending moment diagram is two parabolas, each the same as HAK, but with its apex away from the centre, and in a direction opposite to that on fig. 116. The horizontal distance through which each apex moves is given by the equation $\alpha = \frac{W}{U}c$. Suppose HAK to be two parabolas lying one on the top of the other, and let the prop press up with a greater and greater force, then the two parabolas shift away from each other. When $W = \frac{1}{2}U$, A_1 is over S_1, and A_2 is over S_2, the middle points of OH and OK respectively, and the bending moment at O is zero; this is the best value of W, if the beam may only be bent so that its convex side shall be down. It is evident that $S_1A_1 = \frac{1}{4}OA$; and that the beam might be sawn through at O, when it would be two beams of span c, each uniformly loaded and supported at the two ends.

If the beam may be bent both upwards and downwards, let the central prop press upwards till $W > \frac{1}{2}U$, then OS_1 is greater than $\frac{1}{2}OH$, or in symbols $\alpha > \frac{1}{2}c$; and it can be seen from the figure that the ordinates from H' to K' are *negative*, from H' to H positive, and from K' to K positive, while at H, H', K, and K' the bending moments are zero. Hence the beam will be bent with the convex side downwards from H to H', and from K' to K, and upwards from H' to K'. A hinge might be put on the beam at H' and K', since there is no tendency to bend at these points, which are called *points of contrary flexure.* There are three maxima bending moments, two equal positive ones at S_1 and S_2, and a negative one at O. If the material of the beam may be bent upwards and downwards *equally well*, then the best value of W is that which makes the positive and negative maxima equal, as their common value in this case is less than the greatest value would be in any other; for, suppose them equal, then if W be increased the parabolas move outwards, and the ordinate at O will increase; while, if W be made smaller, the parabolas will approach and the ordinates at S_1 and S_2 will increase. Let the three maxima be equal to each other, then $ON = S_1A_1$, or

$$A_1S_1 : A_1R :: 1 : 2; \qquad \text{hence} \qquad S_1H' : RN :: 1 : \sqrt{2}$$

since the curve is a parabola; or in symbols

$$\beta : \alpha :: 1 : \sqrt{2}.$$

Practically this could be accomplished by causing the prop to press upwards till

$$W = \frac{OS_1}{OH}U = \frac{\alpha}{\alpha + \beta}U = \frac{\sqrt{2}}{\sqrt{2}+1}U = \cdot 586\,U;$$

or by fixing hinges at the two points H' and K' at the proper distances from O, the central prop may be made to bear the above share of U.

If the prop at the centre press upwards so that $W = U$, then $P = Q = 0$, and $OS_1 = OH$; A_1 coincides with H, and A_2 with K; the bending moment is everywhere negative, and its maximum value ON equals OA. OH and OK are cantilevers as on fig. 114.

THE CONTINUOUS BEAM.

Beam uniformly loaded and supported on many props.—Fig. 118. Let BE be a beam bearing an uniform load and supported on many (5 in the figure) props; this is only an extension of the previous case. The end C, fig. 117, instead of resting on a prop, might be hinged to the end of a cantilever, which in turn might have its other end hinged to a beam, &c.: see fig. 118. Here a double cantilever, as $B'C'$, over each intermediate prop is hinged on each side to the end of a beam;

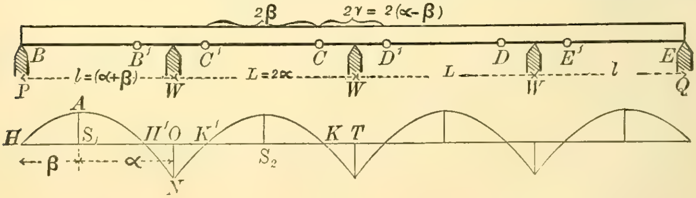


Fig. 118.

each cantilever bears an uniform load over itself, as well as half the load on the beam concentrated at its end, and is therefore in the condition of the cantilever shown on fig. 117. Each intermediate beam as $C'C$ is uniformly loaded, and is supported by a hinge at each end; the two end beams $B'B$ and $E'E$ are supported at one end by a hinge, and at the other by one of the extreme props; each central span L consists of a beam and two cantilevers; the end spans l of a beam and a cantilever.

If the hinges be put in the positions indicated by H' and K' on fig. 117, then the negative maxima bending moments over the props at O, T, &c., are equal to the positive maxima bending moments at S_1, S_2, &c., the centres of the beams; the common value of all these maxima will be *less* than the greatest for the hinges in any other position, and

$$l : L \ :: \ \alpha + \beta : 2\alpha \ :: \ 1 + \sqrt{2} : 2\sqrt{2} \ :: \ \cdot 854 : 1.$$

Cantilever uniformly loaded and with a load at its free end.—Fig. 119. As in the previous case, add the bending moments at the section distant x from O the fixed end, due to the loads separately; thus

$$- M_x = \frac{U}{2c}(c-x)^2 + W(c-x).$$

We consider the bending moments on a cantilever negative as compared with those on a beam, and so they will be represented by ordinates drawn *down* from the span instead of *up* as in the case of beams. If we further put y instead of M_x for the ordinate at the point on the curve corresponding to *any* value of x, then y will be M_x only for values of x from 0 to c; and

$$- y = \frac{U}{2c}(c-x)^2 + W(c-x) = \frac{U}{2c}(c-x)\left(c - x + \frac{2Wc}{U}\right)$$

or

$$y = \frac{U}{2c}(c-x)\left(x - c - \frac{2Wc}{U}\right).$$

This curve is a parabola with its axis vertical and its apex above the span. To find the position of the apex A, it is only necessary to find that value of x which makes y greatest; now y is greatest when

$$(c-x)\left(x - c - \frac{2Wc}{U}\right)$$

is a maximum; and since the sum of these two factors is constant, their product is greatest when they are equal; putting then

$$x - c - \frac{2Wc}{U} = c - x; \quad x = \left(1 + \frac{W}{U}\right)c$$

is the distance of A to the *left* of O, or $\frac{W}{U}c$ is the distance of A to the left of E; that is, the apex of the parabola is beyond the free end by the distance $\frac{W}{U}c$, or the same fraction of the span that W the concentrated load is of U the distributed load.

The height of A above OE is the value of y when we substitute the above value for x; or height of A is

$$\frac{U}{2c}\left(c - \frac{U+W}{U}c\right)\left(\frac{U+W}{U}c - c - \frac{2Wc}{U}\right) = \left(\frac{W}{U}\right)^2 (\tfrac{1}{2}Ul) = \left(\frac{W}{U}\right)^2$$

times the maximum bending moment for the uniform load alone. Since the coefficient of x^2 is $\frac{U}{2c}$, the principal equation to the parabola AEF is

$$Y = \frac{U}{2c}X^2;$$

and it is therefore the *same* parabola as that for the uniform load alone, which would follow at once from the *Theorem A*, fig. 93, p. 131.

The *Bending Moment Diagram* is EFO, formed by the parabola EG, whose apex is at E and which makes the diagram for the uniform load alone, shifted without turning till its apex is at A.

Graphical Solution.—With a scale of feet for horizontals, lay off OE (fig. 119) equal to the length, and produce it to S, so that $ES = \frac{W}{U}c$, or is the same fraction of the length as the load at the end is of the uniform load, and draw a vertical upwards through S; apply the parallel rollers to the line OE; place any parabolic segment against the rollers with its apex on the vertical through S; shift the rollers till the curved edge passes through the free end E, and draw the curve AEF; OEF is the bending moment diagram. The scale of, say, ft.-lbs. for verticals is to be constructed such that OF measures $\left(\frac{U}{2} + W\right)l$, where U and W are in lbs., and l is in feet. The same scale may be constructed as follows:—Place the parabolic segment with its apex on the vertical through the free end E, and move the rollers till the apex comes to E; draw the dotted curve EG, and make a scale of ft.-lbs. for verticals such that OG measures upon it $\frac{1}{2}Ul$, where U is in lbs., and l is in feet.

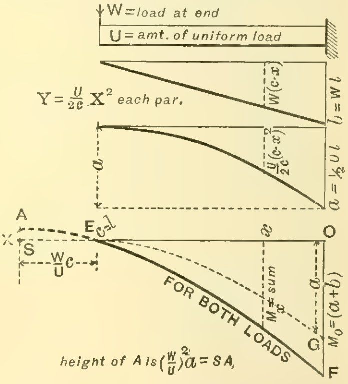


Fig. 119.

Cor.—If W be great compared to U, then EF is sensibly a straight line; because if U be constant AEF is the *same* parabola, no matter what W may be; as W increases, A moves from E, and the arc EF, part of the wing of that parabola further from the apex, becomes flatter and flatter.

Beam Loaded both Uniformly and with Unequal Weights fixed at Irregular Intervals.

Fig. 120.—Let U be the amount of the uniform load; the parabola BA_0O is the bending moment diagram due to it, and a is the height of its apex A_0. Let $Oefgh \ldots K$ be the bending moment diagram due to the loads W_1, W_2, W_3, W_4, &c.; from one end O draw lines parallel to the sides, and meeting the vertical through

b the centre of the span at the points b_1, b_2, &c.; let b_1, b_2, &c., stand for the

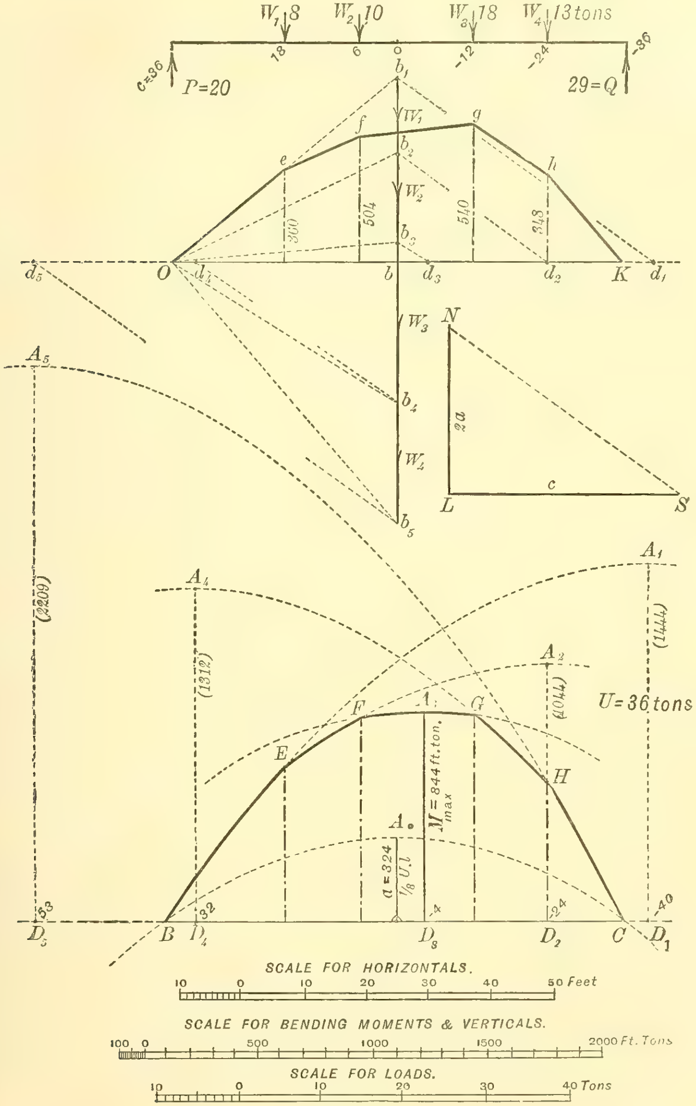


Fig. 120.

distances of these points from b, those which are below being considered negative.

In fact, the bending moment diagram $Oefgh \ldots K$ is drawn with the load line b_1, b_5 through the middle point b of the span, while the origin O is at the left end of the span. The scale used for W_1, W_2, &c., must be such that the intercept $\overline{b_1b_2}$ shall represent $W_1 \times c$ ft.-tons when measured on the vertical scale of ft.-tons for the parabola BA_0C already drawn.

The two diagrams are now to be combined by the application of the *Theorem A*, p. 131, from which it follows that—If to the ordinates of the parabola BA_0 we add those of the straight line Ob_1, the locus is the *same* parabola with its apex to the right of b at a distance given by the equation

$$\overline{bd_1} = \frac{b_1}{2a} c,$$

and the curve still passes through B; let A_1EB be the parabola so placed, then BE, the portion of this parabola from the end of span to the vertical through the first weight, is the bending moment diagram for that portion of the span. If to the ordinates of the parabola BA_0 we add those of the straight line Ob_2, the locus is the *same* parabola with its apex to the right of b at a distance given by the equation

$$\overline{bd_2} = \frac{b_2}{2a} c, \qquad \text{(\textit{Theorem A}, p. 131)}$$

and the curve still passes through the point B; if we now raise the parabola through the vertical distance between the two parallels Ob_2 and ef, which is accomplished by drawing the curve through E instead of B, we have the curve in the position A_2FE; and EF the portion between the verticals through the first and second weights is the bending moment diagram for that portion of the span. Similarly the parabolic arcs FG, GH, and HC are drawn over the remaining portions or "fields" of the span.

A construction equivalent to the above equations is shown on fig. 120. It is much simplified by drawing the bending moment diagram in conjunction with the shearing force diagram. Hence—

Graphical Solution.—The concentrated loads $W_1 = 8$, $W_2 = 10$, $W_3 = 18$, and $W_4 = 18$ tons in the example on fig. 121 are drawn downwards in cyclic order to a scale of tons. Then b, the joint between the reactions at the supports, is to be found either by drawing a link polygon or by taking moments about one support. The supporting forces are $P = 20$ and $Q = 29$ tons. The uniform load is one ton per foot, so that the supporting forces due to it are 18 tons at each end. The dotted lines show the shearing force diagram for the loads separately standing on the common base BC. The thick lines show the two compounded, and crossing the base at d_3 the position of the maximum bending moment. The other sloping lines produced to meet the base at d_1, d_2, d_4, and d_5 determine the points over which must stand the vertices of the parabolic arcs for the other "fields."

Now draw BC for the base of the bending moment diagram, produce it each way, and from $d_1d_2 \ldots$ project down on it the points $D_1D_2 \ldots$, &c. Apply the parallel rollers to BC; place the parabolic segment against them with its apex on the vertical through D_1, move the rollers till the curved edge passes through B, and draw BE meeting the vertical through W_1 in E. Shift the segment till the apex is on the vertical through D_2; move the rollers till the curved edge passes through E, and draw EF. Shift the segment till the apex is on the vertical through D_3; move the rollers till the curved edge passes through F, and draw FG meeting the vertical through W_3 in G, &c. The accuracy of the drawing is checked by observing whether the last curve passes through C the other end of the span. Lastly with the same parabolic segment draw BA_0C standing on the span, and construct a scale so that the height of its apex A_0 shall measure on it $\frac{1}{8}Ul$, that is, one-eighth of the product of the uniform load and the span.

A Diagram of the Square Roots of Bending Moments is shown at the bottom of fig. 121. The parabolic arcs are replaced by

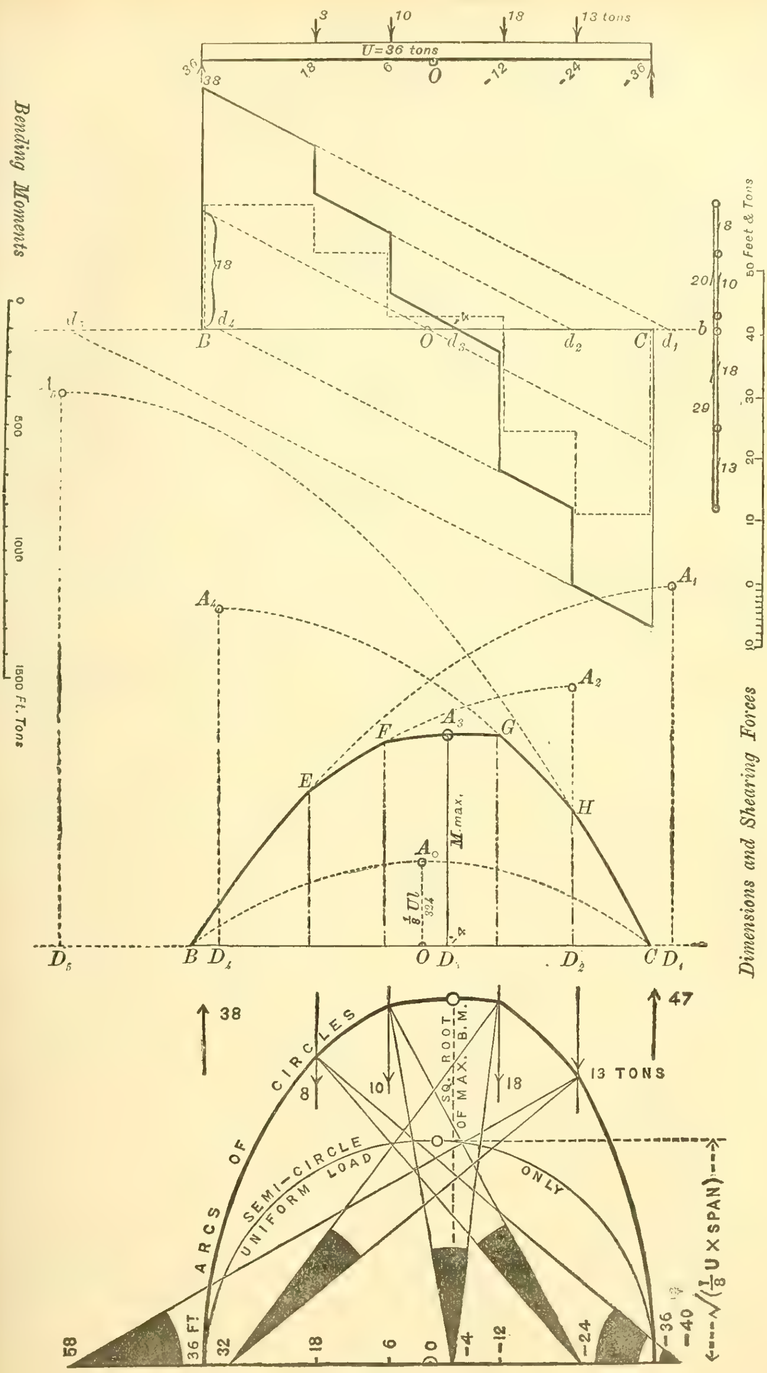

U=36 tons
3
10
18
13 tons
Bending Moments
Dimensions and Shearing Forces
50 Feet & Tons
1500 Ft. Tons
B
C
O
b
d_1
d_2
d_3
d_4
d_5
A_0
A_1
A_2
A_3
A_4
A_5
D_1
D_2
D_3
D_4
D_5
E
F
G
H
M.max.
$\frac{1}{8}Ul$
324
38
47
13 TONS
ARCS OF CIRCLES
SEMI-CIRCLE
UNIFORM LOAD ONLY
SQ. ROOT OF MAX. B.M.
$\sqrt{(\frac{1}{8}U \times SPAN)}$
36 FT.
58
32
18
6
0
-4
-12
-24
-36
-40

Fig. 121.

arcs of circles, the centres being the points D_1, D_2, &c., where the slopes of the shearing force diagram when produced meet the base. See the *Theorem G*, fig. 98, p. 135, as to the parabolic segment degraded.

BEAM UNIFORMLY LOADED ON A PORTION OF THE SPAN.

On fig. 122, let $2c$ be the length of the span, and O its centre; let G be the centre of the load area, and $2k$ the extent of the load. The intensity of the uniform load is w lbs. per foot-run; and, as in the previous case, we take w as the height of the load area in feet, so that every square foot of load area represents one lb. The total load area is a rectangle of height w ft., and length $2k$ ft.; its area is $2wk$ square feet, so that the total load on the span is $2wk$ lbs., which may be supposed to be concentrated at G the centre of gravity of the load area; this gives the supporting forces P and Q as for the actual distribution.

Let $\bar{x} = OG$, the distance from the centre of span to the centre of gravity of the load; then

$$P = \frac{2wk}{2c}(c + \bar{x}) = \frac{wk}{c}(c + \bar{x}).$$

For any section between the left end of the span and of the load area, the only force to the left of the section is P, and the bending moment may be calculated just as if the whole load were at G; for such sections the bending moment increases uniformly from zero at the left end till we come to the section through the left end of the load area, and the bending moment diagram for that part of the span is a straight line sloping up from the left end of span till it meets the vertical through the left end of the load area. Similarly for the other end, the diagram is a straight line sloping up from the right end till it meets the vertical through the right end of the load area; the moments at the two ends of the load area are easily calculated from P and Q.

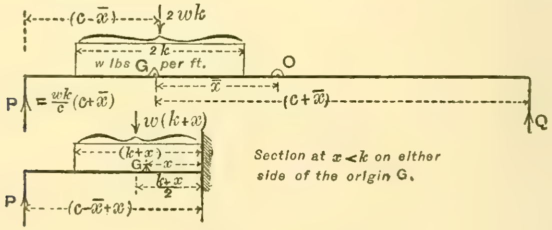


Fig. 122.

For sections within the load area, it will be convenient to choose G as origin; and let x, which must be not greater than k, be the distance to the section which may be either to the right or left of G; in our diagram it is to the right. We will consider x positive in the direction from the origin G *towards* the centre; in the diagram x is positive to the right. The load area standing upon the part of span to the left of our section consists of a rectangle of height w ft., and length $(k + x)$ ft.; its area is $w(k + x)$ lbs. which may be supposed to be concentrated at the middle of the area. To the left of the section there are two forces, P acting upwards with a leverage of $(c - \bar{x} + x)$ ft., and $w(k + x)$ lbs. acting downwards with a leverage of $\frac{1}{2}(k + x)$ feet; hence at this section

$$M_x = P(c - \bar{x} + x) - w(k + x) \cdot \frac{k + x}{2} = \frac{wk}{c}(c + \bar{x})(c - \bar{x} + x) - \frac{w}{2}(k + x)^2;$$

arranging in powers of x,

$$M_x = \left(wkc - \frac{wk\bar{x}^2}{c} - \frac{wk^2}{2}\right) + \frac{w}{2}\left(\frac{2k\bar{x}}{c} - x\right)x.$$

This is the equation to a parabola with its axis vertical and its apex above the span; it is the equation to the bending moment, for values of x from k to $-k$; for any other value of x we may put y for the corresponding ordinate, but y will no longer be M_x. To find the distance of the apex from G, it is only necessary to find that value of x which makes M_x greatest; now M_x is greatest when the product $\left(\frac{2k\bar{x}}{c} - x\right)x$ is greatest; the sum of the two factors of this product is constant, so that the product is greatest when the factors are equal: putting

$$x = \frac{2k\bar{x}}{c} - x, \quad \text{we have} \quad x = \frac{k}{c}\bar{x}, \quad \text{or} \quad \frac{2k}{2c}\bar{x};$$

that is,

$$GS = \frac{\text{length of load}}{\text{span}} GO.$$

It is evident that always

$$\bar{x} < c, \quad \therefore \quad \frac{k}{c}\bar{x} < k; \quad \text{or} \quad \frac{k}{c}x,$$

the horizontal distance that the apex is from G is less than k the half extent of load; that is, the apex is always above a point which is *within* the load area.

To find the height of the apex above the span: substitute the above value of x into the equation to the parabola, and

$$M_{\max} = \left(wkc - \frac{wk\bar{x}^2}{c} - \frac{wk^2}{2}\right) + \frac{w}{2}\left(\frac{2k\bar{x}}{c} - \frac{k\bar{x}}{c}\right)\frac{k\bar{x}}{c}$$

$$= \frac{wk}{2c^2}(c^2 - \bar{x}^2)(2c - k) = \frac{W}{4c}(c^2 - \bar{x}^2)\left(2 - \frac{k}{c}\right).$$

The *position* of this maximum is easily remembered, but as the above expression for its value is complicated, it may be easier to calculate the bending moment at that point directly.

Cor. 1.—If the centre of load be over the centre of span, the maximum bending moment is at their common centre.

Cor. 2.—If k increases, $\bar{x}$ decreases ultimately, so that $(c^2 - \bar{x}^2)$ is increasing; at the same time, the product $k\,(2c - k)$ increases and becomes greatest when $k = c$; hence $M_{\text{max.}}$ is greatest for the whole span loaded.

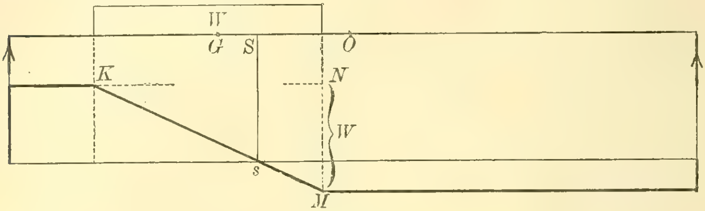


Fig. 123.

The *Bending Moment Diagram*, *BHEC* (fig. 124), consists of the above parabola with its axis vertical and its apex A situated as described above, and drawn both ways till it meets the verticals through the ends of the load at H and E, and of the straight lines HB and EC. In the above equation to the parabola, the coefficient of x^2 is $\frac{1}{2}w$, and the principal equation to the curve is $Y = \frac{1}{2}wX^2$. This is the

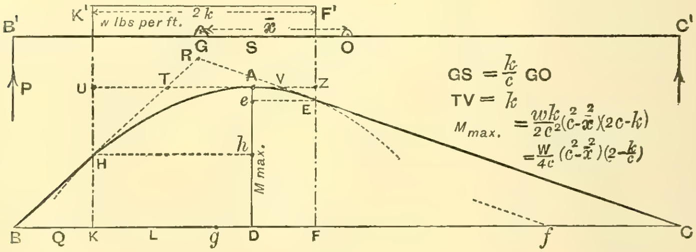


Fig. 124.

principal equation to the curve on fig. 110, so that in the present case the parabola is the *same* as that for an uniform load of intensity w over the whole span. BH and CE are the same as if the whole load were at G; therefore they meet at R, a point on the vertical through G; if the load were concentrated at G, BRC would be the bending moment diagram. At any point as L, the ordinate to HR is greater than that to HA, because the former is the product of P into BL, while the latter is that same product minus the moment of the load area to the left of L; that is, every point in HR is outside the curve. For such a point as Q, by substituting gQ for x in the equation to the parabola, its ordinate is again less than that of BH, so that BH is a tangent at H to the parabola, and EC is a tangent at E; T and V are the middle points of AU and AZ; therefore $TV = \frac{1}{2}UZ = k$, the half extent of load. In short the *oblique* parabolic segment HAE replaces the *scalene* triangle HRE. See both painted on the "distorting table" of the moving model (fig. 151).

Cor.—Since *HAE* is always the same parabola for the same value of *w*, it is evident that *A* will be higher when the parabola passes through *B* and *C*; that is, when the whole span is loaded. This is the same result as corollary 2, page 176, derived in this case by geometry.

Graphical Construction of Shearing Force and Bending Moment Diagrams.—Draw *B'C'* to a suitable horizontal scale of feet to represent the span, and draw the parallelogram *K'F'* to represent the uniform load on a part of the span. This is done twice on figs. 123, 124, over the shearing force diagram and over the bending moment diagram.

Calculate the supporting forces *P* and *Q*, and, with a suitable scale of lbs., draw a horizontal line through *K* at the height *P* above the base of the shearing force diagram, and another through *M* at the depth *Q* below it. These give the shearing on the unloaded "fields" right and left of *W*, while *KsM* is the shearing force on the central loaded field.

To draw the bending moment diagram, rule vertical lines down from *K'* and *F'*, dividing the base *BC* into three "fields." Choose a point *R* directly below *G*, the centre of the load. Join *R* to *B* and *C* by straight lines, meeting the verticals from *K'* and *F'* at *H* and *E*. Then *BRC* would be the bending moment diagram, had the whole of *W* been concentrated at *G*. It only remains, then, to replace *HGE* by a parabolic arc to which *HB* and *EC* shall be tangents. Draw *SAD* vertically downwards from *s*, the point where the shearing force changed sign, and prick *A* at a point *midway* between *h* projected horizontally from *H*, and the point where *BHR* produced meets *hAS*. The two right parabolic quadrants *hAH* and *eAE* are to be constructed. A scale for foot-lbs. is then to be constructed, so that the height of *R* above *BC* shall represent $P \times B'G$.

If *K'F'* be a big fraction of the span, the bending moment diagram may be quickly drawn with any parabolic segment cut in brass or cardboard placed against the blade of the T-square, and pushed up with its vertex on the vertical through *s* and the arc *HAE* drawn. Next, prick a point as high above *A* as *h* is below; this point joined to *H* is the tangent there which is to be produced till it meets the vertical through *B'* at *B*; in like manner, *EC* is drawn, and now *B* and *C* are to be joined when *BC* should be horizontal, which will check the accuracy of the drawing.

The construction is greatly simplified when the load begins at one end of the beam.

BEAM LOADED ON ONE SEGMENT.

NOTE.—This case is of importance, as the half span, only, loaded is required in the treatment of the Iron-Arched Girder: see figs. 249, I., and 251, I. Also the load covering a segment varying in length is required in the treatment of the girder with its ends fixed horizontally: see fig. 232.

Graphical Construction of the Shearing Force Diagram and Bending Moment Diagram (figs. 125, 126). Calculate or construct the supporting forces *P* and *Q*. To a suitable scale, prick *K* at the height *P* above the base of the shearing force diagram at left end, and *M* at the depth *Q* below it at the right end of the load. Then *KsM* is the locus for the loaded segment, and a horizontal through *M* is locus for the unloaded segment.

Place a parabolic template with its axis vertical and its lower edge against the T-square. Push up the T-square and shift along the parabolic set square till its curved edge passes through *B*, the vertex being on the vertical through *s*, then

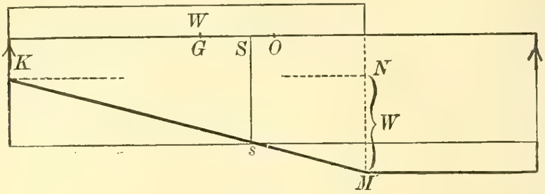


Fig. 125.

draw the arc *BAE*, and join *E* to *C* with a straight line. Construct a scale of foot-tons to suit the calculated height of any point, say that of *E* or of *A*.

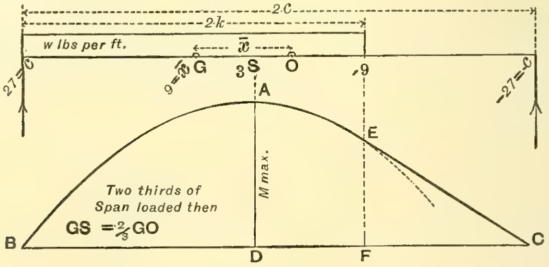


Fig. 126.

Beam uniformly loaded on its Two Segments with loads of different intensities (fig. 127).—This case is not of much practical importance, but serves to illustrate the perfectly general case of fixed loadings which is to follow.

To construct the shearing force diagram, calculate *P* and *Q*, the left and right supporting forces, by taking moments about one end. Prick *K* at *P* tons above the left end of the base, and *h* at *Q* tons below the right end. Draw the straight locus *KM* falling w_1 tons vertically per foot horizontal so that *M* shall be W_1 tons lower than *K*. Draw *Mh* falling w_2 tons per foot. The maximum bending moment must occur at *t* where this locus crosses the base.

To construct the bending moment diagram (fig. 128). If W_2 to the right of *Z* were the only load on the span, then the locus of the bending moment diagram on left segment would be a straight slope *B'K*.

Suppose, again, that the whole span was loaded with w_1 lbs. per foot when the bending moment diagram would be a parabolic locus standing on *B'C'* with its vertex under *O* the middle point of the span, the modulus of the parabola being $\frac{1}{2}w_1$. Now remove completely the load to the right of *Z*, which is the same as adding an upward load; from *B'* to *f* this would require the ordinates of a straight slope to be subtracted from those of the parabolic arc which then, by *Theorem A*, p. 131, would still be an arc *B'ae* of the same parabola of modulus $\frac{1}{2}w_1$ but with its vertex *a* shifted to the left under S_1.

Lastly, adding the ordinates of the loci *B'ae* and *B'K*, we have for the left "field" *BF* the parabolic locus BA_1E of modulus $\frac{1}{2}w_1$ with its vertex over T_1 defined by *t* where the slope *KM* of the shearing force diagram crossed the base.

In the same way the locus of the bending moment diagram for the "field" *FC* is a parabolic locus A_2EC of modulus $\frac{1}{2}w_2$, and with its vertex over T_2 defined by the point on the shearing force diagram where the slope *hM* produced meets the base.

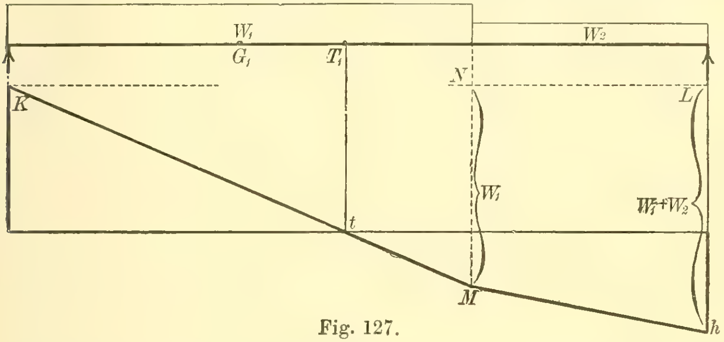


Fig. 127.

Maximum Bending Moment.—It is easy on any numerical example to find BT_1 so that the load on it shall equal the supporting force *P* at *B*. Then $M = T_1A_1$ a maximum is to be calculated directly at that section.

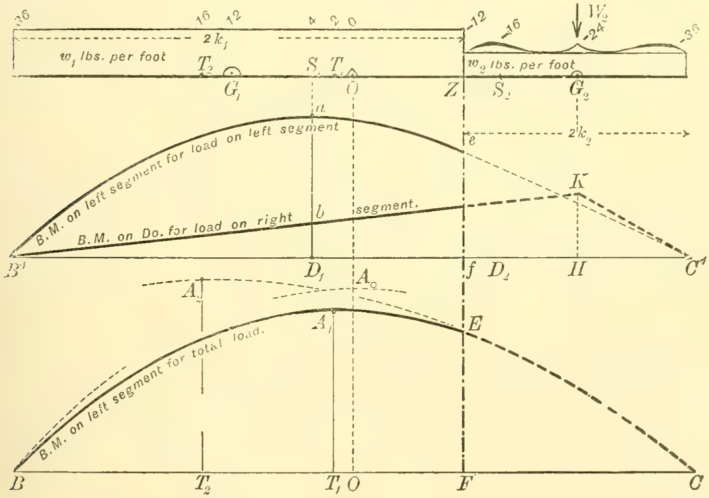


Fig. 128.

Graphical Construction of Bending Moment Diagram.—Having first constructed the shearing force diagram as explained above, construct two parabolic segments standing on a common base, but with the height of their vertexes in the ratio of w_1 to w_2. These are to be cut out in cardboard. The first is to be placed against the

⊤ square and pushed up with its vertex on the vertical through T_1 (where the slope of the shearing force diagram crossed the base), and shifted till the curved edge passes through B, and the arc BA_1E drawn. This same parabolic template is then to be pushed up with its vertex on the vertical through O till its curved edges passes through B and C simultaneously, and its vertex A_0 pricked. There a scale is to be constructed so that OA_0 shall read on it $\frac{1}{8}w_1l^2$ where $l =$ span, when the maximum T_1A_1 can be scaled off. To finish the locus for the right segment the second parabolic template must be used, and pushed up with its vertex on the vertical through T_2 (where the corresponding slope hM produced on the shearing force diagram meets the base) when its curved edge will simultaneously pass through E and C, and the arc EC is then to be drawn.

BEAM LOADED IN ANY MANNER WITH FIXED LOADS.

Graphical Construction of the Shearing Force Diagram.—The supporting forces P and Q at the left and right supports are to be calculated by taking moments about one end of the span. Or all the distributed loads may temporarily be concentrated at their centres of gravity, and P and Q constructed by drawing a load line and link polygon as on fig. 101. To a suitable scale of feet for horizontals, draw a base line for the shearing force diagram, and with a suitable scale of tons lay P upwards from the left end, and Q downwards from the right end. Starting from the top of P, draw a locus which will cross the base once and end at the bottom of Q. This locus will cover "field" after "field" of the span thus. It will cross *unloaded fields* horizontally; at a *concentrated load* it will drop the amount of that load vertically, and will cross *uniformly loaded fields* sloping downwards with a straight slope inclined at the given rate of loading.

Graphical Construction of the Bending Moment Diagram.—If there be "fields" of the span, some loaded at one uniform rate, and some at others, then as many parabolic cardboard templates should be prepared as there are varieties of uniform load. These templates should have a common base and their heights be in a continued proportion the same as that of the different intensities of uniform loads (see fig. 158).

The base for the bending moment diagram being drawn, it is to be divided into "fields"; each concentrated load is the boundary between two "fields"; so also is a point at which the intensity of the uniform load suddenly changes including the ends of unload fields. The locus of the bending moment diagram will begin at one end of the base and end at the other; being a maximum at the same point at which the shearing force changes sign. This locus will cross the fields thus; it will cross *unloaded fields* with a *straight locus sloping up* at a rate given by the absolute height of the shearing force horizontal locus in that field, or *down* at the depth of it. It will cross uniformly loaded fields in a parabolic arc the vertical axis of the parabola being the vertical through the point where the sloping locus of the shearing force diagram for that field crosses the base when produced. Also the modulus of the parabola shall be half the intensity of the uniform load in the field. Lastly, the arc shall at its ends have a common tangent with the arc in adjoining field, or, if the locus in the adjoining field be straight, it shall be a tangent. At points where a load is concentrated, the loci meet at an angle whose tangent is the sudden fall on the shearing force diagram. The locus is readily drawn over field after field, using the template for any loaded field already prepared to correspond to it.

The scale is to be prepared to give the value at some point as calculated there, say at the point where the shearing force diagram crossed the base; or with one of the prepared parabolic segments, a parabolic segment may be drawn standing on the span as base when a scale is to be prepared upon which the height of its vertex shall measure $\frac{1}{8}wl^2$, where w is the intensity corresponding to the template used.

EXAMPLES.

137. A beam is uniformly loaded, and a prop in the centre bears one-third of the load. Find the maximum bending moment (fig. 117).

Let $U =$ the amount of the uniform load,

$W =$ the upward thrust of central prop $= \frac{1}{3}U$.

Here $\alpha = \frac{1}{3}c$; and since $\alpha < \frac{1}{2}c$, the two parabolas intersect above the span, and there are no negative bending moments; there is a positive minimum at the centre, and a positive maximum at α on each side of the centre.

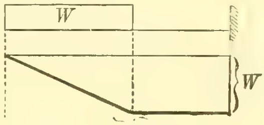


Fig. 129.

The base of the segment

$$HA_1H' = 2\beta = \frac{1}{3}c = \frac{2}{3}l.$$

Height S_1A_1 : Height OA : : $(\frac{2}{3})^2 : 1^2$; therefore

$$S_1A_1 = \frac{4}{9}OA,$$

or

$$M_\alpha = \frac{4}{9} \times \frac{1}{8}U.l = \frac{1}{18}U.l, \text{ maximum.}$$

138. A cantilever 18 feet long is loaded uniformly for two-thirds of its length from the free end, with 10 cwt. per foot-run. Find the bending moments at intervals of two feet. See fig. 130.

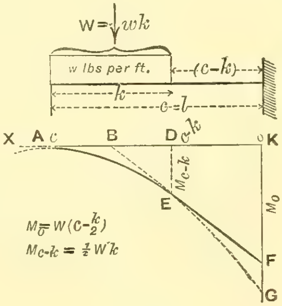


Fig. 130.

Look upon the loaded part as a cantilever uniformly loaded, then

$$M_x = \frac{w}{2}(c - x)^2 = 5\,(18 - x)^2;$$

therefore

$$M_{18} = 0, \quad \text{and} \quad M_6 = 720 = DE.$$

Now consider BK a cantilever loaded at B with 120 cwts.; then

$$M_x = W\left(c - \frac{k}{2} - x\right) = 120\,(12 - x);$$

so that $M_6 = 720 = DE$, and $M_0 = 1440$ ft.-cwts.

Ans. 0, 20, 80, 180, 320, 500, 720; 960, 1200, 1440 ft.-cwts.

A graphical solution is obtained by drawing fig. 130 upon a large scale, and measuring the ordinates at intervals of two feet.

139. A beam 72 feet span is loaded with 8 and 10 tons at points 18 and 6 feet to the left of the centre, and with 18 and 13 tons at points 12 and 24 feet to the right of the centre; there is also an uniform load of half a ton per foot of span. Find the position and value of the maximum bending moment. These data are drawn to scale on figs. 120 and 121.

Taking moments about the right support

$$P \times 72 = 8 \times 54 + 10 \times 42 + 18 \times 24 + 13 \times 12 + 36 \times 36.$$

So that $P = 38$ tons, including the reactions for spread load. At the central point O the shearing force is $F_0 = P - 8 - 10 - 18 = 2$ tons; and to bring this to zero, we must pass to a point 4 feet further to the right; hence $F_{-4} = 0$. The maximum bending moment will be at a point 4 feet to the right of the centre, and

$$M_{-4} = 38 \times 40 - 8 \times 22 - 10 \times 10 - 20 \times 20 = 844 \text{ ft.-tons.}$$

To find the shearing forces at intervals of 4 feet. For the fixed loads alone

$$F_{36 \text{ to } 18} = P = 20, \qquad F_{18 \text{ to } 6} = 12, \qquad F_{6 \text{ to } -12} = 2,$$
$$F_{-12 \text{ to } -24} = -16, \qquad F_{-24 \text{ to } -36} = -Q = -29 \text{ tons.}$$

For the uniform load alone $F_{36} = P = 18$ tons, and by proportion the shearing force at each point is half the abscissa of the point.

For the two loads combined, the shearing forces at points 4 feet apart are

Feet.	Tons.	Feet.	Tons.	Feet.	Tons.	Feet.	Tons.
36	38	16	20	– 4	0	– 24	– 41
32	36	12	18	– 8	– 2	– 28	– 43
28	34	8	16	– 12	– 22	– 32	– 45
24	32	4	4	– 16	– 24	– 36	– 47
20	30	0	2	– 20	– 26		

The shearing force changes sign at the point $x = -4$, and at this point the bending moment is a maximum.

140. A beam 54 feet span is loaded uniformly for two-thirds of its length from the left end with 10 cwt. per foot-run. Find the position and magnitude of the maximum bending moment. (See fig. 126.)

In this case, $c = 27$ ft., and $OG = 9$ ft., measured to the left of O.

From G lay off $GS = 6$ ft. $= \frac{2}{3}GO$, since the load extends over two-thirds of the span, and the maximum moment occurs at S, that is, at 3 ft. to the left of the centre. Suppose the whole load $W = 360$ cwt. is concentrated at G; then

$$P = \frac{360}{54} \times 36 = 240 \text{ cwt.}$$

This, of course, is equal to load up to S, that is, 24×10.

Taking a section at S, the portion of the span to the left is 24 ft., so that the load upon it is 240 cwt. acting downwards, and if supposed to be concentrated at its centre, its leverage about the section is 12 ft.; at the same time P acts upwards with a leverage of 24 ft., and

$$M_3 = 240 \times 24 - 240 \times 12 = 2880 \text{ ft.-cwts. max.}$$

141. The left half of a beam 32 feet span is uniformly loaded with 1 ton per foot-run. Find the position and magnitude of the maximum bending moment.

Ans. The maximum occurs at the section 4 feet to the left of the centre, and its value is $M_4 = 72$ ft.-tons.

142. A beam 50 feet span is uniformly loaded from the right end for an extent of 10 feet, with 2 tons per foot run. Find the position and magnitude of the maximum bending moment.

Ans. The maximum occurs at the section 16 feet to the right of the centre, and its value is $M_{-16} = 81$ ft.-tons.

143. A beam 36 feet span is loaded uniformly from the middle point towards the left to an extent of 12 feet, with 2 tons per foot-run. Find the position and magnitude of the maximum bending moment. (See fig. 124.)

In this case, $OG = 6$ ft., and $GS = \frac{1}{3}OG = 2$ feet, since the extent of load is one-third of span; the maximum bending moment is at S, 4 ft. to the left of the centre. Suppose the whole load W, 24 tons, concentrated at G, we have

$$P = \frac{24}{36} \times 24 = 16 \text{ tons.}$$

Taking a section at S, the extent of the load to the left is 8 ft., and is equivalent to 16 tons acting downwards with a leverage of 4 feet, while P acts upwards with a leverage of 14 feet; hence

$$M_4 = 16 \times 14 - 16 \times 4 = 160 \text{ ft.-tons.}$$

144. A beam 72 feet span is loaded in two segments; the left segment is two-thirds of the span, and is loaded uniformly at the rate of two tons per foot. The right segment is uniformly loaded at the rate of one ton per foot. Find the position and amount of the maximum bending moment.

Fig. 128 is this example drawn to scale.

The data in symbols are

$$2c = 72, \quad 2k_1 = 48, \quad 2k_2 = 24, \quad w_1 = 2, \quad \text{and} \quad w_2 = 1;$$

$$P \times 72 = 96 \times 48 + 24 \times 12 \quad \text{and} \quad P = 68 \text{ tons.}$$

In order that the shearing force at T_1 may be zero, it is necessary that the load between the left end and T_1 should equal P, that is 68 tons; hence T_1 is 34 feet from the left end or 2 feet left of the centre. And

$$M_2 = 68 \times 34 - 68 \times 17 = 1156 \text{ ft.-ton maximum.}$$

145. A beam 50 feet span is loaded with 4 cwts. per foot for a distance of 30 feet beginning at the left end, and the remainder is loaded with 9 cwts. per foot. Find the maximum bending moment.

$$\text{Intensity on left segment} \times \text{square of its length} = 4 \times 30^2 = 3600;$$

$$\text{intensity on right segment} \times \text{square of its length} = 9 \times 20^2 = 3600;$$

since these are equal, the maximum bending moment is at the junction of the segments, that is 5 feet to the right of the centre.

To find P, we have

$$P \times 50 = 120 \times 35 + 180 \times 10; \quad \text{therefore} \quad P = 120.$$

Taking a section at the junction of the segments

$$M_{-5} = P \times 30 - 4 \times 30 \times \tfrac{30}{2} = 1800 \text{ ft.-cwts. maximum.}$$

146. A beam is 64 feet span; its left segment, 40 feet long, is loaded with 3 cwts. per foot, the right segment with 5 cwts. per foot. Find the maximum bending moment.

Ans. At 3 feet to right of centre, $M_{-3} = 1837{\cdot}5$ ft.-cwts. maximum.

147. Draw the shearing force diagrams for the beam and cantilever where a uniform load is combined with concentrated loads (figs. 131 and 132).

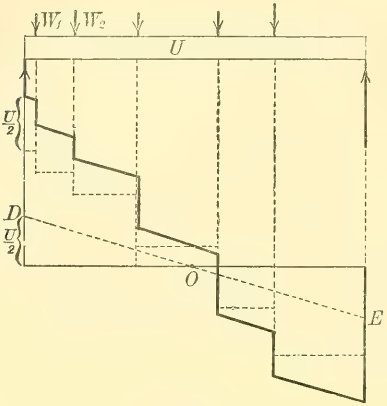


Fig. 131.

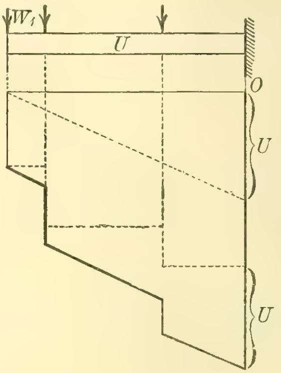


Fig. 132.

Fig. 131 combines figs. 102 and 110 ; fig. 132 combines figs. 104 and 113.

148. A beam 72 feet span supports an uniform load of 2 tons per foot-run, extending from the left end to the point $x = -12$, and an uniform load of 1 ton per foot-run extending from the right end to the point $x = -12$. Find the shearing forces at intervals of 12 feet. (See example No. 144, and figs. 127 and 128.)

$P = 68$ tons ; F decreases at the rate of 2 tons per foot for the left segment, and at the rate of 1 ton per foot for the right segment.

Ans. $F_{36} = 68$; $F_{24} = 44$; $F_{12} = 20$; $F_0 = -4$; $F_{-12} = -28$; $F_{-24} = -40$; $F_{-36} = -52$ $(= -Q)$.

F changes sign at $x = 2$.

149. Find, by analysis, the positions of the apexes in the example solved graphically on fig. 120.

On the shearing force diagram the locus begins at a height 38 ; it then slopes at $\frac{1}{2}$ to 1, so that $Bd_1 = 76$, or $Od_1 = 40$. The second slope being 8 lower, then d_1d_2 is 16 and $Od_2 = 24$. See the numbers engraved at the centres of the circles at foot of fig. 121.

Just as $$OA_0 = 324 = \tfrac{1}{4}(36)^2 = \tfrac{1}{4}(BO)^2,$$

so also $$D_1A_1 = \tfrac{1}{4}(Bd_1)^2 = \tfrac{1}{4}(76)^2 = 1444,$$

measured on the scale of ft.-tons. So also the height of A_2 above E is $\frac{1}{4}$th the square of their known horizontal distance apart. Adding this height to the calculated height of E itself we have $D_2A_2 = 1044$. See the heights of the apexes A_3, A_4, and A_5 engraved on fig. 120.

CHAPTER X.

BENDING MOMENTS AND SHEARING FORCES FOR MOVING LOADS.

In the first Chapter the action of a live load when applied to a tie or strut is described; the action is somewhat similar when a live load is applied to a beam. Thus for a beam loaded at the centre, the load W may at one instant be in contact with the central point of the beam, and yet not be resting any of its weight on the beam; the next instant its whole weight may be resting on the beam. It does not follow directly from Hooke's Law, but is a matter for demonstration, that, for an instant, the strain thus produced is *double* that which the dead load produces, provided the greatest strain does not exceed the proof strain.

One way of applying the actual weight W to the centre as a dead load is, as in the case of a tie, to put it on bit by bit; another way is to put the whole weight W on the end of the beam, when the strain is zero, and then push it very slowly towards the centre, when the strain gradually increases to the full intensity due to W as a dead load. If W, on the other hand, be pushed from the end to the centre in an indefinitely short time, it will be the same as if it had been applied suddenly at the centre; in this case, then, W is applied as a live load.

Definition.—A load which passes along a beam, and which thus occupies at different instants every possible position upon the span, is called a *moving* or *travelling load.*

A moving load may be dead or live or of intermediate importance, but not of greater importance than a live load. A travelling crane, which moves very slowly, and so as not to set the suspended weight swinging, is practically a dead moving load. The action of a moving load on a railway bridge is of intermediate importance; when the span of the bridge is short, say less than 20 feet, this importance is about equal to that of a live load; and when the span is long, say more than 200 feet, it may be considered as about midway between a dead load and a live load.

The Commissioners on the Application of Iron to Railway Structures at p. xviii of their report say:—"That as it has appeared that the effect of velocity communicated to a load is to increase the deflection that it would produce if set at rest upon the bridge; also that the dynamical increase in bridges of less than 40 feet in length is of sufficient importance to demand attention, and may, even for lengths of 20 feet, become more than one-half of the statical deflection at high velocities, but can be diminished by increasing the stiffness of the bridge; it is advisable that for short bridges especially, the increased deflection should be calculated from the greatest load and highest velocity to which the bridge may be liable; and that a weight which would statically produce the same deflection should, in estimating the strength of the structure, be considered as the greatest load to which the bridge is subject."

In the same way the shearing strain produced by a moving load is greater than that produced by the same load when fixed. In the cases which follow, it is to be understood that the loads as given are dead loads, or the equivalent reduced dead loads. When the load is partly fixed and partly moving, the equivalent dead load is the sum of the actual dead load and the dead load equivalent to the actual moving load.

Definition.—For any point x, the *Range of Shearing Force* due to a moving load is its extent; and the limits of this extent are the maximum positive and maximum negative values which F_x assumes during the transit of the moving load.

Classes of Moving Loads.—An uniform load coming on at one end of the span, covering an increasing segment till it is *all* on, then moving to a central position on the span, and passing off at the other end, is called an *advancing load.* A train of trucks, shorter than the span of a bridge, coming on at one end, travelling across and going off at the other end of the bridge, is an approximate example of, and is generally to be reckoned as, an advancing load. The reason that it is called approximate, is that although the weight of the trucks may be uniform per foot of length, yet they are not continuously in contact with the bridge but transmit the load thereto by means of wheels at a number of points. An advancing load may be equal in length to the span; in which case, in passing across, it covers the whole span for an instant. If the load be longer than the span, it will continue to cover it for a definite time while passing, but as time does not come into our consideration, it will be included in the advancing load equal in length to the span.

A load concentrated at a point, and which moves backwards and forwards on the span, is called a *rolling load*; a wheel which rolls along a beam is a practical example of this. In reality the load is distributed over a small area, and if now the load be taken to be uniformly distributed over this small area, it may be considered as an advancing load of small extent; on the diagrams it is represented by a wheel or circle.

A Travelling Load System is a load transmitted to the beam in definite amounts at points fixed relatively to each other, the whole load moving into all possible positions on the span; a locomotive engine is a practical example of such a system, and a rolling load is its simplest form. On the diagrams, the load is represented by a number of circles or wheels with their centres fixed on a frame (see fig. 79), or for ease in drawing by a number of vertical arrows connected by a thick horizontal line (see fig. 139).

It will not be necessary to consider moving loads upon cantilevers, as in practice there is seldom such a thing. It is only necessary to suppose the load fixed in the position most remote from the fixed end; this, it is evident, gives the greatest bending moment at each point, the maximum being at the fixed end.

Bending Moments on a beam under an advancing load equal in length to the span.—Suppose the load to come on from the left end and cover a segment of the span, the bending moment diagram is shown on fig. 125; when the whole span is covered, on fig. 110; and when the load is passing off, by fig. 125 reversed. Since the parabolas in these two figures are the *same*, it is evident that the apex A is higher on fig. 110 than upon fig. 125, because on the former the base of the parabolic segment is the whole span; the ordinate, not only for A, but for every point on fig. 110, p. 154, is greater than the corresponding ordinate upon fig. 125, p. 178. Hence the maximum bending moment at each point of the span occurs when the whole span is loaded; of these maxima, the maximum is at the centre, and this case resolves into that of a beam uniformly loaded.

Shearing Forces on a beam under an advancing load of uniform intensity (fig. 133).—At any point x, the positive maximum shearing force occurs when the front of the load is at the point. Suppose the load to be in the position shown at the top of the diagram, then the shearing force is positive and equal to P. If the load move towards the right, P will decrease, and F_x will equal that decreased value. If the load move towards the left, and if the total load be not yet on the span, then the new load is exactly the same as the first with an additional load to the left of x; this additional load is shared in some manner between the two supports, so that the increase of P is only a *fraction* of that added load; in reckoning F_x, however, we subtract from this increased value of P, the whole of the added load, that is, we subtract more than we add. Again, if the load be shorter than the span and be wholly on the span, then, after the advance, the new load is the same as the first with a portion added to the left of x and an equal portion taken off at the tail of the load; the portion added increases by a fraction of itself the value of P, while the portion taken off decreases by a smaller fraction of itself the value of P; and in reckoning F_x, we substract the whole of the added portion from this increased value of P. Hence, whatever be the length of the load, F_x is a positive maximum when the front of the load is at x. Similarly, by calculating the negative shearing forces from the supporting force Q, it can be shown that the shearing force is a negative maximum when the tail of the load is at the point. At each point, during the transit of the load, the shearing force assumes all values between the two maxima, and passes gradually through the whole range in the same time that the load takes to make a transit.

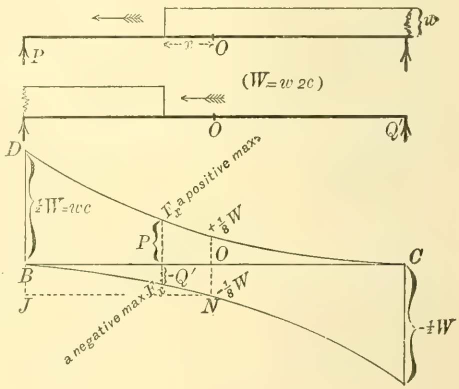


Fig. 133.

Shearing Force Diagram.—Length of load equal to, or greater than, span (fig. 133). When the front of the load is at the left end of span, the whole span is covered, and $P = \frac{1}{2}W$, where W is the load which covers span; hence the positive maximum at the left end is $\frac{1}{2}W$. When the front of the load is at the right end of span, P is zero as no load is on the span; hence the positive maximum at the right end is zero. When the front of load is at any intermediate point, the right segment is loaded, and $F_x = P$ a positive maximum; the value of this maximum increases as the point approaches the left end of span, because the length of the loaded segment is increasing, and because its centre of gravity is nearer the left end; or in symbols, this positive maximum

$$F_x = \frac{\text{load on span}}{\text{span}} \times \frac{\text{length of loaded segment}}{2} = \frac{w}{4c}(c + x)^2.$$

Hence, the locus giving F_x the positive maximum shearing force at each point is a parabola with its axis vertical and its apex at the right end of the span, and whose ordinate at the left end is $\frac{1}{2}W$; the locus giving the negative maximum is a similar parabola below the base and with its apex at the left end of span; the *range* at each point is given by the double ordinate.

Graphical Solution.—Length of load equal to, or greater than, span. With any parabolic segment draw a curve with its apex at the right end meeting the vertical through the left end at D (fig. 133); draw another below the base, and construct a scale for verticals upon which

$$\overline{BD} = \tfrac{1}{2}W,$$

where W is the load which covers the span.

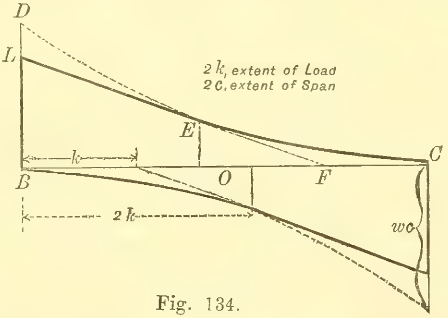


Fig. 134.

Shearing Force Diagram.—Length of load less than span (fig. 134). Let $2k$ be the length of the load. While the load advances from the right end, and so long as it completely covers the right segment, that is up to a distance $2k$ from C, the diagram is a portion of the parabola (fig. 133); for the further advance of the load, the maximum shearing force is increasing, because the centre of gravity of the load is approaching the left end, and the remainder of the locus is therefore a straight line. The straight portion of this locus and the locus for an equal rolling load (see fig. 138) are parallel, and are separated from each other by a distance k measured horizontally; this straight portion, when produced, cuts the base at a point F such that $CF = k$, and it is therefore a tangent to the parabola.

Graphical Solution.—Length of load less than span (fig. 134). Draw the parabola CD as in the previous case; construct a vertical scale such that $BD = wc$, that is equal to half the load which would be on the span supposing the whole span covered; ink in CE the portion of this parabola extending from the apex C through a horizontal distance equal to $2k$; and draw for the remainder of the span, a tangent to the parabola at the point E; this tangent is drawn by laying off $CF = k$, drawing FE, and producing it to L.

Bending Moment for a Beam under an advancing load less in length than the Span (fig. 135).—Let $2c$ = span; $2k$ = extent of load; w = intensity of load; C the origin, and centre of span; G the centre of load; and let the load be upon the span in any position.

Here

$$k < c, \quad \text{and} \quad W = 2wk = \text{total load.}$$

To find P, we may suppose the whole load concentrated at G, and we have

$$P = \frac{2wk}{2c}\,(c + x - y),$$

where x is the abscissa of any point of the span reckoned positive to the left of C, and y is the distance of the same point reckoned positive to the left of G. Taking a section at the point x, we have two forces acting on the portion of the span to the left of the section, viz. P acting upwards with a leverage $(c - x)$, and a load area equivalent to a force $w\,(k - y)$ acting downwards with a leverage $\frac{1}{2}(k - y)$; hence the bending moment at this section is

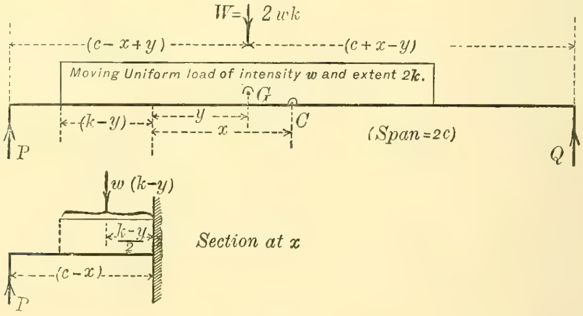


Fig. 135.

$$M_x = P(c - x) - w\,(k - y) \times \frac{k - y}{2} = \frac{w}{c}\,(c + x - y)(c - x) - \frac{w}{2}\,(k - y)^2$$

$$= \left(\frac{wk}{c}\,(c^2 - x^2) - \frac{wk^2}{2}\right) + \frac{w}{2}\left(\frac{2kx}{c} - y\right) y. \qquad (1)$$

As the load moves about, y varies and the bending moment M_x at the section x depends upon the position of the load, that is, upon the value of y. To find the position of the load which gives the greatest bending moment at the point x, it is only necessary to find the value of y which makes M_x a maximum; now M_x is greatest when when the product $\left(\frac{2kx}{c} - y\right) y$ is greatest; and as the sum of the two factors of this product is constant, the product is greatest when the factors are equal to each other, that is when

$$\frac{2kx}{c} - y = y;$$

so that we have

$$y = \frac{kx}{c}, \tag{2}$$

or

$$y : k \ :: \ x : c. \tag{3}$$

This proportion expressed in words gives the following:—

Rule.—The greatest bending moment at any point of the span occurs when there is directly over it, that point in the load which is situated in the extent of the load in a position similar to that in which the point is situated in the extent of the span.

Substituting in (1) the value of y in (2), we have

$$\text{max.}M_x = \left(\frac{wk}{c}(c^2 - x^2) - \frac{wk^2}{2}\right) + \frac{w}{2}\left(\frac{kx}{c}\right)^2 = \frac{wk}{2c^2}(c^2 - x^2)(2c - k)$$

$$= \frac{2wk}{4c}(c^2 - x^2)\left(2 - \frac{k}{c}\right) = \frac{W}{4c}(c^2 - x^2)\left(2 - \frac{k}{c}\right) \tag{4}$$

This is the equation to the maxima bending moments, and may be written thus

$$\text{max.}M_x = C_0(c^2 - x^2),$$

where C_0 is a constant quantity; the locus is therefore a parabola with its apex above the centre of span, and the maximum of these maxima—that is, the maximum for the whole span—is at the centre, or where $x = 0$;

$$\text{max.}M_0 = \frac{Wc}{4}\left(2 - \frac{k}{c}\right) = \tfrac{1}{8}\left(2 - \frac{k}{c}\right)Wl = \tfrac{1}{4}W(l - k). \tag{5}$$

For the maximum bending moment, the coefficient

$$m = \tfrac{1}{8}\left(2 - \frac{k}{c}\right); \qquad \text{(See page 151)}$$

and the principal equation to the parabola is

$$Y = \frac{W}{4c}\left(2 - \frac{k}{c}\right)X^2.$$

Compare the above with the result of investigation at p. 174, from which this might have been deduced. See also fig. 124.

By the preceding rule the maximum bending moment at the centre of the span occurs when the centre of the load is over the centre of the span.

Graphical Solution.—Fig. 136. With a scale of feet lay off the span and draw a vertical upwards through O the centre; apply the rollers to the span; place any parabolic segment against the rollers with its apex on the vertical through O; shift the rollers till the curved edge passes through the ends of the span, which it will do simultaneously, and draw the curve. Construct a scale of ft.-lbs. for verticals such that $OA = \frac{1}{4}W(l-k)$, where W is in lbs., and l and k are in feet.

Note that to give the maximum bending moment at any point, the load assumes a different position for each point according to the above rule, and that it is possible to fulfil the condition of the rule for every point without any of the load going off the span.

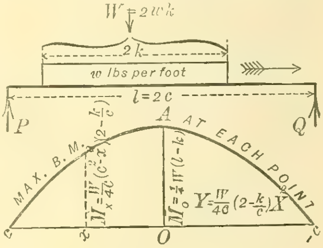

W = 2wk
2k
w lbs per foot
l = 2c
P
Q
A
MAX. B. M. AT EACH POINT
$M_x = \frac{W}{4c}(c^2 - x^2)(2 - \frac{k}{c})$
$M_0 = \frac{1}{4}W(l-k)$
$Y = \frac{W}{4c}(2 - \frac{k}{c})X$
c
x
O
c

Fig. 136.

Cor. 1.—Suppose the extent of load equal to the span; then

$$k = c. \quad \left(2 - \frac{k}{c}\right) = 1,$$

and we have

$$\text{max.}M_x = \frac{W}{4c}(c^2 - x^2), \quad \text{and} \quad \text{max.}M_0 = \tfrac{1}{8}W.l,$$

the same as for the span uniformly loaded (fig. 110), and as shown in the preceding case. Note further, that the rule for fixing the position of the load so as to give the maximum bending moment at any point is fulfilled simultaneously for *all* points of the span; as it is evident that when the centre of load is over the centre of span, every point of the load is over the corresponding point of the span.

Cor. 2.—Suppose the extent of load to be zero; then

$$k = 0, \quad \left(2 - \frac{k}{c}\right) = 2,$$

and the load is a rolling load for which

$$\text{max.}M_x = \frac{W}{2c}(c^2 - x^2), \quad \text{and} \quad \text{max.}M_0 = \tfrac{1}{4}W.l.$$

When the rule for finding the position of the load which gives the maximum bending moment at any point is applied to this case, it is found that the maximum occurs at any point of span when the rolling load is at that point.

As this is an important case, and leads to cases still more important, we will give a separate investigation.

Bending Moments for a beam under a rolling load (fig. 137).—Consider any point of the span at the distance x from the centre, distances to the left being reckoned positive. Let R be the amount of the rolling load, and suppose it over the point in consideration.

We may calculate the bending moment M_x from either of the two equations

$$M_x = P(c - x), \quad \text{or} \quad M_x = Q(c + x).$$

If the load moves to the right, then the upward supporting force P' is less than P, and

$$M'_x = P'(c - x) < M_x;$$

if now the load moves to the left, the supporting force Q' is less than Q, and

$$M''_x = Q'(c - x) < M_x;$$

thus M_x decreases whether R moves to the right or left, that is, M_x, the bending moment at any point x, is greatest when R the rolling load, is over the point.

Let R be over the point x, then

$$_{\text{max.}}M_x = P(c - x) = \frac{R}{2c}(c + x)(c - x) = \frac{R}{2c}(c^2 - x^2).$$

This is the equation to the maxima bending moments; the bending moment diagram is a parabola, with its axis vertical and its apex above the centre of span; and the maximum of these maxima, that is the maximum bending moment for the whole span occurs at the centre when the load is over the centre; putting $x = 0$, we have

$$_{\text{max.}}M_0 = \frac{R}{2}c = \tfrac{1}{4}R \,.\, l,$$

the value of the constant $m = \frac{1}{4}$, and the principal equation to the parabola is

$$Y = \frac{R}{2c}X^2.$$

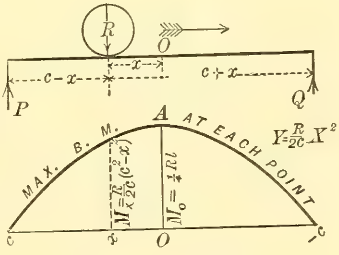


Fig. 137.

Graphical Solution (fig. 137).—With a scale of feet for horizontals lay off the span, and draw a vertical upwards through O the centre; apply the rollers to the span, place any parabolic segment against the rollers with its apex on the vertical through O; shift the rollers till the curved edge passes through the ends of the span, which it will do simultaneously, and draw the curve. Construct a scale of foot-lbs. for verticals, such that $OA = \frac{1}{4}Rl$.

Bending Moment for a beam under both a rolling and an uniform dead load.—Let R be the dead load equivalent to the actual rolling load, and U the uniform load. For each load separately the bending moment diagram is a parabola with its apex over the centre. For the combined load, the diagram is a parabola whose apex is also over the centre, and whose modulus is the sum of their moduli; hence

$$M_x = \left(\frac{U}{4c} + \frac{R}{2c}\right)(c^2 - x^2) = \frac{U + 2R}{4c}(c^2 - x^2), \text{ maxima,}$$

and

$$M_0 = \frac{c}{4}(U + 2R) = \tfrac{1}{8}(U + 2R)\,.\,l, \text{ max. of maxima,}$$

the bending moments being in terms of a *dead* load throughout.

Graphical Solution.—With any parabolic segment draw a parabola with its apex over the centre and passing through the two ends, and make a scale for verticals upon which the middle ordinate $OA = \frac{1}{8}(U + 2R)\,.\,l$.

Shearing Forces in a beam under a rolling load (fig. 138).—At any point of the span, F_x the shearing force is positive and equal to P, so long as R is to the right of the point; since P increases as R moves towards the left support, F_x is evidently a positive maximum when R is indefinitely close to, and on the right side of, the point. When R passes to the left of the point, $F_x = P - R = - Q$; since Q increases as R comes closer to the right support, it is again evident that F_x is a negative maximum when R is indefinitely close to, and on the left side of, the point. When R is indefinitely close to the point, P and Q have sensibly the same values as for R exactly at the point, and we have the following:—

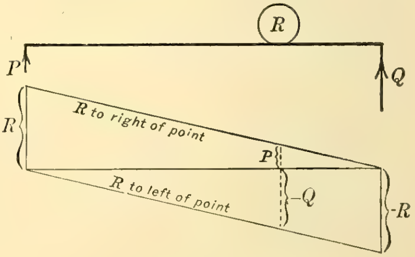


Fig. 138.

To find the maximum shearing force at any point x, place R over the point, and calculate P and Q for that position of the load; these are respectively the maximum positive and maximum negative values of F_x.

During the passage of the load, the shearing force assumes all values lying between these maxima; and it is important to observe that the shearing force not only changes sign at any point as R passes over the point, but that it changes from its greatest positive to its greatest negative value, or *vice versa*, according as R is moving to left or right, and does so even although the load be moving slowly. If moving quickly, this sudden change produces what is called a hammer blow.

The positive maximum at each point is the value of P as R comes to the point; and since P is proportional to the remote segment, it follows that the positive maximum at each point is proportional to the distance of the point from the right end of span; it is zero for the right end, and increases uniformly till it is R for the left end; for, when R is just to the right of the right end of span, P is zero, no load being on the span; and again, when R is just to the right of the left end of span, P is sensibly equal to R.

For a rolling load the range at each point is constant and is equal to R.

Graphical Solution.—From the left end of the base draw upwards a vertical equal to R, and join its extremity to the right end of the base; similarly from the right end, draw downwards a vertical equal to R, and join its extremity to the left end of the base; at each point the ordinate upwards gives the maximum positive, and the ordinate downwards the maximum negative, shearing force; while the double ordinate gives the range.

TWO-WHEELED TROLLY CONFINED TO A GIRDER LIKE A TRAVELLING CRANE.

Bending Moments for a beam under a travelling load system of two equal weights at a fixed interval apart (fig. 139).—Let R be the total load, and $W_1 = W_2$ the weights numbered from the left end; let $4s$ be their distance apart, so that if G be the origin for loads, the abscissæ of W_1 and W_2 are $2s$ and $-2s$, respectively; the origin for the span is O the centre, and $\bar{x}$ is the distance from O to G.

First, let W_1 be over the point x, the whole load being on the span; then P may be calculated as if the whole load R were at G, that is,

$$P = \frac{R}{2c}(c - \bar{x}), \quad \text{and} \quad {}_1M_x = P(c - x) = \frac{R}{2c}(c - \bar{x})(c - x). \quad (1)$$

If the load travels a little to the right, P diminishes, and therefore the bending moment at x diminishes; if the load travels until W_1 is at a small distance a to the left of the section, then (fig. 140)

$$P' = \frac{R}{2c}(c - \bar{x} + a),$$

and, if x is positive,

$$M'_x = P'(c - x) - W_1 a = \frac{R}{2c}(c - \bar{x} + a)(c - x) - \frac{R}{2}a$$

$$= \frac{R}{2c}(c - \bar{x})(c - x) - \frac{R}{2c}ax = {}_1M_x - \frac{Ra}{2c}.x < {}_1M_x.$$

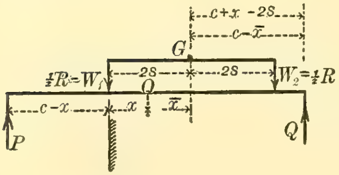


Fig. 139.

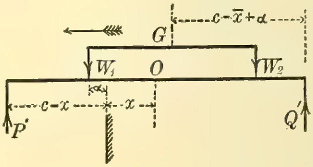


Fig. 140.

Hence ${}_1M_x$ is the maximum bending moment for values of x from 0 to c; that is, for any point in the left half of the span the maximum occurs when W_1 is over it, provided the whole load be then on the span. By symmetry the maximum for any point in the right half of the span occurs when W_2, the right weight, is over it.

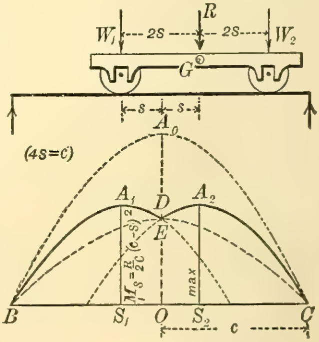


Fig. 141.

Bending Moment Diagram (fig. 141).—Substituting for $\bar{x}$ its value $(2s - x)$, we have

$${}_1M_x = \frac{R}{2c}(c - 2s + x)(c - x), \quad (2)$$

the equation to the maxima bending moments for first half of span. The locus is a parabola with its axis vertical, and the principal equation is $Y = \frac{R}{2c} X^2$; it is therefore the *same* parabola as for a rolling load R (see fig. 92, p. 130). To find the horizontal distance to the apex A_1, find that value of x which makes the product $(c - 2s + x)(c - x)$ greatest; since the sum of the factors is constant, this occurs when they are equal; thus

$$c - 2s + x = c - x, \quad \text{or} \quad x = s. \tag{3}$$

That is, the apex A_1 lies to the left of O at a distance s, one quarter of the distance between the two weights. To find the height of A_1, put

$$x = s, \quad \text{and} \quad {}_1M_s = \frac{R}{2c}(c - s)^2, \tag{4}$$

the maximum of maxima for first half of span.

It is evident that A_2 will lie to the right at a distance s, and that the two parabolas will intersect at D on the vertical through the centre. It is convenient to call the first half of the span *field* 1, and to say that this field is *governed* by W_1; and we observe that the maximum in field 1 occurs when W_1, being in its own field, lies as far to one side of O the centre, as G lies to the other.

If it be possible for W_1 to occupy every point in its field without W_2 going off the span, we say that W_1 can *overtake* its field. In the present problem it is necessary that $4s$, the distance between the weights, be not greater than c the half-span, in order that each weight may be able to overtake its field. The problem divides into two cases.

Case 1, $4s =$ or $< c$.—On fig. 141, $4s = c$; and it is evident that one weight $\frac{1}{2}R$ may be at any point of the half-span, while the other weight is not on the span. Hence the locus of the maximum bending moment at each point when only one weight is on the span is the parabola BEC, due to a rolling load $\frac{1}{2}R$ as in fig. 137.

The apex E of this parabola coincides with D, the intersection of the pair of parabolas. This may be seen thus:—Shift the load until W_2 is over the centre; then, since W_1 is over B the extremity of the span, we may either consider it not yet on the span when OE is the bending moment, or we may consider that W_1 is just on the span when OD is the bending moment. Also note that $OE = \frac{1}{2}OA_0$, as BEC is the parabola due to the rolling load $\frac{1}{2}R$, while BA_0C is the parabola which would be due to a rolling load R.

The apex A_1 is higher than the apex E; the two parabolic arcs BA_1 and BE intersect at B, and every point on the arc BA_1D is outside of BEC. Hence the locus BA_1DA_2C is everywhere outside of BEC, and gives the maximum bending moment for each point of span.

Again, on fig. 142, $4s < c$; that is, A_1 and A_2 are closer together than on fig. 141; hence D is higher than E and BA_1DA_2C is again outside of BEC. BF and HC, parts of the parabola BEC, are shown by heavy dotted lines, and they indicate the bending moments, when one weight only is on the span.

Graphical Solution; $4s \gtreqless c$ (figs. 141, 142).—With a scale of feet lay off BC equal to the span, and upon each side of the centre lay off $OS_1 = OS_2 = s$; apply the rollers to BC; place any parabolic segment against the rollers with its apex on the vertical through S_1; shift the rollers till the curved edge passes through B, and draw BA_1D; similarly draw DA_2C. Then BA_1DA_2C is the diagram of maximum bending moment at each point, the common height of A_1 and A_2 being maximum for the whole span.

Place the segment with its apex on the vertical through the centre, and move the rollers till the curved edge passes through B and C; mark A_0, and construct a scale of ft.-lbs. for verticals such that $OA_0 = \frac{1}{4}Rl$.

Case 2, $4s > c$, fig. 143.—In this case the apexes A_1 and A_2 are farther apart than on fig. 141; hence D is below E; hEf, the central portion of the parabola BEC due to only one weight on the span, lies outside of BA_1DA_2C and gives the maximum at each point for that portion. It will be seen that

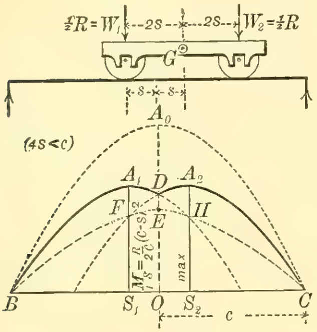


Fig. 142.

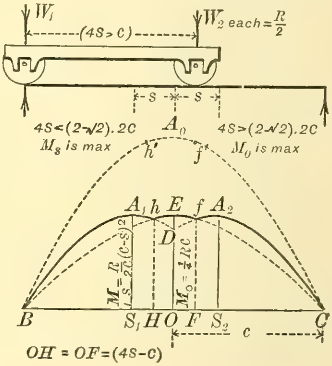


Fig. 143.

$$OH = OF = 4s - c;$$

for, suppose W_2 at F, then W_1 is at B, and Ff, representing the bending moment at F, is the ordinate of BEC or of DA_2C, according as we consider that W_1 is not yet on, or is just on, the span. In this case, W_1 cannot overtake the portion HO, neither can W_2 overtake the portion OF, of their respective fields. There are two equal maxima at A_1 and A_2, and a third maximum at E; and the greatest of these is the maximum for the whole span. The point E may thus be the same height as, or higher or lower than, A_1.

Suppose that E is of the same height as A_1, then A_0 will be twice as high as A_1; and we will have

$$OB : S_1B :: \sqrt{2} : 1; \quad \text{that is,} \quad c : (c - s) :: \sqrt{2} : 1,$$

or

$$4s : 2c :: (2 - \sqrt{2}) : 1; \quad \text{hence} \quad 4s = (2 - \sqrt{2}) \,.\, 2c,$$

or the distance between the weights $= (2 - \sqrt{2})$ times the span.

Thus if the distance between the weights equals or exceeds the quantity $(2 - \sqrt{2}) \cdot 2c$, or about $\frac{7}{12}$ths of the span, the maximum bending moment for the whole span is at the centre.

Graphical Solution: $4s > c$ (fig. 143.—Lay off BC equal to the span; on each side of the centre lay off $OS_1 = OS_2 = s$, and $OH = OF = 4s - c$; draw vertical through S_1, H, O, F, S_2; apply the rollers to BC; place any parabolic segment against the rollers with its apex on the vertical through S_1; shift the rollers till the curved edge passes through B, and draw BA_1h, stopping at the vertical through H: similarly draw CA_2f. Shift the segment till the apex is on the vertical through the centre; move the rollers till the curved edge passes through B and C, and draw BA_0C. Bisect OA_0 in E; and plot as many points in hEf as may be necessary, by bisecting the ordinates of $h'A_0f'$, or in any manner construct the parabola BEC. Then BA_1hEfA_2C is the diagram of maximum bending moment at each point. Construct a scale for verticals such that $\overline{OA_0} = \frac{1}{4}R \cdot l$.

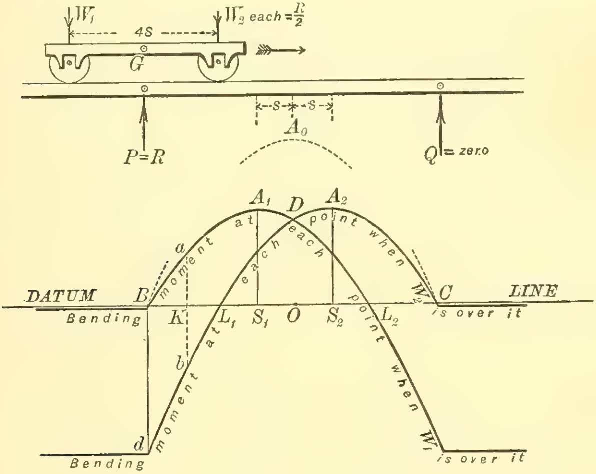

W_1
W_2 each $= \frac{R}{2}$
4S
G
S
S
A_0
P=R
Q = zero
A_1
D
A_2
a
DATUM
B
C
LINE
K
L_1
S_1
O
S_2
L_2
b
d
Bending moment at each point when W_2 is over it
Bending moment at each point when W_1 is over it

Fig. 144.

The complete interpretation of the locus BA_1DA_2C is shown in fig. 144. Suppose the beam to extend beyond the supports at B and C, and to be fixed at these supports so that P and Q may act upwards or downwards. In the figure the travelling load is standing with G over B, so that $P = R$, and Q is zero; hence at L_1, the point under W_2, the bending moment is zero, and this is the point at which CA_2D meets BC. Let the load move towards the left until W_2 is over any point as K; Q now acts downwards; at the point K, the beam is bent upwards and the bending moment is negative; the value of this *negative* moment is $Q \cdot \overline{KC}$, and it is given by the *downward* ordinate $\overline{Kb}$. When W_2 arrives over B, the bending moment at B is negative; and its value, $Q \cdot \overline{BC} = W_1 \cdot 4s$, is given by the ordinate $\overline{Bd}$.

As the load moves farther to the left, the bending moment at each point, as W_2 comes over it, is of the constant value Bd; BC is now a cantilever under the downward load Q, and the bending moment at each point of BC is now negative, increasing indefinitely as the load moves towards the left. For all positions of the load, with no restriction on the value of $4s$, BA_1DA_2C gives the maximum positive bending moment at each point; and the height of A_1 is the greatest positive bending moment that can possibly be produced by the load system.

Bending Moments for a beam under a travelling load system of two unequal weights at a fixed interval apart (fig. 145).—Let R be the total load; W_1 and W_2 the weights numbered from the left end; let G their centre of gravity be the origin for loads, $2h_1$ and $-2h_2$ being the abscissæ of W_1 and W_2, so that

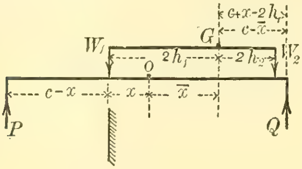


Fig. 145.

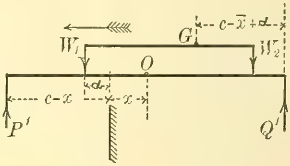


Fig. 146.

the distance between the weights is $2h_1 + 2h_2$. Let W_1 be over any point of the span whose abscissa is x measured (positive to left) from the centre as origin, and let it be understood all through that the whole load is on the span.

As in the previous case,

$$ {}_1M_x = \frac{R}{2c}(c - \bar{x})(c - x) \qquad (1)$$

is the equation to the bending moment at any point x when W_1 is over it.

If the load travels a little to the right, P diminishes, and therefore M_x diminishes; if the load travels until W_1 is at a small distance a to the left of the section, then (fig. 146)

$$M'_x = \frac{R}{2c}(c - \bar{x} + a)(c - x) - W_1 a$$

$$= \frac{R}{2c}(c - \bar{x})(c - x) + \left\{\frac{R}{2c}(c - x) - W_1\right\} a$$

$$= {}_1M_x - \left\{x - \left(c - \frac{W_1}{R} . 2c\right)\right\} \frac{Ra}{2c} < {}_1M_x, \quad \text{if} \quad x > \left(c - \frac{W_1}{R} . 2c\right).$$

That is, the bending moment at x any point of the span, when

W_1 is over it, is greater than when the load is in any other position, provided that the point itself is situated between B the left end, and F a point whose distance from the centre is

$$\overline{OF} = c - \frac{W_1}{R} \cdot 2c,$$

or from the left end is

$$\overline{BF} = \frac{W_1}{R} \cdot 2c.$$

Further,

$$\overline{CF} = \frac{W_2}{R} \cdot 2c,$$

and the bending moment is greatest at any point of $\overline{CF}$ when W_2 is over it. $\overline{BF}$ and $\overline{CF}$ are fields 1 and 2, and they are commanded by W_1 and W_2, respectively.

Bending Moment Diagram (fig. 147).—For $\bar{x}$ substitute $2h_1 - x$ in equation (1), and we have

$$_1M_x = \frac{R}{2c}(c - 2h_1 + x)(c - x), \tag{2}$$

the equation to the maxima bending moments for field 1.

The locus is a parabola whose axis is vertical, and principal equation is

$$Y = \frac{R}{2c} X^2;$$

it is therefore the same parabola as for a rolling load R.

The abscissa of the apex A_1, that is OS_1, is found as before by equating the factors of equation (2), thus

$$c - 2h_1 + x = c - x, \quad \text{or} \quad x = h_1.$$

That is, the apex A_1 lies to the left of O at a distance $OS_1 = h_1$; similarly A_2 lies to the right of O at a distance $OS_2 = h_2$, each being half the distance between W and G. Putting $x = h_1$, we have

$$S_1A_1 = \frac{R}{2c}(c - h_1)^2. \tag{3}$$

Similarly,

$$S_2A_2 = \frac{R}{2c}(c - h_2)^2. \tag{4}$$

If the point S_1 does not lie in field 1, the ordinates which are the bending moments for field 1 continually increase from zero at B the left end to their greatest value at F the other end of the field; if S_1 lies in field 1, then

$$_1M_{h_1} = S_1A_1 = \frac{R}{2c}(c - h_1)^2 \tag{5}$$

the maximum of maxima for field 1.

Similarly, if S_2 be situated in field 2, the height of A_2 will be the maximum bending moment for field 2.

Suppose $W_2 > W_1$; then, since both parabolas are the *same* as that for the rolling load R, and are therefore the *same* as each other, A_2 is higher than A_1 because the quadrant CA_2S_2 stands on a longer base than BA_1S_1; and

$$_2M_{h_2} = S_2A_2 = \frac{R}{2c}(c - h_2)^2 \tag{6}$$

the maximum of maxima for field 2, and maximum for whole span.

Now F is both in field 1 and field 2; and when W_1 arrives at F, the ordinate of the first parabola gives the maximum bending moment at F; again, when W_2 arrives at F, the ordinate of the second parabola also gives the maximum bending moment at F; that is, the ordinates at F are equal, or the two parabolas intersect at D a point on the vertical through F.

It is well to observe that the maximum in either field occurs when the weight commanding the field, while lying in its own field, is as far from the centre of the span upon one side as G is upon the other. If it be impossible in one of the fields for the weight so to lie, then for that field the bending moment continuously increases towards the end of the field not coinciding with the end of the span.

Case 1, $2h_1 + 2h_2 \leqq \dfrac{\text{smaller weight}}{\text{total weight}} \times$ span, or distance between the weights $\leqq$ shorter field.—In this case it is evident that each weight can overtake its field.

Suppose now that the distance between the weights is *equal to* the smaller field, and that the load stands with the greater weight just over F the junction of the fields, then the smaller weight is directly over the support. We may consider that the whole load is on the span, so that FD, the ordinate of the intersection of the parabolas, gives the bending moment at F; or we may consider that the greater load alone is on the span, so that the ordinate at F to the parabola for the greater load alone as a rolling load also gives the bending moment at F; that is, the last-named parabola passes through D. The parabola BDC for the greater load alone can be derived from BA_0C by taking the same fraction of its ordinates as the greater load is of the total load; it is therefore everywhere flatter than BA_0C; and since BA_1D and DA_2C are the *same* parabola as BA_0C, it follows that the locus BA_1DA_2C is everywhere outside of the parabola for either load alone.

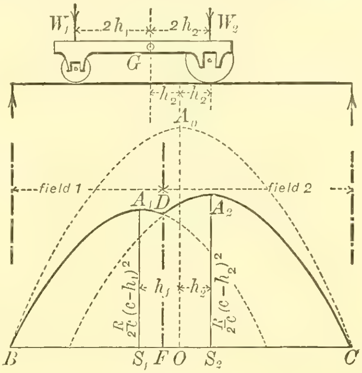


Fig. 147.

Again, suppose the distance between the weights to be decreased, then the apexes A_1 and A_2 approach, and D their point of intersection rises; here again the locus BA_1DA_2C is outside of the parabola for either load alone. Therefore, for *Case* 1, the locus BA_1DA_2C gives the maximum bending moment at each point of span due to the load coming on at either end, passing across, and going off at the other end.

Graphical Solution (fig. 147).—Find G the centre of gravity of the loads, by dividing the total distance between the loads inversely as the loads by arithmetic or by the construction on fig. 101. Divide the span BC in F, so that $\overline{BF} = \dfrac{W_1}{R} 2c$; or draw parallels in opposite directions from B and C, and of lengths equal to the loads respectively, when the line joining their extremities will cut $\overline{BC}$ in F; and observe that the distance between the weights is not greater than BF. From the centre of the span lay off $OS_1 = h_1$ towards the left, and $OS_2 = h_2$ towards the right; draw verticals through O, F, S_1, and S_2; on the diagram, the vertical through F is drawn with dash and dot, to indicate that it separates two fields; those through S_1 and S_2 shown by fine continuous lines give maxima bending moments, and that through O shown by a dotted line is only constructive. Apply the rollers to BC; place any parabolic segment against the rollers with its apex on the vertical through S_1; shift the rollers till the curved edge passes through B, and draw BA_1D: similarly, draw CA_2D, and the segment should now pass through C and D simultaneously if the drawing has been accurately constructed. The figure BA_1DA_2C is the diagram of maximum bending moment at each point of the span. Shift the segment till its apex is on the vertical through the centre; move the rollers till the curved edge passes through B and C, and mark A_0; construct a scale for verticals such that $OA_0 = \frac{1}{4}R \,.\, l$.

Case 2, $2h_1 + 2h_2 > \dfrac{\text{smaller weight}}{\text{total weight}} \times$ span, or distance between the weights > shorter field (fig. 148).—The two apexes A_1 and A_2 are further apart than in the previous case; D occupies a lower position and is no longer on BEC, of which a portion kEh is above BA_1DA_2C; and the diagram showing the maximum bending moment at each point is now $BkEhA_2C$.

To find the points h and k; let W_1 be over B, then W_2 is at H, where BH is the distance between the weights. We may either consider that W_1 is not yet on the span, when the ordinate at H to BEC gives the bending moment there; or that W_1 is just on the span, when the ordinate at H to DA_2C gives the same moment; that is, the parabolas intersect at h. Again, equating the value of the ordinate of the parabola BA_1D to that of BEC, we find the abscissæ of the points of intersection; thus

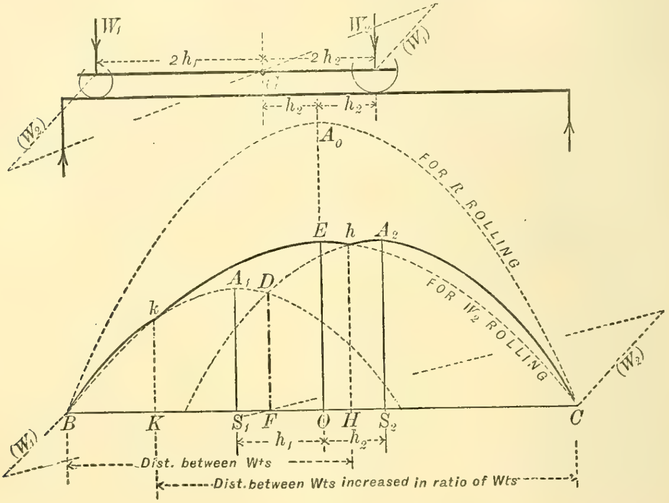


Fig. 148.

$$\frac{R}{2c}(c + x - 2h_1)(c - x) = \frac{W_2}{2c}(c^2 - x^2);$$

$$(c - x)\{R(c + x - 2h_1) - W_2(c + x)\} = 0; \qquad (7)$$

putting $c - x = 0$, then $x = c$, or the parabolas intersect at B, which we already know; putting

$$R(c + x - 2h_1) - W_2(c + x) = 0,$$

then

$$W_1(c + x) = R \cdot 2h_1,$$

and

$$x = \frac{R}{W_1} \cdot 2h_1 - c = OK, \qquad (8)$$

which gives K the point on OB under the other point of intersection of the parabolas.

Now

$$CK = CO + OK = \frac{R}{W_1} \cdot 2h_1;$$

but

$$R : W_1 : W_2 \ :: \ h_1 + h_2 : h_2 : h_1;$$

therefore

$$CK = \frac{h_1 + h_2}{h_2} \cdot 2h_1 = \frac{h_1}{h_2}(2h_1 + 2h_2) = (2h_1 + 2h_2)\frac{W_2}{W_1} \tag{9}$$

= the distance between the weights increased in the ratio of the weights.

It may happen that E and A_2 are of the same height, in which case there are two equal maxima for the whole span, one at the centre, the other at S_2; again, E may be higher than A_2 when the maximum for whole span is at the centre. To investigate this, we have (equation 4)—

$$S_2A_2 = \frac{R}{2c}(c - h_2)^2: \quad \text{and} \quad OE = \tfrac{1}{4}W_2 \cdot l = \tfrac{1}{2}W_2 \cdot c;$$

equating these quantities, we have

$$\frac{R}{2c}(c - h_2)^2 = \tfrac{1}{2}W_2 \cdot c;$$

or

$$(c - h_2)^2 = \frac{W_2}{R}c^2; \quad \text{or} \quad c - h_2 = \sqrt{\frac{W_2}{R}} \cdot c;$$

therefore

$$h_2 = \left(1 - \sqrt{\frac{W_2}{R}}\right)c. \tag{10}$$

Also

$$h_1 = \frac{W_2}{W_1}h_2 = \frac{W_2}{W_1}\left(1 - \sqrt{\frac{W_2}{R}}\right)c; \tag{11}$$

therefore

$$h_1 + h_2 = \left(1 + \frac{W_2}{W_1}\right)\left(1 - \sqrt{\frac{W_2}{R}}\right)c;$$

or

$$\frac{h_1 + h_2}{c} = \frac{W_1 + W_2}{W_1} \cdot \frac{\sqrt{R} - \sqrt{W_2}}{\sqrt{R}};$$

or

$$\frac{2h_1 + 2h_2}{2c} = \frac{R}{R - W_2} \cdot \frac{\sqrt{R} - \sqrt{W_2}}{\sqrt{R}};$$

or,

$$\text{distance between weights} = \frac{\sqrt{R}}{\sqrt{R} + \sqrt{W_2}} \cdot 2c = \frac{R - \sqrt{RW_2}}{W_1}2c. \tag{12}$$

That is, if the height of A_2 be equal to the height of E the apex of the parabola due to the rolling load W_2, the distance between the weights W_1 and W_2 is equal to the fraction $\frac{\sqrt{R}}{\sqrt{R} + \sqrt{W_2}}$ multiplied by the span; W_2 being the greater weight, and R the total load. This fraction is always greater than $\frac{1}{2}$, and therefore it is possible for W_2 to be at the centre without W_1 being on the span. If the weights are fixed at this fraction of the span apart, then during their transit there are two equal maxima for the whole span, one at O when W_2 is over the centre, and another at S_2 when W_2 is over S_2.

If the distance between the weights is increased, the apexes A_1 and A_2 are lowered, and A_2 becomes lower than E the apex of the parabola for W_2 alone; so that during the transit of the load, the maximum for the whole span would occur at the centre while W_2 is over it.

It is thus shown that, when the distance between the weights equals or exceeds the above fraction of the span, the maximum bending moment for the whole span occurs at the centre when the greater weight is over it, and equals the maximum due to the greater load alone as a rolling load, viz.

$$\max.M_0 = \tfrac{1}{4} W_2 \,.\, l. \tag{13}$$

By taking W_1 small enough, the fraction $\dfrac{\sqrt{R}}{\sqrt{R}+\sqrt{W_2}}$ may be made to differ from one-half by a quantity as small as we please; on the other hand, this fraction is greatest when W_2 takes its smallest possible value. Since W_2 is the greater weight, it is always greater than $\frac{1}{2}R$; putting W_2 equal to $\frac{1}{2}R$, we find that the greatest value of the fraction is equal to $(2 - \sqrt{2})$ or $\cdot586$, and we have the following:—

If the distance between the unequal weights exceeds the half-span by $\frac{86}{1000}$ths of the span, or about $\frac{1}{12}$th part, then during the transit of the load, the maximum bending moment for the whole span occurs at the centre when the greater weight is over it; should the weights be very unequal, the maximum will occur at the centre if the distance between the weights be only a little (less than a twelfth of the span) greater than the half span.

Graphical Solution (fig. 148).—Construct BA_1DA_2C as in *Case* 1, and observe that the distance between the weights is greater than BF; this determines that it belongs to *Case* 2. Lay off

$$BH = 2h_1 + 2h_2, \quad \text{and} \quad CK = (2h_1 + 2h_2)\,\frac{W_2}{W_1}.$$

Draw verticals through H and K; ink in the portions Bk and Ch of the locus BA_1DA_2C, and construct a number of points on kEh by cutting the ordinates of BA_0C in the ratio $\dfrac{W_2}{R}$. If K does not lie on CB, the arc Bk is inadmissible, and the locus is $BEhA_2C$. Construct the scale for verticals such that $OA_0 = \frac{1}{4}R \,.\, l$.

Note.—During the transit W_2 really commands from C to K.

Cor. 1.—If CK equals CB, the points k and B coincide, and the parabolas touch at B; if CK exceeds CB then k is beyond B. In either case BEC is outside of BA_1D, and W_2 commands the whole span.

Cor. 2.—If the beam extends past the supports as in fig. 144, and if P and Q are able to act upwards and downwards, then the locus BA_1DA_2C gives the maximum positive bending moment at each point for all cases; each curve drawn as on that figure gives the bending moment at each point when the corresponding weight comes over it.

Cor. 3.—Let both weights (fig. 148) be confined to the span, then

1°. If $FC \geqq (2h_1 + 2h_2)$, the locus for maximum bending moment at each point is the arc BD, chord Dh, and arc hA_2C;

2°. If $FC < (2h_1 + 2h_2)$, the locus is arc Bh', chord $h'D$, chord Dh, and arc hA_2C. The point h' is above H' and on the arc BA_1D, and $CH' = (2h_1 + 2h_2)$. H' and h' are not shown on the figure.

The maximum will be at S_2 or H.

Travelling load concentrated on two wheels together with fixed loads.—Let the travelling load be such as is described in *Case* 1, fig. 147. With any parabolic segment draw the diagram as on fig. 147; on this, to the same scales, draw the diagram for fixed loads as on fig. 120. Through S_1 draw a vertical, and find the points d_1, d_2, &c., as described for, and shown on fig. 120; in this case $2S_1A_1 = LN$, and $BS_1 = LS$. With the same parabolic segment, placing its apex successively on the verticals through d_1, d_2, &c.: draw the arcs BE, EF, &c., as on fig. 120, until the vertical dividing the two fields is reached. Again, draw a vertical through S_2 and find the points d_5, d_4, &c.; in this case $2S_2A_2 = LN$, and $CS_1 = LS$; with the same parabolic segment draw the arcs CH, HG, &c., until the point formerly fixed is reached.

For *Case* 2, fig. 148, the method to be employed is somewhat similar:—Draw the diagrams as on figs. 148 and 120. Draw the arcs BE, EF, &c., of fig. 120, until the vertical through K is reached, and the arcs CH, HG, &c., also of fig. 120, until the vertical through H is reached, all with the parabolic segment used for drawing BA_1DA_2C of fig. 148, as for *Case* 1. Complete the figure by drawing the the arcs between the verticals through K and H with the same parabolic segment as BEC (fig. 148) is drawn with; here, $2OE = LN$, and $c = LS$, where c equals the half-span.

One case is of special interest, viz., that of case fig. 147, together with a fixed load W at F the junction of the fields. The effect of adding this weight is to make the apexes A_1 and A_2 move horizontally toward each other. Some particular value of W will make them come on the same vertical when it is evident that they must coincide; any value of W greater still will cause them to pass each other as on fig. 116; and it can easily be shown that the value of W which makes them coincide is

$$W = \frac{h_1 + h_2}{c} R.$$

Hence, if on fig. 147, besides the load shown on the beam, there be added at F the junction of the fields, a weight which is the same fraction of R that the distance between the wheels is of the span, then BA_0C is the bending moment diagram, just as if R were concentrated on one wheel as a rolling load. And conversely, if R be concentrated as a rolling load, and at the same time the weight above-mentioned act *upwards* at F then BA_1DA_2C is the bending moment diagram for the combined load; that is, we may construct a bending moment diagram for case fig. 147 by drawing the parabola BA_0C, and above the same base a triangle corresponding to the *upward* weight at F; the distances between the triangle and parabola give the bending moments, but there is the disadvantage of having a figure which consists of *two* irregular outlines, and which fails to show directly the positions of the maxima A_1 and A_2.

Beam under a travelling load concentrated on two wheels, together with an uniform dead load.—This divides into the same cases as for the travelling load alone (figs. 147 and 148).

The locus BDC still has D on the vertical through F, but the modulus of the curves is the sum of the moduli for the loads separately, that is, for the uniform load and the rolling load; the abscissa of each apex is the same fraction of the abscissa of the corresponding apex for the rolling load alone, OS_1 and OS_2, figs. 147 and 148, that the modulus is of the sum of the moduli (see *Theorem B*, fig. 94).

Sum of moduli $= \dfrac{U + 2R}{4c}$ and the fraction mentioned above is

$$\left(\frac{R}{2c}\right) \div \left(\frac{R}{2c} + \frac{U}{4c}\right) = \frac{2R}{U + 2R};$$

hence the equations to the maxima bending moments are

$$ {}_1M_x = \frac{U+2R}{4c}\left(c + x - \frac{2R}{U+2R}\,.\,2h_1\right)(c-x) \text{ for first field ;} \tag{1}$$

$$ {}_2M_x = \frac{U+2R}{4c}\left(c - x + \frac{2R}{U+2R}\,.\,2h_2\right)(c+x) \text{ for second field.} \tag{2}$$

Such portions of the fields as were bounded by the chord instead of the arc are still so bounded. In the case where for the transit *kEh* superseded *BDC* (fig. 148), the same central portion is now bounded by the locus

$$M'_x = \frac{U + 2W_2}{4c}(c^2 - x^2). \tag{3}$$

Graphical Solution.—For an uniform load U combined with any of the cases, figs. 141, 142, 143, 147, and 148, the graphical solution is the same in all particulars, case for case, except that OS_1 and OS_2 are now to be laid off a $\frac{2R}{U+2R}$th part of what they were laid off for the travelling load alone; the vertical scale is to be constructed so that

$$OA_0 = \tfrac{1}{8}(U + 2R)\,.\,l,$$

where R, as previously stated, is the dead load equivalent to the actual rolling load.

EXAMPLES.

150. An advancing load, as long or longer than the span, and of intensity $1\frac{1}{2}$ tons per foot, comes upon a beam 36 feet long. Find the maximum bending moment for the whole span for all positions of the load

$$\text{max.}M_0 = \tfrac{1}{8}W\,.\,l = \tfrac{1}{8}54 \times 36 = 243 \text{ ft.-tons.}$$

Find the maxima at intervals of six feet

$$\text{max.}M_x = \frac{W}{4c}(c^2 - x^2) = \tfrac{3}{4}(18^2 - x^2)\,;$$

therefore

$$M_{\pm 18} = 0, \quad M_{\pm 12} = 135, \quad M_{\pm 6} = 216, \quad \text{and} \quad M_0 = 243 \text{ ft.-tons,}$$

all maxima.

151. For the same beam of 36 feet span, with the same intensity of load $1\frac{1}{2}$ tons per foot, the whole length of the load being now only 12 feet. Find the maximum for the whole span

$$\text{max.}M_0 = \tfrac{1}{4}W(l - k) = \tfrac{1}{4} \times 18 \times (36 - 6) = 135 \text{ ft.-tons.}$$

Find now the maxima at intervals of 9 feet

$$\text{max.}M_x = \frac{W}{4c}(c^2 - x^2)\left(2 - \frac{k}{c}\right) = \tfrac{5}{12}(324 - x^2)\,;$$

and

$$M_{18} = 0, \quad M_9 = 101\tfrac{1}{4}, \quad M_0 = 135 \text{ ft.-tons,}$$

all maxima.

152. In Example 151, find $\max.M_9$ by placing the load in its proper position upon the span, and taking a section at $x = 9$.

Nine feet to the left of the centre is one-fourth of the span to the left of the centre; to give the maximum bending moment, the load is to be placed so that the point which is one-fourth of the extent of the load to the left of its centre shall be over the point. That is, the point of the load 3 ft. to the left of G is to be over the point of the span 9 feet to the left of C; hence G is 6 feet to the left of C, and

$$P = \tfrac{18}{36} \times 24 = 12 \text{ tons.}$$

Taking a section at $x = 9$, we have upon its left a load area extending for 3 feet and equivalent to $4\tfrac{1}{2}$ tons; hence

$$\max.M_9 = 12 \times 9 - 4\tfrac{1}{2} \times 1\tfrac{1}{2} = 101\tfrac{1}{4} \text{ ft.-tons.}$$

153. Find the principal equation to the parabola which is the curve of maxima bending moments in example 151.

Ans. $Y = \dfrac{W}{4c}\left(2 - \dfrac{k}{c}\right) X^2 = \tfrac{5}{12} X^2.$

154. A beam 42 feet span is subject to an advancing load of 3 tons per foot and 12 feet long. Find the maximum bending moment at 7 feet on either side of the centre. Where is the centre of the load situated when this maximum is produced?

Ans. $\max.M_{\pm 7} = 288$ ft.-tons. Five feet from centre of span.

155. In the previous example, find the maximum bending moment for the whole span, for all positions of the above load.

Ans. $\max.M_0 = 324$ ft.-tons.

156. A beam 30 feet span is subject to a rolling load of 40 tons; find the maximum bending moment for whole span. At what point does it occur, and how is the load then situated?

Ans. $\max.M_0 = \tfrac{1}{4}R \,.\, l = 300$ ft.-tons. R is at centre.

157. In the previous example, find the maxima bending moments at intervals of 5 feet. How must the load be situated in each case?

Ans. $\max.M_x = \dfrac{R}{2c}(c^2 - x^2) = \tfrac{4}{3}(225 - x^2);$

therefore

$$M_{\pm 15} = 0, \quad M_{\pm 10} = 166\tfrac{2}{3}, \quad M_{\pm 5} = 266\tfrac{2}{3}, \quad \text{and} \quad M_0 = 300 \text{ ft.-tons,}$$

all maxima. The above load of 40 tons in each case is over the point.

158. If the load of 40 tons in the above examples be spread uniformly over 3 inches, instead of being concentrated at a point, how much are the above results in error?

$$\max.M_x = \frac{R}{4c}(c^2 - x^2)\left(2 - \frac{\text{extent of load}}{\text{extent of span}}\right):$$

this differs from the above expression by the factor

$$\tfrac{1}{2}\left(2 - \frac{2k}{l}\right) = \tfrac{1}{2}\left(2 - \frac{\cdot 25}{30}\right) = \frac{239}{240},$$

hence the results above would be in excess by a $\tfrac{1}{240}$th part, or by $\tfrac{5}{12}$ths per cent.

159. A beam 36 feet span, bears a travelling load of 18 tons concentrated on two wheels 12 feet apart, there being 12 tons on the left and 6 tons on the right wheel. Find the maximum bending moment; the equations to, and amounts at intervals of 6 feet of, the maxima bending moments.

Data: $W_1 = 12$, $W_2 = 6$, $R = 18$ tons, $c = 18$, $2h_1 = -4$, $2h_2 = 8$ feet.

Dividing 36 directly as 1 and 2, we have BF and FC, fields 1 and 2, equal respectively to 24 and 12. Since the distance between the wheels is not greater than the shorter field, the example comes under Case I. (see fig. 147), and the maximum bending moment at each point is the locus BDC, whether the load is confined to the span or makes a transit.

The maximum bending moment for whole span occurs at $x = h_1 = -2$; its value is to be found by supposing the load standing with the left wheel two feet to the left of the centre as on the model (fig. 149) to which the text now refers, and then calculating the moment at that point,

$$P = 8 \text{ tons, and } {}_1M_{-2} = 128 \text{ ft.-tons max.}$$

The equations to the maximum bending moment at each point are,

For field 1, ${}_1M_x = \frac{1}{2}(14 - x)(18 + x)$, for values of x from -18 to 6.

For field 2, ${}_2M_x = \frac{1}{2}(10 + x)(18 - x)$, for values of x from 6 to 18.

And evaluating at intervals of six feet, we have

$M_{-18} = 0$, $M_{-12} = 78$, $M_{-6} = 120$, $M_0 = 126$, $M_6 = 96$, $M_{12} = 66$ ft.-tons, &c.

Note.—The height of A_2 may be calculated from fig. 147 thus:—$OS_1 = 2$, the base of quadrant then is 16; the modulus is the load divided by span, that is $\frac{1}{2}$; hence the height of apex, or modulus into base squared, is 128.

To make the graphical solution, lay off $OS_1 = 2$, $OS_2 = 4$, and draw BDC as previously described; make a vertical scale upon which $OA_0 = \frac{1}{2} \times 18 \times 18 = 162$ (see model, fig. 151).

160. A beam, 36 feet span, bears a travelling load of 18 tons concentrated in equal portions on two wheels 12 feet apart. Find the maximum bending moment.

In this case, $W_1 = W_2$, and $4s < c$, so that (fig. 142) $OS_1 = OS_2 = s$. The maximum occurs at 3 feet on either side of the centre, whether the load is confined to the span like a travelling crane or makes a transit like a truck; its amount is found by assuming the load to be standing with the left wheel 3 feet to left of centre.

Ans. $P = 7\frac{1}{2}$ tons; ${}_1M_3 = {}_2M_{-3} = 112\frac{1}{2}$ ft.-tons, maximum for whole span. That is, half of the square of fifteen (see the model, fig. 151).

161. If the load of Ex. 160 shift till the left-wheel is 6 feet from the left end of the beam, calculate the bending moment 9 feet from the left end,

$$P = 12 \text{ tons, } M_9 = 12 \times 9 - 9 \times 3 = 81 \text{ ft.-tons.}$$

This is the position of the load on the model (fig. 151), and is the bending moment on the girder under a point 3 feet to the right of the left wheel. Also $\frac{3}{4}(6^2 - 3^2)$ or $20\frac{1}{4}$ is the bending moment at the same point on the wheel base or 12 foot beam, joining the two wheels due to the uniform load of $1\frac{1}{2}$ tons spread over it. These two added together give $81 + 20\frac{1}{4} = 101\frac{1}{4}$ ft.-tons. This is the maximum bending moment on the girder at the point 9 feet from the left end, if the uniform load rested directly upon it without the intervention of the two-wheeled trolly.

Compare this Example with the preceding, and with Exercises 151, 152.

162. A beam 56 feet span bears a travelling load of 16 tons concentrated on two wheels 32 feet apart, 7 tons being on the left wheel and 9 tons on the right. Find the maximum bending moment during the transit.

From equation 12 at fig. 148,

$$\frac{R - \sqrt{R \, . \, W_2}}{W_1} 2c = 32 \text{ feet.}$$

which happens to be the distance between the weights; hence there will be two equal maxima, one at 7 feet to the right of the centre when the greater load is over it; the other, at the centre also when the greater load is over it, the smaller load not being then on the span. Place the load in those positions respectively, and calculate the moments.

Ans. ${}_2M_{-7} = {}_2M'_0 = 126$ ft.-tons.

163. A travelling load of 5 tons concentrated on two wheels 10 feet apart, 1 ton being on the left wheel and 4 tons on the right, passes over a beam of 40 feet span. Find the maxima bending moments at intervals of 4 feet, and the maximum for the whole span.

Distance between weights × ratio of weights = 40 feet = span, so that k coincides with B (fig. 148), and therefore arc BhE lies everywhere above arc BD; that is, the maximum at each point of span occurs when the greater weight is over it. Further, the height of A_2 is greater than that of E, since the distance between the weights does not exceed $\frac{7}{12}$ths of the span. Placing the greater load over points at intervals of 4 feet and calculating the bending moments, or substituting into the equations to the loci Bh and hA_2C, we have

$${}_2M'_{20} = 0, \quad {}_2M'_{16} = 14 \cdot 4, \quad {}_2M'_{12} = 25 \cdot 6; \quad {}_2M_8 = 35, \quad {}_2M_4 = 42, \quad {}_2M_0 = 45,$$

$${}_2M_{-4} = 44, \quad {}_2M_{-8} = 39, \quad {}_2M_{-12} = 30, \quad {}_2M_{-16} = 17, \quad {}_2M_{-20} = 0 \text{ ft.-tons.}$$

Since the greater weight lies 2 feet to the right of the centre of gravity of the load, the maximum lies at 1 foot to the right of the centre of span, and

$${}_2M_{-1} = 45\tfrac{1}{8} \text{ ft.-tons maximum for whole span during transit.}$$

The equations to the loci from which these may be calculated are for Bh,

$${}_2M'_x = \tfrac{4}{40}(c^2 - x^2) = \tfrac{1}{10}(400 - x^2),$$

for values of x from 20 to 10; and for hA_2C,

$${}_2M_x = \tfrac{5}{40}(c + 2h_2 - x)(c + x) = \tfrac{1}{8}(18 - x)(20 + x)$$

for values of x from 10 to $-$ 20.

CHAPTER XI.

MOVING MODEL ILLUSTRATING THE BENDING MOMENTS AND SHEARING FORCES ON A GIRDER BRIDGE DUE TO A LOCOMOTIVE PASSING OVER IT.

A MODEL appeals to some minds to which analysis and descriptive geometry are tiresome.

The model consists of two boards about 18 inches square, with a girder a foot long at the top of each. The rails are directly over the girders, so that the boards are spaced about 2 inches apart, the gauge of the rails (fig. 149). The locomotive is supposed to weigh 36 tons, and it has three wheels, the two wheels on the near side, axles 12 feet apart, ride on the near girder and transmit to it a total load of 18 tons, 12 by the leading and 6 by the trailing wheel. The axle of the single wheel on the remote side is at the centre of gravity, being 4 feet from the leading and 8 feet from the trailing axle, and rides on the remote girder, transmitting 18 tons directly to it.

On the face of the model are three hands which turn upon pivots as the locomotive is pushed along the bridge. The side hands are pivoted at the ends of a horizontal line representing the span of the girder, 36 feet to a scale of 3 feet to an inch. This line is the base of the bending moment diagram. The central hand is pivoted directly over the point lying 6 feet to the right of the centre of the span, the point dividing the span in the ratio 2 to 1, just as the two wheels share the load. The height of the pivot is $5\frac{1}{3}$ inches.

As the locomotive is shoved along the bridge, the hands turn so that their intersections are always directly under the wheels. At the same time the pole, at the extreme left of the model, moves up and down, keeping the three threads attached to it always parallel to the three hands, one to each. The other ends of the threads pass through three eyelet holes in the face of the model, and little weights are attached to them to take up the slack out of sight. The eyelet holes are ranged on a vertical line, which is the load line to a scale of 3 tons to an inch. The polar distance is 6 feet, the scale for bending moments is consequently 18 foot-tons to an inch.

While the locomotive stands still, the three hands form a polygon of three sides standing upon the base. This polygon is the instantaneous bending moment diagram. It also is a balanced frame for the loads fixed in that position, and the threads from the pole form the reciprocal figure giving the thrusts along its sides.

Between the boards a pantagraph is pivoted loose on the pivot of the central hand. One point of the pantagraph is driven by the locomotive, and therefore the middle point of the pantagraph describes a horizontal path, with half the travel of the locomotive. From it a thread passing over a pulley makes the pole move up and down with half the speed of the locomotive. From the central point of the pantagraph *three wires* radiate out—the middle one slides in a tube fixed at right angles on the pivot of the central hand, and rocks the pivot about; the other two wires in the same way rock two pivots (not seen on the woodcut), which are placed horizontally right and left of the central one, just as the end eyelet-holes are placed above and below the central one. These end pivots transmit the rocking motion to the pivots of the end hands by means of cranks and connecting rods.

The mechanism will be readily understood by supposing the threads from the poles to be wires transpiercing journals at the eyelet-holes, and causing them to rock as the pole goes up and down ; and then supposing these geared to the pivots of the hands, one to each, by pulleys and endless bands between the boards, thus compelling each hand to be always parallel to the corresponding thread.

The intersection of the hands below the leading wheel sweeps out a parabola which is painted on the face of the model. It is at the vertex for the position of the locomotive shown on the woodcut. The leading wheel being 16 feet from the left abutment, the centre of gravity is 16 feet from the right abutment. Now the supporting force at the left end is proportional to the second mentioned 16, being in fact *one-half* of it, this being the ratio of total load to span. And the lever of the left supporting force to cause bending on the girder under the leading wheel is the first mentioned 16 feet. So that the bending moment there, for the position of locomotive shown, is

$$M_{-2} = \tfrac{1}{2} \times 16 \times 16 = 128 \text{ foot-tons, a max.}$$

(Compare Example No. 159, p. 210.)

Should the locomotive move one foot right or left, one of the factors 16 becomes 15, and the other 17, giving a lesser product. Should it move two feet right or left of the position shown on woodcut, the factors become 14 and 18, with a still smaller product by a proposition of Euclid. Hence the locus of the intersection of left and central hands is a parabolic right segment, half base 16 feet, and passing through the pivots of those two hands.

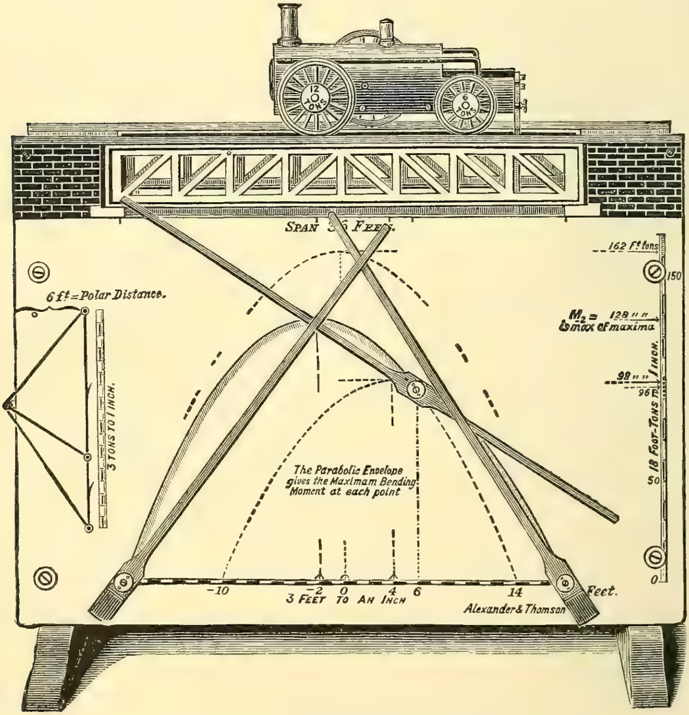


Fig. 149.

When the trailing wheel is 14 feet from the right abutment, the centre of gravity is 14 feet from the left abutment, and the intersection of the central and right hands is at a height,

$$M_4 = \tfrac{1}{2} \times 14 \times 14 = 98 \text{ foot-tons, a max.,}$$

and is at the vertex of its path, which is again a parabolic right segment, half base 14 feet, modulus *one-half*, and passing through the pivots of those two hands. The position of the pivot of the central hand is then on the intersection of those two parabolas. The height of that pivot is readily found thus: Observe that the perpendicular dropped from the pole upon the load line divides it into two segments which are the supporting forces. If the locomotive be moved till the central hand is horizontal, then the central thread being horizontal is perpendicular to the load line, and so we have the left supporting

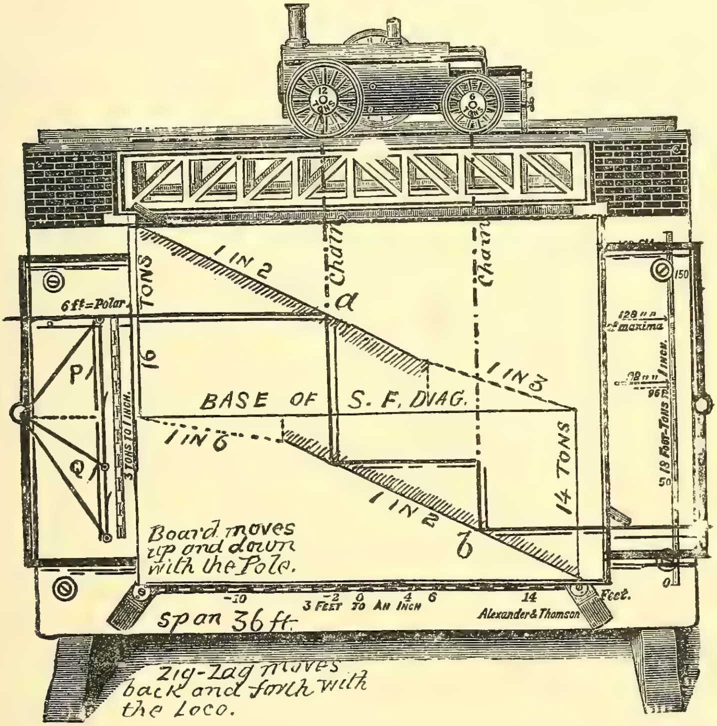


Fig. 150.

force 12 tons, two-thirds of the whole load; hence the centre of gravity of the locomotive must now trisect the span, and be 12 feet from the left abutment. The leading wheel then will be 8 feet from left abutment, and the bending moment under it is 12 × 8, or 96 foot-tons. But the central hand being horizontal, this also is the height of its pivot, namely, $5\frac{1}{3}$ inches.

On the other face of the model not shown there is no central hand nor thread, and the triangle formed over the base with the two end hands is the instantaneous bending moment diagram for the remote girder, upon which only a single load, 18 tons, rolls. This intersection sweeps out a parabolic right segment, having 18 feet for its half base, with the same modulus as before, namely, *one-half*; it passes through the pivots of the end hands, and its vertex is over the centre of span.

The fact that the polar distance remains *constant* shows that the central parabola (giving the bending moments on the remote girder due to the 18 tons rolling on one wheel) and the pair of parabolas (giving the bending moments for the near girder) constitute a pair of diagrams to one common scale. Of course the scale for the single central parabola, which alone appears on the back face of the model, is readily constructed. The scale must be such that the height of the vertex of the central parabola shall measure on it $\frac{1}{4}WL = 162$ foot-tons. Otherwise, as already stated, the polar distance being 6 feet makes the scale for bending moment 18 foot-tons to the inch, that is, 6 times finer than the scale for tons.

The height to the central parabola at a point 6 feet to one side of the centre is *one-half* of the product of the two segments into which that point divides the span or it is $\frac{1}{2} \times 24 \times 12 = 144$ foot-tons. But the height of the central pivot is 96 foot-tons, so that the depth of the central pivot from the central parabola is 48 foot-tons. Now 48 foot-tons is the statical moment or product of 12 tons, the leading wheel into 4 feet its distance from the common centre of gravity.

Another way then of stating the position of the central pivot is to say that the vertical through it must divide the span in the ratio 2 to 1 in which the leading and trailing wheels share the total load, and that it must lie on this vertical at a depth below the central parabola by an amount given by the statical moment of either wheel about their common centre of gravity.

The construction follows by induction for a locomotive with many wheels. That is, the angles of the polygonal instantaneous bending moment diagram for the locomotive at rest, sweep out, when it moves, a series of parabolic right segments with a common modulus, viz. the ratio of the total load to the span. They intersect in pairs at the pivots on vertical lines, which divide the span into segments, having the same ratio to the span that the loads on the wheels have to the total load. The vertices of the parabolas are over points on the span, right and left of its centre, in the same way, but by half the absolute amounts that the wheels are right and left of the centre of gravity of the locomotive. They can readily be drawn with one parabolic template, the scale being constructed so that the height of the parabolic segment, drawn with that template, and standing on the span, shall be *a fourth of the product of the total load and the span.*

A still simpler process is to draw semicircles instead of parabolic right segments. The locus will then be a diagram of the square roots of maximum bending moments, and the scale must be constructed so that the height of the semicircle standing on the span shall be the square root of a fourth of the product of the total load and the span. In the next chapter these diagrams will be shown drawn to scale.

To exhibit the shearing force diagram, a board is put on front being supported by the pole and a point on the opposite side moving with it. This board (fig. 150) moves up and down with the pole at half the rate at which the locomotive moves back and forth. From the centres of the two wheels a zigzag piece of wire hangs by two light chains. On the face of the board is painted the locus of *a* the point where the zigzag drops by 12 tons, and also the locus of *b* where it has dropped 6 tons. The slope of these loci is 1 in 2.

The model is made by Messrs. Yeates and Son, Grafton-street, Dublin, and may be seen in Dublin at the Engineering School, Trinity College, or at the College of Science, Poona.

The description of this model is published here for the first time, except that the figure appeared on the correspondence in Mr. Farr's paper on "Moving Loads on Railway Under-Bridges" in the Transactions of the Institute of Civil Engineers, 1900.

A model of ruder construction, with sliding frames, and called "A Bending Moment Delineator," is described in the Transactions of the Engineers and Ship Builders of Scotland, November, 1889, and was exhibited at the Munich Exhibition in 1893, and described in the *Katalog.*

We add here a photograph of the "Delineator" from a large black-board model (fig. 151). It will be seen that the three arms are constrained by slots to move exactly as the hands on the model (fig. 149). The loads on the wheels are equal, being 9 tons each, and are due to a uniform load of 1½ tons per foot-spread on the wheel base. The bending moments on the wheel base itself is shown by the parabolic segment painted on a "Distorting Table," which consists of a number of vertical strips of wood slightly spaced, and connected to each other by two horizontal lazytongs across their backs, each strip being pinned to a joint of the upper and under lazytongs. This distorting table has holes, one at each corner, which slip over four pegs connected to the cross-arm which bears the two knob-handles at the ends. If the distorting table be pulled forward off the pins it can then be stretched like an accordion, but, when on the pins, it can only be subjected to *pure distortion.*

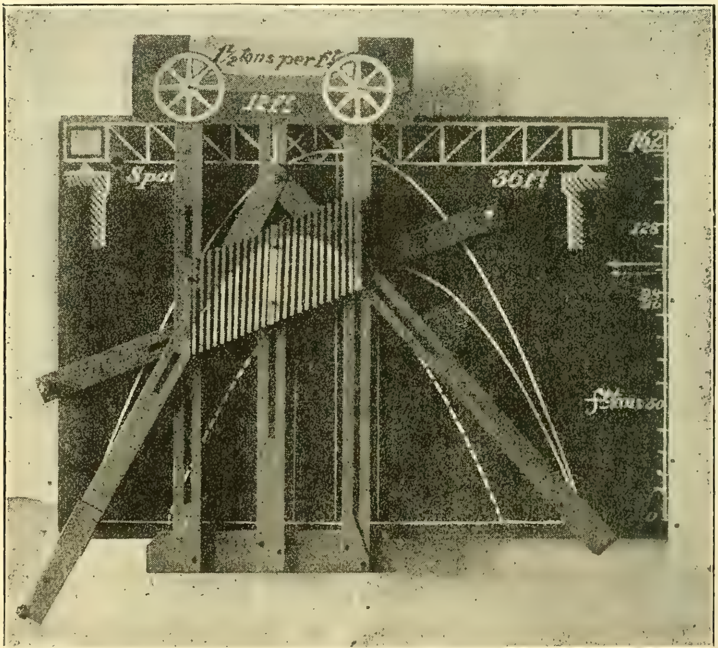


Fig. 151.

In the middle position the figure painted on the distorting table is a *right* parabolic segment; but, when the locomotive is shoved to one side as shown on the photograph, it is oblique, but of course the lengths of the white strips are unaltered. The two oblique slotted bars are always tangents at the ends of the parabolic segment.

If the uniform load drop directly down upon the girder, and move about on it without the intervention of the trolly, then the instantaneous bending moment, as shown on the photograph, is the parabolic locus for the loaded central segment of the span and the tangents to it for the two side unloaded segments. Compare fig. 124, p. 176, which this model illustrates when the load is at rest.

In the photograph the load is so disposed on the girder that the left-hand quartering point of the load is directly over the left-hand quartering point of the span. Consider the bending moment at the left quartering point of the span for this position of the load. It is given by the height above the base of the top of the left quartering white stripe painted on the distorting table. We know that this is the maximum bending moment at this point of the span (see figs. 135 and 136). Now the model exhibits this beautifully; for if we move the load a little to the right, the "white strip" is replaced by one shorter than itself for which reason the top is *lower*, but the base of the parabolic segment has *risen* for the cross-bar on which it stands has rotated clock-wise about a pivot under the centre of the girder which exactly neutralises the effect of the shorter "white strip" replacing the original one. That is for a small motion of the load to the right-hand the bending moment at the quartering point of the span (as given by the new instantaneous diagram) is unaltered. Similarly it is unaltered for a slight motion of the load in the other direction. At that point, then, the *variation* of the bending moment is zero for a small movement which is the criterion for a maximum.

This "distorting table" also serves to illustrate the addition of the parabolic slope to the straight slope (see theorems in Chapter VII., p. 131).

It also serves to show how the shearing force diagram painted on it for any moving load is to be distorted and superimposed upon the shearing force diagram for the dead load. It can also illustrate many points in pure lineal strain and pure distortion.

CHAPTER XII.

BENDING MOMENTS AND SHEARING FORCES DUE TO A TRAVELLING LOAD SYSTEM.

Bending Moments for a beam under a travelling load system of unequal weights fixed at irregular intervals, the load being confined to the span so that no weight passes off (fig. 152).—Let R be the total load; W_1, W_2, . . . W_r . . . W_n, the weights numbered in order from the left end; G the centre of gravity and origin for the weights; $2h_1$, $2h_2$ . . . $2h_r$ . . . $2h_n$, the abscissæ of the

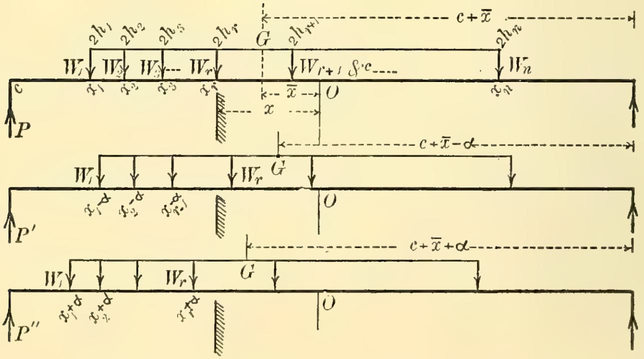


Fig. 152.

weights, those to the right of G including their negative sign; O the origin for, and $2c$ the length of, the span, distances towards the left being positive; and let x be the abscissa of any section.

First.—Let the load be in a position such that the r^{th} weight is over the section; x_1, x_2 . . . x_{r-1}, the abscissæ of the $(r-1)$

weights, measured from O; $\bar{x}$ the abscissa of G, and c that of the left end of span; then

$$P = \frac{R}{2c}(c + \bar{x}),$$

and

$$_rM_x = \frac{R}{2c}(c + \bar{x})(c - x) - W_1(x_1 - x) - W_2(x_2 - x)$$
$$- \ldots - W_{r-1}(x_{r-1} - x) - W_r(x_r - x); \quad (1)$$

the last term is zero, since $x_r = x$, and it may either be expressed as above or omitted.

Second.—Let the load be situated at a short distance a to the right of its former position; then

$$P' = \frac{R}{2c}(c + \bar{x} - a);$$

and

$$M'_x = \frac{R}{2c}(c + \bar{x} - a)(c - x) - W_1(x_1 - a - x) - W_2(x_2 - a - x)$$
$$- \ldots - W_{r-1}(x_{r-1} - a - x)$$
$$= {}_rM_x - \frac{R}{2c}(c - x)a + (W_1 + W_2 + \ldots + W_{r-1})a. \quad (2)$$

$$< {}_rM_x, \text{ if } \frac{R}{2c}(c - x) > \Sigma_1^{r-1}(W), \text{ or if } c - \frac{\Sigma_1^{r-1}(W)}{R}.2c > x;$$

that is, the bending moment for the first position is the greater, if the distance of the section to the left of the centre is less than $c - \frac{\Sigma_1^{r-1}(W)}{R}.2c$; or what is the same thing, if the distance of the section from the left end is greater than $\frac{\Sigma_1^{r-1}(W)}{R}.2c$.

Third.—Let the load be situated at a short distance a to the left of the first position; then

$$P'' = \frac{R}{2c}(c + \bar{x} + a),$$

and

$$M''_x = \frac{R}{2c}(c + \bar{x} + a)(c - x) - W_1(x_1 + a - x) - W_2(x_2 + a - x)$$
$$- \ldots - W_r(x_r + a - x)$$
$$= {}_rM_x + \frac{R}{2c}(c - x)a - (W_1 + W_2 + \ldots + W)a. \quad (3)$$

$$< {}_rM_x, \text{ if } \frac{R}{2c}(c - x) < \Sigma_1^{r}(W), \text{ or if } c - \frac{\Sigma_1^{r}(W)}{R}.2c < x;$$

that is, the bending moment for the first position is greater than for the third, if the distance of the section to the left of the centre is greater than $c - \frac{\Sigma_1{}^r(W)}{R} . 2c$; or, what is the same thing, if the distance of the section from the left end is less than $\frac{\Sigma_1{}^r(W)}{R} . 2c$.

Hence there is a portion of the span, lying between the point whose distance from the left end is $\frac{\Sigma_1{}^r(W)}{R} . 2c$, and the point whose distance from the left end is $\frac{\Sigma_1{}^{r-1}(W)}{R} . 2c$, such that the bending moment at any section in that portion is greater when the r^{th} weight is over it than for any other position of the load, if for each position all the weights are on the span as premised. This portion of the span we call the r^{th} *field*, and we say it is *commanded* by the r^{th} weight. The extent of this r^{th} field is

$$\frac{\Sigma_1{}^r(W)}{R} . 2c - \frac{\Sigma_1{}^{r-1}(W)}{R} . 2c = \frac{W_r}{R} . 2c, \tag{4}$$

the same fraction of the span as the r^{th} weight is of the total load; hence, in order to mark the fields, the span is to be divided into as many portions as there are weights, these portions being proportional to the weights and in the same order. The maximum bending moment at any point occurs when the weight, which commands the field in which the point lies, comes over that point. Since no weight is to go off the span, sometimes there is a part of a field which the commanding weight cannot occupy, and then the weight is said not to be able to *overtake* that part of its field; as will be proved when we come to the graphical solution, the maximum bending moment for such points occurs when the commanding weight is as close thereto as it can be brought.

Into the expression for ${}_rM_x$ the maximum at any point of the r^{th} field, substitute as follows:—

$$\bar{x} = (x - 2h_r), \quad (x_1 - x) = (2h_1 - 2h_r), \quad (x_2 - x) = (2h_2 - 2h_r), \text{ \&c.,}$$

$$\ldots (x_{r-1} - x) = (2h_{r-1} - 2h_r);$$

and we have

$$
\begin{aligned}
{}_rM_x &= \frac{R}{2c}(c + x - 2h_r)(c - x) - W_1(2h_1 - 2h_r) - W_2(2h_2 - 2h_r) \\
&\qquad - \ldots - W_{r-1}(2h_{r-1} - 2h_r) \\
&= \frac{R}{2c}(c + x - 2h_r)(c - x) + 2h_r\,\Sigma_1^{r-1}(W) - \Sigma_1^{r-1}(W.\,2h); \quad (5)
\end{aligned}
$$

this is the equation to the maxima bending moments for the r^{th} field. The locus is a parabola; its axis is vertical; its apex is above the span, and may lie to either side of O, the centre of span; its modulus is $\frac{R}{2c}$, a quantity which is the same for all fields, and is the same as the modulus of the parabola due to R as a rolling load. The abscissa of A_r, the apex of the r^{th} parabola, is that value of x which makes ${}_rM_x$ or $(c + x - 2h_r)(c - x)$ greatest, that is where

$$x = h_r; \qquad (6)$$

hence the apex of each parabola lies on the same side of, and horizontally half as far from, the centre of the span as the commanding weight is from G, the centre of gravity of the load. The apexes for some of the fields may lie in their own fields; and in such fields the maximum of the maxima is given by the ordinate of the apex; for other fields, the apexes may lie outside of their own fields, and in these there is no maximum of maxima, but the maxima increase continuously from one end of the field to the other.

Bending Moment Diagram (fig. 153).—The locus is the polygon BD_1D_2, &c., formed with parabolic arcs BD_1, D_1D_2, &c.; each parabola being the same as BA_0C that for the rolling load R, but lying with their apexes at the distances $OS_1 = h$, $OS_2 = h_2$, &c., where h_1, h_2, &c., are half the respective distances of W_1, W_2, &c., from G the centre of gravity of the load. The parabolas intersect in pairs on the verticals through the junctions of the fields; that is, through F_1, F_2, &c., points such that

$$BF_1 : F_1F_2, \text{ \&c.} : BC :: W_1 : W_2, \text{ \&c.} : R;$$

for, if F_r be the junction between the r^{th} field and the $(r + 1)^{th}$ field, then F_r is the last point in the r^{th} field; the maximum at

F_r occurs when W_r is over it and is given by the ordinate at F_r to the r^{th} parabola. Again, F_r is the first point in the $(r+1)^{th}$ field; the maximum at F_r occurs when W_{r+1} is over it and is given by the ordinate at F_r to the $(r+1)^{th}$ parabola. But the maximum at F_r is some one definite quantity; hence the two parabolas have a common ordinate at F_r.

Maximum Bending Moment for whole span.—In fig. 153, A_3 and A_4 lie respectively in their own fields; and the one which has the greater ordinate gives the maximum for whole span. In the general case one or more apexes will lie in their own fields; one at least, as we cannot conceive of such a series of curves lying so that every apex is inside of another of the curves. The ordinate of the apex that lies in its own field, if only one is so situated, is maximum for the whole span; the ordinate of the highest apex, if more than one be so situated.

Suppose the first curve continued to the right, past D_1; the second past D_2, &c.; then each parabola is the locus of the bending moment at each point as the corresponding weight comes over it, the whole load being on the span. If the load stands still in any position, as for instance that in which the load is drawn in fig. 153, the bending moment at the point where W_1 stands is the ordinate there of the first parabola, at the point where W_2 stands the ordinate of the second parabola, &c. If then the ordinates at these points be drawn each to the proper parabola, and the tops of the ordinates be joined, we will have the bending moment diagram for that set of fixed loads; we will have in fact the diagram shown in fig. 100, because the load is now the fixed load shown in fig. 79.

Further, for the position of the load shown in fig. 153, the straight line joining the tops of the two ordinates, one drawn to parabola 2 from the point where W_2 stands, and the other to parabola 3 from the point where W_3 stands, will pass through D_2 the intersection of these parabolas, because the horizontal projection of that joining line is constant, being equal to the distance between W_2 and W_3; and we know that one end of this joining line coincides with D_2 when either of the weights W_2 or W_3 is over F_2; hence by the theorem (fig. 95) it will always pass through D_2. Now, for the position of the load shown, the joining line gives the bending moments at all intermediate points, so that the ordinate of D_2 is the bending moment at F_2 for that position of the load; and similarly for any other position for which F_2 lies between the weights W_2 and W_3. In other words, the bending moment at F the junction of two fields is the same, whether the weight commanding the field on either side is over it, or whether the load stands in any position for which F lies between those weights. When the load stands in a position where those two weights are both to one side of F, then the joining line *produced* still passes through D, but the ordinates of the produced line are not bending moments. For instance, the line joining the tops of the ordinates on fig. 153, one drawn to parabola 4 from the point where W_4 stands, and the other to parabola 5 from the point where W_5 stands, will, when produced, pass through D_4, the junction of those parabolas; however, not the ordinate at F_4 to that produced line, but the ordinate to the line which terminates at C, gives the bending moment there.

On fig. 153, observe that if the load moves more than one foot to the left, W_1 goes off the span; and there is a portion of field 3 at its left end, which W_3 cannot overtake, and for which the corresponding portion of parabola 3 is dotted, being inadmissible. For the position of the load shown, we saw that the bending moment at each point of that portion of field 3 is given by the ordinate to a straight line from D_2 to the top of the ordinate of parabola 3, at the point where W_3 is standing; that is, by the chord of parabola 3, from D_2 to the top of that ordinate. Now, the closer W_3 comes to F_2 the steeper will that chord be, and consequently the greater the bending moments at all these points. Hence the chord of the dotted or inadmissible part of parabola 3 gives the maximum bending moment at each point of the portion of field 3 which W_3 cannot overtake. Similarly for any portion of any field which the commanding weight cannot overtake, the chord of the parabola, instead of the arc, gives the maxima bending moments for the whole load on the span. Fig. 148 has already been quoted as an example of this, when we assumed the arc Bk, chord kD, chord Dh, and arc hC to be the diagram of maxima bending moments for load confined to the span.

Graphical Solution for Bending Moment Diagram (fig. 153).—Lay off the wheel base and find G the centre of gravity of the load either by analysis as at fig. 79, or graphically as in fig. 101 and indicated in fig. 153. Lay off BC equal to the span; divide it at the points F into fields proportional to the weights, either by arithmetic or as indicated on the diagram, and draw vertical lines through them to separate the fields. Lay off OS_1 equal to half the distance of W_1 from G, OS_2 equal to half the distance of W_2 from G, &c.—each point S being on that side of the centre O on which the corresponding weight lies with respect to G; draw verticals through the points S; apply the parallel

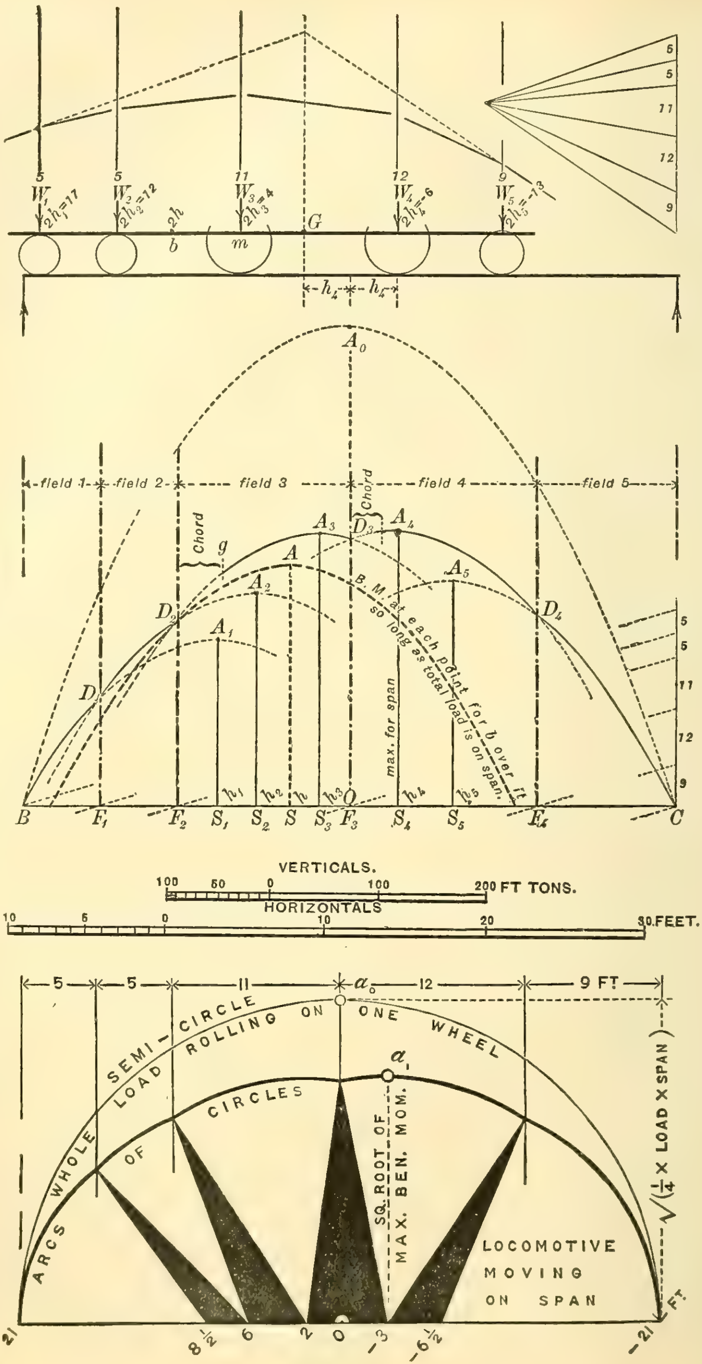

5
5
11
12
9
W_1
$2h_1=17$
W_2
$2h_2=12$
$2h$
b
W_3
$2h_3=4$
m
G
W_4
$2h_4=-6$
W_5
$2h_5=-13$
h_4
h_4
A_0
field 1
field 2
field 3
Chord
field 4
field 5
Chord
g
A_3
D_3
A_4
A
A_2
A_5
D_2
A_1
D_4
D_1
B. M. at each point for b over it so long as total load is on span.
max. for span
h_1
h_2
h
h_3
O
h_4
h_5
B
F_1
F_2
S_1
S_2
S
S_3
F_3
S_4
S_5
E_4
C
VERTICALS.
100
50
0
100
200 FT TONS.
HORIZONTALS
10
5
0
10
20
30 FEET.
5
5
11
a_0
12
9 FT
SEMI-CIRCLE
WHOLE LOAD ROLLING ON ONE WHEEL
a_1
ARCS OF CIRCLES
SQ. ROOT OF MAX. BEN. MOM.
LOCOMOTIVE MOVING ON SPAN
$\sqrt{(\frac{1}{4} \times \text{LOAD} \times \text{SPAN})}$ FT.
21
$8\frac{1}{2}$
6
2
0
-3
$-6\frac{1}{2}$
-21

Fig. 153.

rollers to BC; place any parabolic segment against the rollers with its apex on the vertical through S_1; shift the rollers till the curved edge passes through B, and draw the arc BD_1, stopping at the vertical through F_1; shift the segment till the apex is on the vertical through S_2, move the rollers till the curved edge passes through D_1, and draw the arc D_1D_2, stopping at the vertical through F_2. Similarly draw arc after arc in succession for each field, and if the arc for the last field passes through C the extremity of the span, it checks the accuracy of the drawing. Lastly, shift the segment till the apex is on the vertical through the centre; move the rollers till the curved edge passes through the two extremities, mark A_0 and construct a scale for verticals and bending moments such that $\overline{OA_0} = \frac{1}{4}R.l$. Find by inspection the portions of the fields which the commanding weights cannot overtake, and over such portions replace the arcs by chords. Then the locus gives the maximum bending moment at each point for all possible positions of the load, the load being confined to the span.

Graphical Solution for a Diagram of the Square Roots of Bending Moments.—Divide the span into "fields" proportional to the loads on the wheels. From the middle point of the span prick points at distances *one half* of those at which the wheels lie from the centre of gravity of the load. From those points as centres draw circular arcs one over each field and we have a diagram of the square roots of the maximum bending moments. It only remains to construct a scale. On the span draw a semicircle and construct a scale such that a_0, the height of the crown of the semicircle, shall measure on it the square root of a *fourth* of the product of the *total load* and *total span.*

The height of a_1 may be scaled off the drawing, and its value as compared to the known height of a_0 found directly by the rule of three without constructing a scale.*

Shearing Force Diagram (figs. 154 and 157).—At any point the shearing force increases as each weight in succession approaches from the right; and when a weight passes the point, it suddenly diminishes by an amount equal to that weight. At each point there is a maximum when a weight is just to the right of the point, and a minimum when it is just to the left.

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* This simple and elegant graphical solution of so complex a problem by means of circular arcs only was published in a note in the Transactions of the Institute of Civil Engineers for 1900.

In order to find for any point the maximum and minimum corresponding to a particular weight:—Place the load system so that this weight is over the point; from P subtract the weights to the left of that weight for the maximum, and further subtract that weight for the minimum. In figs. 154 and 157, a locus giving the maximum at each point due to a particular weight approaching, is shown by a full line; the locus giving the minimum due to the same weight is evidently parallel to the first, and below it at a constant distance equal to that weight; this parallel locus is shown by a dotted line. In each of these diagrams there are five loci drawn in full lines, and giving the maximum at each point due to the approach from the right of each of the five weights respectively; of the five dotted loci giving the minimum at each point due to the receding of each weight respectively, only one is shown, and it is drawn parallel to the locus shown by the lowest full line, and at a constant depth below it equal to the weight at the right end of the load. Having determined the full lines or positive loci, it is easy to draw the others.

The first locus Aa is the value of P when W_1 is over any point; so that as the load comes on from the right end, and so long as W_1 alone is on the span, Aa is a portion of the diagram due to W_1 as a rolling load; that is, it slopes at an angle whose tangent is $W_1 \div 2c$, or, in other words, at a rate proportional to W_1; further, Aa extends over a horizontal distance equal to that between W_1 and W_2. When W_2 comes over any point, W_1 and W_2 alone being on the span, then F_x is calculated by finding P and subtracting the constant quantity W_1; P is now increasing as for a rolling load $(W_1 + W_2)$ concentrated at its centre of gravity, so that the second link aa slopes at an angle whose tangent is $(W_1 + W_2) \div 2c$, or, in other words, at a rate proportional to $W_1 + W_2$; further, aa extends from the point where the preceding link ended, and continues through a horizontal distance equal to that between W_2 and W_3. Thus, the locus $Aaaa$ . . . begins at the right end of the span, each link sloping more and more at rates in direct proportion to the sum of the weights on the span, and extending respectively over horizontal distances equal to those between the weights; the *last* link extends over a distance which is the excess of the span over the extent of the load. On fig. 157, the first four links are short and equal, the last one is long. The other loci Bbb . . ., Ccc . . ., &c., consist of links sloping more and more as weights come on at the right end, and less and less as they go off at the left; for instance, Ccc . . ., fig. 157, consists of two equal short links and a long one increasing in slope, and two equal short ones decreasing in slope; the rate of slope of each link is directly proportional to the sum of the weights on the span at the time corresponding.

After having drawn the first locus *Aaa* . . ., the initial points *B*, *C*, *D*, &c., of the other loci are found as follows:—The ordinate of *B* represents the value of the shearing force at the right end of span when W_2 is just to the right of that point; if the load be placed in this position, the shearing force at the right end of span, and in the interval between that end and the point where W_1 stands, is constant; the value of the shearing force just to the left of W_1 is given by the ordinate of *a* the left extremity of the first link *Aa*, and this quantity diminished by W_1 is the value required. Hence *B* is on the vertical through the right end of span, and at a depth W_1 below the level of the first joint *a* on the locus *Aaa* . . .; similarly *C* is on the same vertical and at a depth W_2 below the level of the first joint *b* on the locus *Bbb* . . . .

Graphical Solution for Shearing Force Diagram (fig. 154).—The example shown in the figure is the same as that for which

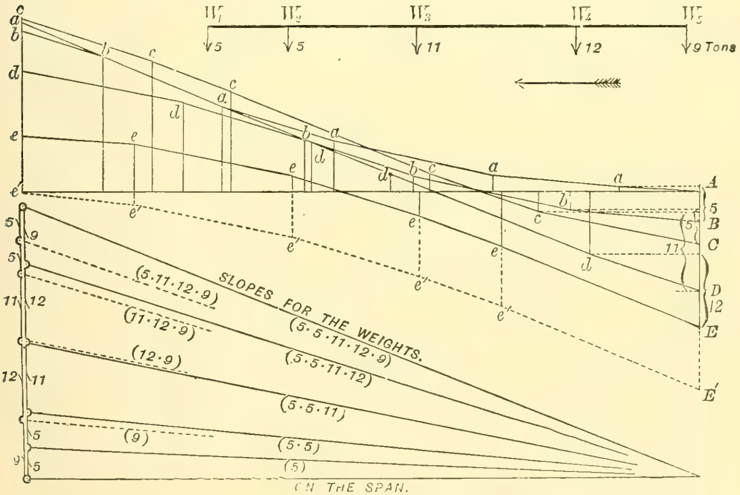

Fig. 154.

the bending moment diagram is given in fig. 153. Lay the weights up, in order, on the vertical through the left end of span, and then down again in order; draw a ray from the right end of span to each junction; rays to joints in ascending order are drawn in full lines, and to joints in descending order are drawn in dotted lines. Draw another line Ae' equal to the span, so as not to complicate the figure. Draw the locus Aaa ... link after link parallel to the slopes in ascending order, each link extending in order for a horizontal distance equal to that between the weights in pairs, the last link completing the locus; in the figure, the first link Aa extends for a distance equal to that between W_1 and W_2; the second link aa to that between W_2 and W_3, and so on; the last link extends for a distance equal to the difference between the span and the length of the load. Plot B on the vertical through the right end, at a depth equal to W_1 below the level of the first joint a in the locus already drawn; draw the locus Bbb ... parallel to the respective slopes in ascending order; the first link Bb is parallel to the slope for $(W_1 + W_2)$, each link extends till a new weight comes on, the second last link extends till W_1 goes off, and the last link is drawn parallel to the slope for $(W_2 + W_3 + W_4 + W_5)$. Plot C at a depth equal to W_2 below the level of the first joint b, and draw the locus Ccc ...; the first link Cc is parallel to the slope for $(W_1 + W_2 + W_3)$; the other links are drawn parallel respectively to the slopes in ascending order, and when these are exhausted the remaining links are drawn parallel to the slopes in descending order; the extent of each link is determined as each weight after W_3 comes on, and then as weight after weight goes off. In the same way, for each weight, a locus is drawn in full lines, consisting of as many links as there are weights; the highest ordinate at any point gives the maximum positive shearing force thereat, for transit of load. Dotted loci are drawn, one parallel to each locus shown by a full line, and below it at a distance equal to the weight to which it corresponds; the deepest ordinate at each point gives the negative maximum. On the diagram, $e'e'$ ... E' is drawn parallel to ee ... E; and since, in this case, the locus $e'e'$ ... E' gives the maximum shearing force for every point of span, the other four dotted loci are not shown.

Bending Moments for a Beam under a travelling load system of equal weights fixed at equal intervals and confined to the span (figs. 155, 156).—The locus $BD_1D_2D_3C$ drawn as in the general case, will be symmetrical about the centre. If the number of weights be n, and their distance apart be $\frac{1}{n}$th of the span, then it is evident that the span will be divided into n equal fields whose common extent is the same as the distance between two weights, and that each weight will just be able to overtake its field; if the distance between two weights be less than $\frac{1}{n}$th of the span, each weight is still able to overtake its own field. Therefore, for the common distance between the n weights equal to or less than $\frac{1}{n}$th of the span, the bending moment diagram is the locus $BD_1D_2D_3 \ldots C$ everywhere following the curves; the maximum bending moment is at the centre, or at one quarter of the common interval on either side of the centre, according as n is odd or even. On the other hand, when the common interval between the n weights is greater than $\frac{1}{n}$th of the span, each weight will always be in its own field, and will only be able to overtake a portion of its field; the bending moment diagram is the locus $BD_1D_2D_3 \ldots C$ following the arcs for portions of fields overtaken by the weights commanding, and the chords for the remainder. If n be odd, the middle weight can always be placed at the centre of the span, and at that point the maximum bending moment will occur; if n be even, and it be possible for a weight to come as close to the centre as or closer than a quarter of the common interval, then the maximum bending moment will be on both sides of the centre, and at one quarter of the common interval therefrom; if a weight cannot come so close, the maximum will still be on both sides of the centre, and at points as near thereto as the weight on either side of it may approach.

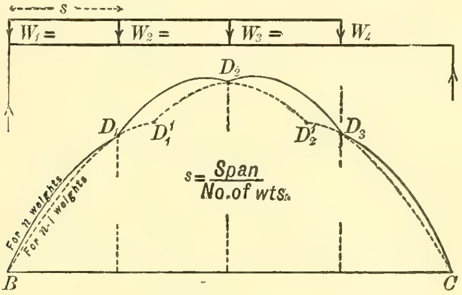

Fig. 155.

For n even, the proof that the maximum is at the point, a quarter of the common interval on either side of the centre may be shown thus:—Let G be over the centre then the weight nearest to the centre will be distant one-half of an interval; if the end weight be distant from the end at least a quarter of an interval, it will be possible for a weight to approach the centre, a quarter interval; hence, the span must equal the $\overline{n-1}$ equal intervals, and at least two quarter intervals more, that is

$$2c \geqq (n-1)\,s + \tfrac{1}{2}s\,; \quad 2c \geqq \frac{2n-1}{2}\,s.$$

Theorem.—If a system of n equal weights at a common interval not greater than $\frac{1}{n}$th of the span be confined to the span, then the locus of the maximum bending moment at each point will entirely include the loci due to any smaller number of the same equal weights at the same common interval.

First.—Let the common interval equal $\frac{1}{n}$th of the span. Place the load (fig. 155), with one weight over the support, then the other weights are exactly over the $n-1$ junctions of the fields. We may either consider that the n weights are on the span, or that $n-1$ weights only are on the span; hence the two loci, the one for n weights and the other for $n-1$, have common ordinates at the points where the weights are; that is, the locus for $n-1$ weights passes through the points D_1, D_2, D_3, &c., of, and lies entirely within, the locus for n weights.

On the figure, the locus drawn with full lines is that due to n weights, while the locus drawn with dotted lines is that due to $n-1$ weights.

Second.—Let the common interval decrease by 2δ. In the new locus for the n weights, each pair of consecutive apexes will be closer by the amount δ, so that every point in the locus will be higher than before; in the new locus for the $n-1$ weights, any two consecutive apexes will be closer by δ, so that each point in the locus will be higher than before. In the locus for the n weights, the parabolas have a greater modulus than those in the locus for the $n-1$ weights, so that the increase of height due to the approach of the apexes by δ in the one case, will be greater than that due to the approach of the apexes by the amount δ in the other; that is, if the distance between the weights be decreased, every point in the locus drawn with full lines (fig. 155), will rise more than the corresponding point in the dotted locus, so that much more will the new dotted locus be entirely within the other.

It follows, then, that if the common interval between n equal weights be equal to or less than $\frac{1}{n}$th of the span, or in other words, if each of the n weights can overtake its field, then the loci due to the n weights, to $n-1$ of the same weights at the same common interval, to $n-2$, &c., are, in order, each entirely within the one preceding it, and therefore all the others entirely within the first.

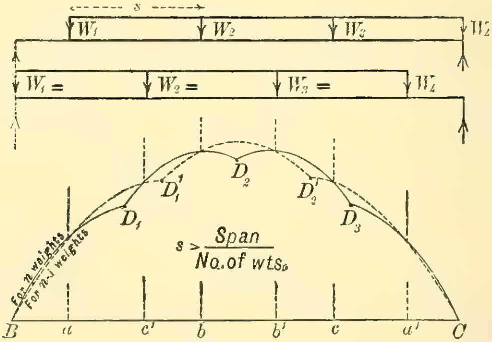

Fig. 156.

Cor.—If the interval between the n weights be greater than $\frac{1}{n}$th of the span, the locus due to these weights confined to the span will intersect that due to $n-1$ of the same equal weights, also confined to the span; for reasoning as before, when the common interval is increased, the locus drawn with full lines on fig. 155, is lowered more than the other. From fig. 156, it will be evident that the points of intersection a, b, c, a', b', c' are found by laying off the common intervals from each end of the span, and that these points are the junctions between the alternate arcs and chords on the diagram for the n loads confined to span. In order that the n weights may be all on the span at one time, the common interval must not be greater than $\frac{1}{n-1}$th of the span; so that, for the case of $n-1$ of these weights, each could overtake its field; and the loci for $n-2$ weights, $n-3$, &c., are all within the locus for $n-1$ weights shown by dotted lines on fig. 156.

Graphical Solution for Shearing Force Diagram (fig. 157).—Lay the weights up, in order, on the vertical through the left end of span, and draw a ray from the right end of span to each junction; each slope is evidently the shearing force diagram for a rolling load equal to one of the weights, the sum of two, the sum of three, &c., respectively. Draw verticals at intervals from the right end equal to the common interval between the weights, and the same in number; draw the first link *Aa* parallel to the slope for one weight, the next link *aa* parallel to the slope for two weights, &c., similarly draw the locus $e'e' \ldots E'$ below the span. The ordinates to $Aaa \ldots$ and $e'e' \ldots E'$ give respectively, the maximum shearing force positive and negative at each point, during the transit of the load system; and in this case it is not necessary to draw the other loci.

Beam under a travelling load system confined to the span, together with a uniform dead load.—In the bending moment diagram, the locus consists of the same number

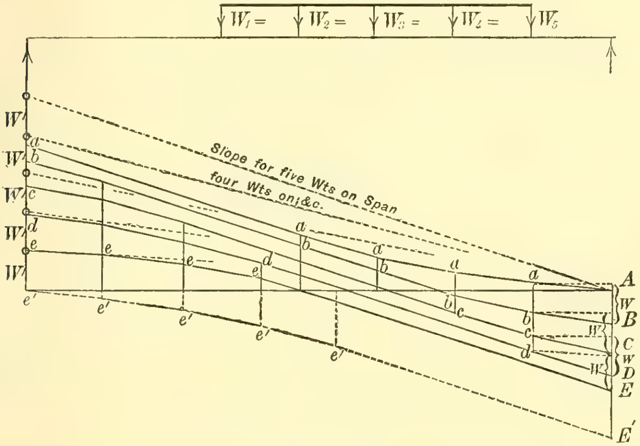

Fig. 157.

of parabolas as for the travelling load alone, and the fields are the same; any portion of a field bounded before by a chord is still bounded by the chord of the new parabola. The modulus of each parabola is the sum of the moduli for R the dead load equivalent to the actual travelling load and for the uniform load, namely,

$$\frac{R}{2c} + \frac{U}{4c} = \frac{U + 2R}{4c}.$$

The distance OS of any apex from the centre of the span is now only a fraction of what it was for the travelling load alone; the fraction being the modulus for R rolling divided by the sum of the moduli, that is,

$$\left(\frac{R}{2c}\right) \div \left(\frac{R}{2c} + \frac{U}{4c}\right) = \frac{2R}{U + 2R}. \quad \text{(Theorems at fig. 94.)}$$

For the travelling load alone, we had as the equation to the maxima bending moments for the r^{th} field,

$${}_rM_x = \frac{R}{2c}(c + x - 2h_r)(c - x) + 2h_r \Sigma_1^{r-1}(W) - \Sigma_1^{r-1}(W \cdot 2h);$$

altering the modulus and putting in the fraction, we have, for the combined load for the r^{th} field,

$${}_rM_x = \frac{U + 2R}{4c}\left(c + x - \frac{2R}{U + 2R} \cdot 2h_r\right)(c - x) + 2h_r \Sigma_1^{r-1}(W) - \Sigma_1^{r-1}(W \cdot 2h). \quad (1)$$

Graphical Solution.—Exactly the same as that for the travelling load alone (fig. 153), excepting that

$$OS_1 = \frac{2R}{U + 2R}\, h_1, \qquad OS_2 = \frac{2R}{U + 2R}\, h_2, \text{ \&c.},$$

and the scale for verticals is such that $OA_0 = \frac{1}{8}(U + 2R) \,.\, l$.

Beam under a travelling load system confined to the span, together with fixed loads at points. Graphical Solution.—Construct the diagram for moving load, as on fig. 153, and above it construct the polygon for the fixed loads, as on fig. 120, to the same scales. Draw a vertical through the first apex A_1 up through the polygon, and produce those sides of the polygon that lie wholly or in part over field 1, to make intercepts on that vertical as on fig. 120; make LS equal to the semi-base, and LN equal to twice the height of the apex of the parabolic segment whose apex is A_1; and otherwise proceed exactly as on fig. 120, till the locus for the combined load over field 1 is completed; next draw a vertical through A_2 up through the polygon, and produce such sides of the polygon as lie wholly or in part over field 2 to make intercepts on that vertical; complete the construction of the locus for the combined load over field 2 exactly as on fig. 120, except that LS is now to be the semi-base, and LN the height of the apex of the parabolic segment whose apex is A_2. In like manner draw for each field till the locus for the combined load is completed for the whole span.

One particular case is of interest—BA_0C, fig. 153, is the diagram for a rolling load R. Suppose combined with this load a system of fixed loads, one at each junction between two fields—on fig. 153, four loads, one at each of the points F_1, F_2, F_2, F_4—then to obtain the diagram for the combined load, we add to the ordinates of the parabola BA_0C those of the polygon for the fixed loads; and, as on fig. 120, the result will be five parabolas like BA_0C, and the apex of each will move from the centre *away from* the corresponding point F.

Suppose, on the other hand, that the fixed weights at F_1, F_2, &c., act upwards; then subtracting the ordinates of the polygon from those of BA_0C, the result will be five parabolas like BA_0C, but each apex will now have moved from the centre *towards* the corresponding point F.

For a certain set of values of these negative forces at F_1, F_2, &c., the five parabolas will assume the exact position of the parabolas on fig. 153, and it can easily be shown, that the set of weights which will produce this effect is such that the weight at each junction is the same fraction of R, that the distance between the weights commanding the fields of which the point is the junction is of the span. Hence we have the following

Alternative method of drawing a bending moment diagram for a travelling load system confined to the span.—On a base equal to the span, construct a parabolic segment for the bending moment diagram due to the total load R as a *rolling load*; upon the same base, and on the same side of it, construct the polygon which is the bending moment diagram due to a set of fixed loads $W_{1\cdot 2}$, $W_{2\cdot 3}$ &c., one at the junction of each pair of fields (the suffixes denoting the pair of fields between which the load is applied), and of magnitudes given by the following equations:—

$$W_{1\cdot 2} = \frac{2h_1 - 2h_2}{l} R; \qquad W_{2\cdot 3} = \frac{2h_2 - 2h_3}{l} R; \text{ \&c.}$$

Then the ordinate between the parabola and the polygon is the maximum bending moment at any point; the maximum for the whole span is found by drawing a tangent to the parabola parallel to that side of the polygon which *slopes least*, and taking the ordinate at the point of contact.

EXAMPLES.

164. A beam 42 feet span supports the five wheels of the locomotive shown on fig. 79: find the locus of the maximum bending moment at each point for all positions of the load, it being understood that no wheel moves off the span (fig. 153).

Loads	5,	5,	11,	12,	9 tons	= 42 tons.
Intervals		5,	8,	10,	7 feet	= 30 feet.
Distances from G,	17,	12,	4,	− 6,	− 13 feet.	

Dividing the span in the ratio of the weights, we have the extent of the fields as follows :

1st field from 21 to 16 ; 2nd field from 16 to 11 ; 3rd field from 11 to 0 ; 4th field from 0 to − 12 ; 5th field from − 12 to − 21.

Substituting in the general equation

$$ {}_rM_x = \frac{R}{2c}(c+x-2h_r)(c-x) + 2h_r\Sigma_1^{r-1}(W) - \Sigma_1^{r-1}(W \,.\, 2h), $$

we have

$$ {}_1M_x = \frac{R}{2c}(c+x-2h_1)(c-x) = (4+x)(21-x), $$

for values of x from 21 to 16 ;

$$ \begin{aligned} {}_2M_x &= \frac{R}{2c}(c+x-2h_2)(c-x) + 2h_2 \,.\, W_1 - W_1 \,.\, 2h_1 \\ &= (9+x)(21-x) - 25, \end{aligned} $$

for values of x from 16 to 11 ;

$$ \begin{aligned} {}_3M_x &= \frac{R}{2c}(c+x-2h_3)(c-x) + 2h_3(W_1+W_2) - (W_1 \,.\, 2h_1 + W_2 \,.\, 2h_2) \\ &= (17+x)(21-x) - 105, \end{aligned} $$

admissible for values of x from 8 to 0.

$$ \begin{aligned} {}_4M_x &= \frac{R}{2c}(c+x-2h_4)(c-x) + 2h_4(W_1+W_2+W_3) \\ &\qquad - (W_1 \,.\, 2h_1 + W_2 \,.\, 2h_2 + W_3 \,.\, 2h_3) \\ &= (27+x)(21-x) - 315, \end{aligned} $$

admissible for values of x from − 2 to − 12 ;

$$ \begin{aligned} {}_5Mx &= \frac{R}{2c}(c+x-2h_5)(c-x) + 2h_5(W_1+W_2+W_3+W_4) \\ &\qquad - (W_1 \,.\, 2h_1 + W_2 \,.\, 2h_2 + W_3 \,.\, 2h_3 + W_4 \,.\, 2h_4) \\ &= (34+x)(21-x) - 546, \end{aligned} $$

for values of x from − 12 to − 21.

To find the ordinates of the apexes; substitute $x = h_1 = 8{\cdot}5$ in the first equation, $x = h_2 = 6$ in the second, and so on, and we have

$$ {}_1y_{8\cdot5} = 156{\cdot}25, \quad {}_2y_6 = 200 \quad {}_3M_2 = 256 \quad {}_4M_{-3} = 261, \quad {}_5y_{-6\cdot5} = 210{\cdot}25. $$

Only two are expressed by the letter M, as they alone of the five are bending moments; the others do not lie in their own fields. Hence the maximum bending moment for the whole span is

$$ {}_4M_{-3} = 261 \text{ ft.-tons.} $$

165. Calculate the maxima positive and negative shearing forces at intervals during the transit of the load (fig. 154).

The slopes of the links are 5, 10, 21, 33, 42; 37, 32, 21, and 9 tons per 42 feet; 42 feet being the span.

The co-ordinates of the joints of the links are—

Links.	Ft.	Tons.	Ft.	Tons.	Ft.	Tons.	Ft.	Tons.	Ft.	Tons.	Ft.	Tons.
Aaa . . .	– 21	– 00·0	– 16	0·6	– 8	2·5	2	7·5	9	13·0	21	25·0
Bbb . . .	– 21	– 4·4	– 13	– 2·5	– 3	2·5	4	8·0	16	20·0	21	24·4
Ccc . . .	– 21	– 7·5	– 11	– 2·5	– 4	3·0	8	15·0	13	19·4	21	25·5
Ddd . . .	– 21	– 13·5	– 14	– 8·0	– 2	4·0	3	8·4	11	14·5	21	19·5
Ee'e' . . .	– 21	– 29·0	– 9	– 17	– 4	– 12·6	4	– 6·5	14	– 1·5	21	00·0

For positive maxima, the locus *Ccc* . . . is to be taken between the left end of span and the point $x = -1$, and the locus *Aaa* . . . from that point to the right end of span; for negative maxima, the locus *E'e'e'* . . . is to be taken for the whole span.

166. Find the equation to the maxima bending moments in the 3rd field (Ex. 164) directly, and without using the general formula

Choose any point whose abscissa is x, such that when the load is placed with the 3rd weight over it, the whole load will be on the span. The distance of G the centre of gravity from the right end will be $(c + x - 2h_3)$, or $(17 + x)$, and

$$ P = \tfrac{42}{42} \times (17 + x) = (17 + x). $$

Taking a section at x and considering the forces on the left side of it, we have

$$ \begin{aligned} {}_3M_x &= P(21 - x) - W_1(5 + 8) - W_2 \times 8 \\ &= (17 + x)(21 - x) - 105. \end{aligned} $$

167. Find the maximum bending moment at any given points, say $x = 8$, and $x = 10$, in Ex. 164; and find the maximum for the whole span without using the general equations (fig. 153).*

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* These Rules were first published in "Engineering," Jan. 10th and July 25th, 1879.

1°. To find the maximum bending moment at any given point, divide the span into fields proportional to the weights; consider which field the point lies in, and place the load so that the corresponding weight is over the point. Then if the whole load be on the span, the bending moment calculated at the point for the load fixed in this position is a maximum. If when the load is so placed, some weights are off the span, move the load the least distance which will bring them all on the span, and calculate the bending moment at the point for the load fixed in that position.

2°. To find the maximum bending moment for the whole span.—By inspection, place the load so that any particular weight and the centre of gravity may be upon different sides of the centre of span and at equal distances therefrom; then if the whole load be on the span and the weight be in its own field, calculate the bending moment at the point where the weight stands for the load fixed in this position, and it will be the maximum at the point. For each weight which can be placed so as to fulfil these conditions, there is a maximum, and having calculated these maxima, the greatest is the maximum for the whole span.

In Ex. 164, the point $x = 8$ lies in the 3rd field; place the load so that W_3 is over it, and it will be found that all the wheels are on the span; calculating the bending moment at $x = 8$ with the load in this position, we have

$${}_3M_8 = 220 \text{ ft.-tons maximum at point.}$$

The same result may be obtained by substituting $x = 8$ into ${}_3M_x$.

Again, the point 10 lies in field 3, but when W_3 is placed over it, W_1 is not on the span; moving the load two feet to the right brings W_1 just on the span, while W_3 is at 8; that is, the load is in the very same position as previously. Calculating the bending moment at $x = 10$, with the load in this position, we have

$$M'_{10} = 190 \text{ ft.-tons maximum at point.}$$

The same result may be obtained by substituting $x = 8$, and $x = 11$, into ${}_3M_x$, and then adding one-third of their difference to the smaller.

In Ex. 164, we find that by placing W_3 at the point $x = 2$, it is in its own field while G is at the point $x = -2$, and the whole load is on the span; calculating the bending moment at 2 for the load in that position we have ${}_3M_2 = 256$. Again, by placing W_4 at the point $x = -3$, it is in its own field while G is at $x = 3$, and the whole load is on the span; calculating the bending moment at -3 for the load in that position, we have (see example at fig. 100) ${}_4M_{-3} = 261$. Now since no other weight can be placed according to the rule, it follows that ${}_4M_{-3} = 261$ ft.-tons is the maximum for span not only for the position shown on fig. 100, but for all possible positions of the load on the span.

168. Find the maxima bending moments at the junctions of the "fields." That is, find the heights of the points D_1, D_2, &c., on fig. 153.

$$F_1D_1 = {}_1M_{16} = 100 \text{ ft.-tons}; \qquad F_2D_2 = {}_2M_{11} = 175 \text{ ft.-tons};$$

$$F_3D_3 = {}_3M_0 = 252 \text{ ft.-tons}; \qquad F_4D_4 = {}_4M_{-12} = 180 \text{ ft.-tons.}$$

Where ${}_1M_{16}$ means the bending moment at the point 16 feet left of centre when the first wheel is over it; ${}_2M_{11}$ the bending moment at 11 feet left of centre when the second wheel is over it, &c.

The equation to the dotted parabola BA_0C is

$$\frac{R}{2c}(c^2 - x^2) = (21^2 - x^2),$$

so that the ordinates at F_1, F_2, F_3, and F_4 to this dotted parabola are to be found by substituting for x the values 16, 11, 0, and -12, respectively. They are therefore 185, 320, 441, and 297 ft.-tons.

Hence the *depths* of the points D_1, D_2, D_3, and D_4 (which are the junctions of the "fields") below the dotted parabola got by subtracting are 85, 145, 189, and 117 ft.-tons.

It will now be seen that the depth of D_1 below the dotted parabola is

$$W_1 \,.\, 2h_1 = 5 \times 17 = 85\,;$$

of D_2 is $$W_1 \,.\, 2h_1 + W_2 \,.\, 2h_2 = 5 \times 17 + 5 \times 12 = 145\,;$$

of D_3 is $$W_1 \,.\, 2h_1 + W_2 \,.\, 2h_2 + W_3 \,.\, 2h_3 = 145 + 11 \times 4 = 189\,;$$

and of D_4 it is $$W_1 \,.\, 2h_1 \;.\;.\;.\; W_4 \,.\, 2h_4 = 189 - 12 \times 6 = 117.$$

A general proof can readily be made of this remarkable theorem. That the junctions of the parabolic arcs (fig. 153) lie on vertical lines dividing the span in segments which are proportional to the individual load on the wheels, and that the *depths* of any one of these junctions D *below* the parabola which is the bending moment diagram for the whole load rolling on one wheel are given by $\Sigma(W \,.\, 2h)_1^r$, that is by the *geometrical moment* of the weights about their centre of gravity from the first up to and including all those that command fields left of that junction.

BEAM SUBJECT TO THE TRANSIT OF A TRAVELLING LOAD SYSTEM.

Graphical Solution.—We already solved this problem for the transit of two weights, and saw that it then had two cases. The general problem will have many cases, and we shall illustrate the solution by solving a particular problem, viz., for the data shown on fig. 153, where the solution for the load confined to the span is already given.

Suppose the load to come on at the left end and pass off at the right. First, the weight W_5 alone is on the span; then W_5 and W_4, &c.; then the whole load; then W_5 passes off; and so on, till W_1 alone is on; this finally passes off, and a transit has been made. Figs. 159, 160, 161, 162 give the maxima at each point, while one only, two only, three only, four weights only are on the span; fig. 153 gives the maximum at each point, while all five are on; and figs. 163, 164, 165, 166 give the maximum after one, two, three, and four weights respectively have passed off.

The manner of constructing those figures is as follows:—Prepare a number of parabolic segments (fig. 158), one corresponding to each of the manners of loading, the highest one being drawn with the parabolic segment that was used in the construction of fig. 153; divide its height so that the heights of the various apexes will be in the ratio of the loads, viz., 9, 21, 32, 37, 42, 33, 21, 10, 5. Draw a number of ordinates to the curve; divide these similarly with proportional compasses, and draw curves through the points thus found. Separate pieces of cardboard should then be cut out so as to correspond with each of these curves.

A description of the method for constructing fig. 161 will suffice for all. This figure is for the weights 11, 12, 9 tons on the span; and since the total load is 32 tons, select the cardboard segment whose height is proportional to 32; this ensures that the verticals are drawn to the same scale as those on fig. 153. Lay off BC equal to the span on the same horizontal scale as that used in fig. 153.

Draw with a dotted line the locus BD_1D_2C, as for the load confined to the span and capable of moving into all positions on the span. This, however, is not the locus required, since it is evident that the range of motion of the load is limited by the next weight, 5 tons, not being allowed to come on to the span. Hence, place the load as far on the span as possible, but so that the next weight, 5 tons, may not be on the span; that is, place it so that the next weight is over B the extremity; from each weight draw an ordinate to the corresponding parabola, for instance, from the weight 11 tons to the parabola whose apex is A_1. Ink in the curves for the portion of each field which has been travelled over by the weight commanding; the remainder of the locus is made by drawing the polygon whose angles are the tops of the ordinates where the weights stand. Thus, on fig. 161, the locus is the arc Ba of the first parabola since the weight 11 tons has travelled over that part of its field; and the arc D_1b of the second parabola since the weight 12 tons has travelled over that part of its field; the remainder of the locus consists of parts of the polygon $BabdC$, that is the polygon for the load fixed in the position shown on figure; it may be observed that aD_1 is a part of ab (*Theorem D*, fig. 95). Hence the locus (fig. 161) is arc Ba, chord aD_1, arc D_1b, and chords bd, dC.

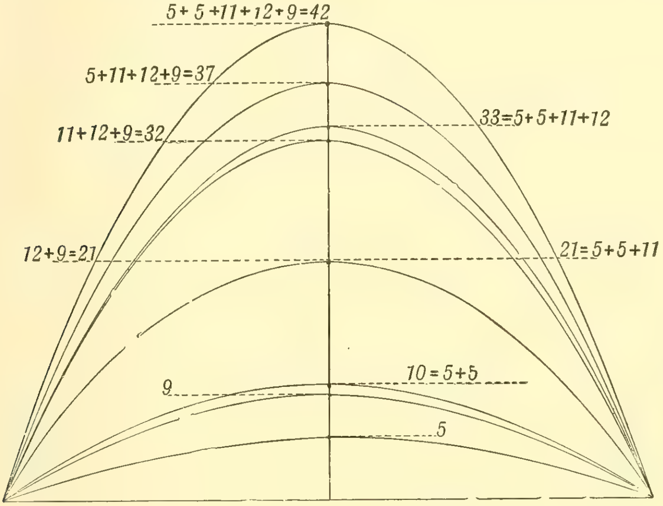


Fig. 158.

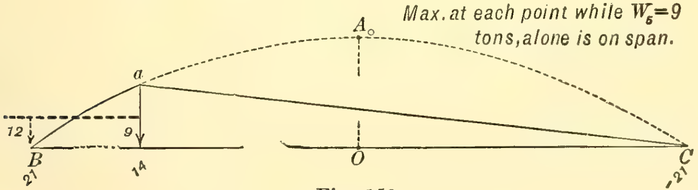


Fig. 159.

It is evident that the arcs named give the maxima for the portions over which they stand for all the possible positions of the three weights on the span; and, that the chord aD_1 gives maxima for the portion of span over which it stands will be seen as follows:—The load cannot shift to the right; suppose it shifts to the left, then the new position of ab will still pass through D_1 (*Theorem D*, fig. 95), but the point a will be lower down on the first parabola; that is, as the load moves to the left, the chord D_1a turns about D_1 in the opposite direction from the hands of a watch, so that the ordinate of every point in it is decreasing. Again, bd produced passes through the point D_2; and if the load moves, the new position of bd still passes through D_2; that is, as the load moves to the left, the chord bd turns about D_2 in the opposite direction from the hands of a watch, so that the ordinate of every point in it is decreasing. Similarly dC turns about C in the same direction as the others, and the ordinate of every point in it is also decreasing.

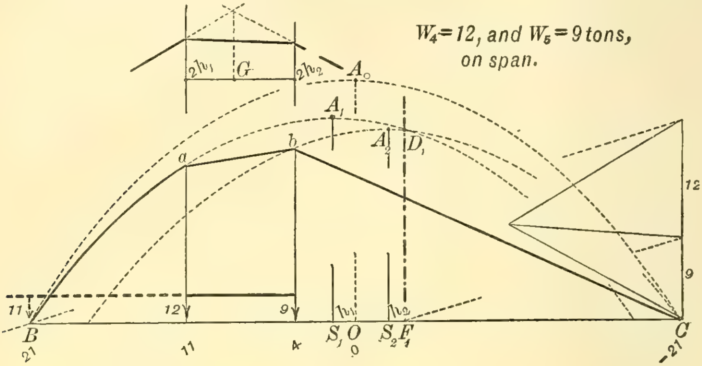


Fig. 160.

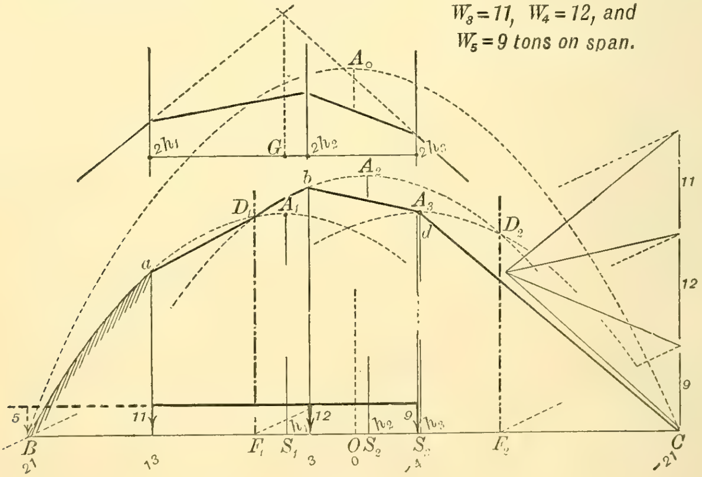


Fig. 161.

If the eight figures, for the complete transit of the load, be superimposed upon fig. 153 for the load confined to the span, the portions which are shaded on figs. 161 and 162 will lie *outside* of the locus on fig. 153; so that, replacing the corresponding portions of fig. 153 by those shaded parts, that is, everywhere following the highest locus, we have a graphical solution for the transit of the load. The labour of this graphical solution is great if we have to prepare the parabolic segments, but the solution can be drawn by means of *one* parabolic segment from the following considerations.

Suppose in the example that all the eight diagrams were drawn with the *same* parabolic segment, then it is evident that each would be to a *different* vertical scale, and that nothing could be determined by superposition. However, it can be seen by inspection that fig. 163 lies entirely within fig. 153; for, since the weight

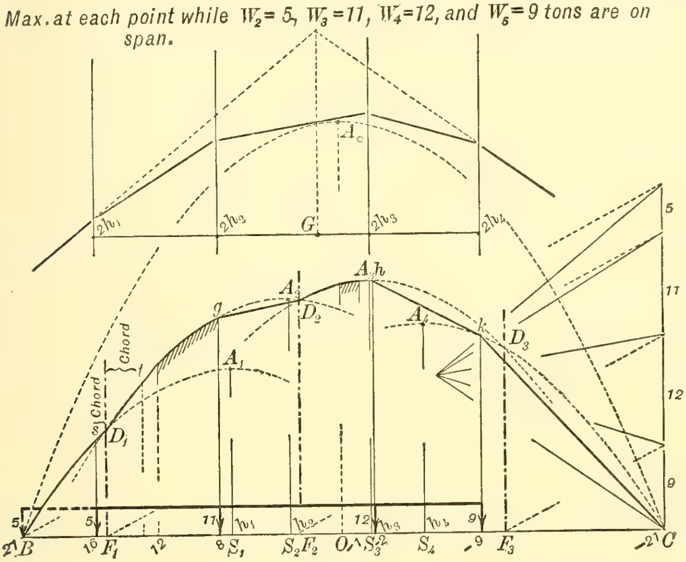


Fig. 162.

9 tons on fig. 163 is over the point of support B, we may consider that it is not on the span as we did when constructing fig. 163; on the other hand, if we consider it to be on the span, then the total load is on the span; and for that position of the total load we may plot the point a upon fig. 153 by drawing an ordinate to the first parabola from the point whose abscissa is 9, that is, from the point where the weight 5 tons stands on fig. 163. Similarly the point b is plotted on fig. 153 by drawing an ordinate to the second parabola from the point whose abscissa is 4, that is, from the point where the second weight, 5 tons, stands on fig. 163; likewise the points d and e are plotted. If the one figure be drawn exactly above the other, it will only be necessary to draw verticals from the points a, b, d, e (fig. 163) to meet the 1st, 2nd, 3rd, 4th parabolas respectively on fig. 153, and we have these points plotted on fig. 153. It will then be seen that these points are inside the locus on fig. 153; so that the whole locus, fig. 163, is inside that of fig. 153. Similarly, by drawing verticals from the points a, b, d (fig. 164) to meet respectively

the 1st, 2nd, 3rd parabolas on fig. 163, these points are plotted on fig. 163 and will be found to lie inside of it. Hence the locus (fig. 164) lies inside the locus (fig. 163), and much more will it lie inside of the locus (fig. 153). In this way it will be seen that the loci (figs. 166, 165, 164, 163, 153) lie each within the other in order, and therefore that all lie within the locus (fig. 153).

In the same way it can be shown that the loci only on figs. 162 and 161 lie partly outside of the locus (fig. 153). On fig. 162 are shown the points *s*, *g*, *h*, *k* transferred from fig. 153 in the manner described. It is evident that *g* and *h* are the extremities of the chords that were parts of the locus for the load confined to the span; because, the reason we followed the chord at D_2 (fig. 153) was that the commanding weight W_3 could not overtake that part; in other words, when W_3 came to *g*, then W_1 was over *B* and about to go off the span; that is, the load was in the position shown on fig. 162.

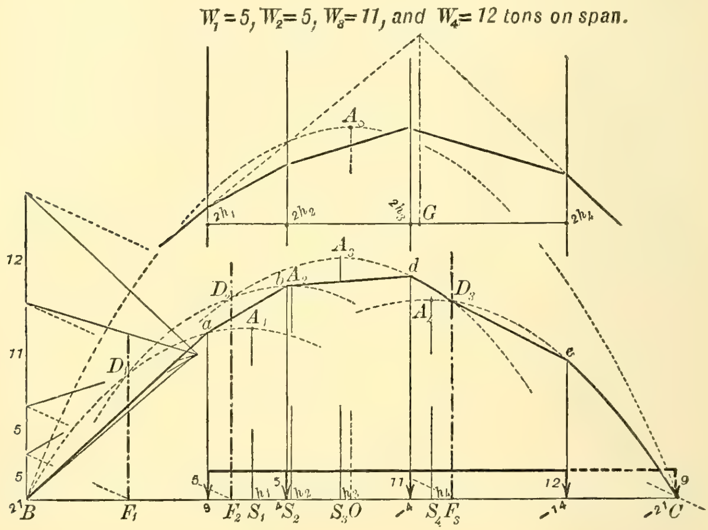


Fig. 163.

It will therefore always be the case that when, for the load confined to the span, we have chords for portions of the locus, then the diagram for one load off the span lies in part outside the locus; and the ends of such chords which are not junctions of fields are points of intersection.

Having ascertained by inspection the shaded parts of figs. 161 and 162 which alone lie outside of fig. 153, it will be necessary to plot points at intervals upon fig. 153 with the proportional compasses; the points thus got from fig. 162 have their ordinates reduced in the ratio of 42 and 37, those from fig. 161 in the ratio of 42 and 32, since the diagrams here spoken of are supposed to be drawn with the same parabolic segment, and therefore to different vertical scales.

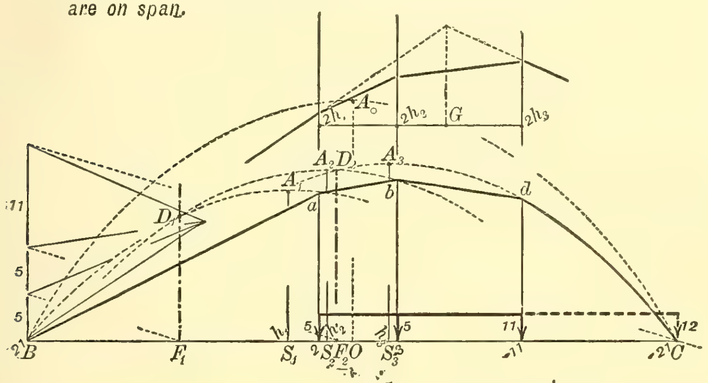
We will complete the analytical solution for the transit of the load in this example. To find the equation to *Ba* (fig. 161); suppose the weight 11 tons to be under this arc and at any point whose abscissa is x, then it will be found by taking moments round Q, that

$$P = \frac{399 + 32x}{42}.$$

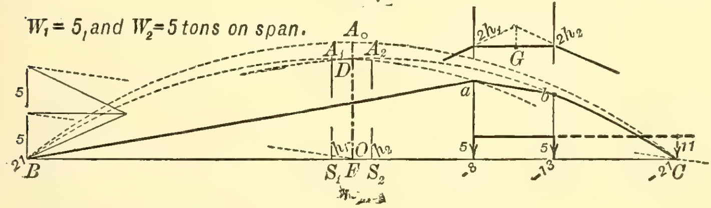
$$M_x = P(c - x) = \tfrac{1}{42}(399 + 32x)(21 - x),$$

for values of x from 21 to 13.

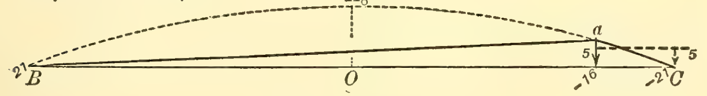
Max. at each point while $W_1 = 5$, $W_2 = 5$, and $W_3 = 11$ tons are on span.



$W_1 = 5$, and $W_2 = 5$ tons on span.



$W_1 = 5$ tons on span.



Figs. 164, 165, 166.

For the arc tg (fig. 162); let the weight 11 tons be under this arc at any point whose abscissa is x, then

$$P = \frac{544 + 37x}{42}.$$

$$M_x = P(c - x) - 5 \times 8 = \tfrac{1}{42}(544 + 37x)(21 - x) - 40,$$

for values of x from 12 to 8.

For the arc D_2h; let the weight 12 tons be under this arc, and at any point whose abscissa is x, then

$$P = \frac{914 + 37x}{42}.$$

$$M_x = P(c - x) - 5 \times 18 - 11 \times 10 = \tfrac{1}{42}(914 + 37x)(21 - x) - 200,$$

for values of x from 0 to -1.

The limits in each case are found by making the equation to the arc simultaneous with that to the arc on fig. 153, which is now superseded; the limits as given above are to the nearest whole number.

During the transit, it is to be observed that in this case the weight $W_3 = 11$ tons commands exactly the left half of the span, while the right half is commanded by $W_4 = 12$ and $W_5 = 9$ tons, just as on fig. 153 for the load confined to the span.

Calculating the ordinate at each foot of span from the equation just given for the values of x there specified, and from the equations already given at fig. 153 for other values of x, we find the maximum bending moment at each foot of span during the transit to be

Abs.	B. M.	Abs.	B. M.	Abs.	B. M.	Abs.	B. M.
20 ..	24·7	9 ..	210·6	− 0 ..	257·0	− 11 ..	197
19 ..	48·0	8 ..	220·0	− 1 ..	259·4	− 12 ..	180
18 ..	69·7	7 ..	231·0	− 2 ..	260·0	− 13 ..	168
17 ..	89·8	6 ..	240·0	− 3 ..	261·0	− 14 ..	154
16 ..	108·5	5 ..	247·0	− 4 ..	260·0	− 15 ..	138
15 ..	125·6	4 ..	252·0	− 5 ..	257·0	− 16 ..	120
14 ..	141·2	3 ..	255·0	− 6 ..	252·0	− 17 ..	100
13 ..	155·3	2 ..	256·0	− 7 ..	245·0	− 18 ..	78
12 ..	171·7	1 ..	255·0	− 8 ..	236·0	− 19 ..	54
11 ..	186·5	0 ..	257·0	− 9 ..	225·0	− 20 ..	28
10 ..	199·4			− 10 ..	212·0		

The maximum for the whole span is still $M_{-3} = 261$ ft.-tons.

Restricted case of the transit of a system of equal weights at equal intervals.

1. *Load shorter than span.*

Let n be the number of weights, and s their common distance apart.

(a) Let the distance between the weights be less than or equal to $\frac{1}{n}$th part of the span, that is let $s \leqq \frac{2c}{n}$.

With any parabolic segment draw the series of n parabolas, as shown with the full lines fig. 155, and the maximum bending moment will be at the centre, or on both sides of the centre and at a quarter interval therefrom, according as n is odd or even. In fact the solution is the same as for the load confined to the span.

(*b*) Let the distance between the weights be greater than $\frac{1}{n}$th of the span, that is let $s > \frac{2c}{n}$.

In this case the double locus, as shown on fig. 156, is to be drawn; one for n weights with any parabolic segment, the other for $\overline{n-1}$ weights with another parabolic segment prepared as on fig. 158, and so that its modulus and the modulus of the first are in the ratio $(n-1):n$. Since, however, the points of intersection $a, b, c \ldots a', b', c'$ may be defined as shown on fig. 156, both loci may be drawn with the *same* parabolic segment, inking in the arcs over Ba, ac', &c., alternately on the two loci; a separate vertical scale will be required for each; or if considered necessary, the one locus may be reduced to the same vertical scale as the other at a number of points and then drawn.

The maximum bending moment will be the highest apex in the one locus or in the other; it must therefore be at the centre, at a quarter interval from the centre, or equally at these three points. The maximum is readily found as follows:—if n be *odd*, place the central weight over the centre of span, and calculate the bending moment at that point; this will be the maximum required: if n be *even*, place either of the two central weights, first at the centre of span, and afterwards at a point in its own field a quarter interval distant from the centre; calculate the bending moment for these two points, and the greater will be the maximum for transit required.

2. *Load longer than span.*

Let n be the greatest number of weights that can be on the span at once.

(*a*) Let the common interval be an aliquot part of the span.

The series of n parabolas as in 1 (*a*) is the locus for transit, fig. 155.

(*b*) Let the interval be not an aliquot part of the span.

The number of weights on the span will be alternately n and $n-1$; while n are on, *no* weight can overtake its field before a weight goes off; while $(n-1)$ are on, *no* weight can overtake its field before another weight comes on; it is evident therefore that the locus for n, and the locus for $(n-1)$ weights are to be drawn as in 1 (*b*) as shown on fig. 156.

NOTE.—In all the above cases of transit, during the passing on of the front of the load and the passing off of its rear, fewer weights than n or $(n-1)$ are for some time on the span; by the theorem immediately preceding, the loci for such smaller numbers of weights are all wholly within the dotted loci on figs. 155 and 156.

Theorem.—The locus of the points of intersection D_1, D_2, &c., of the regular locus $BD_1D_2 \ldots C$ (figs. 155 and 156) for equal weights at equal intervals is a parabola.

Let x be the abscissa of D_r the junction between the r^{th} and $(r+1)$th fields, and let the load stand with the r^{th} weight over that point, then

Remote segment $= c + x -$ distance of r^{th} weight from G,

$$= c + x - \left(\frac{n+1}{2} - r\right) s;$$

therefore

$$P = \frac{nw}{2c}\left(c + x + rs - \frac{n+1}{2} s\right);$$

and

$$M_x = \frac{nw}{2c}\left(c + x + rs - \frac{n+1}{2} s\right)(c - x) - ws\,(1 + 2 + 3 + \ldots + \overline{r-1}), \quad (1)$$

for such values of x as are abscissæ of junctions of fields. Putting y for the ordinate of this locus at *any* point x,

$$y = \frac{nw}{2c}\left(c + x + rs - \frac{n+1}{2}\,s\right)(c - x) - \frac{ws}{2}\,r\,(r - 1), \tag{2}$$

but

$$c - x = r \,.\, \frac{2c}{n} \quad \text{so that} \quad r = \frac{n\,(c - x)}{2c}.$$

Substituting this value for r in equation 2, arranging the terms and putting R for wn, we have

$$y = \frac{R}{4c}\,(c^2 - x^2)\left(2 - \frac{ns}{2c}\right). \tag{3}$$

Hence the locus of B, D_1, D_2, &c., C is a parabola, axis is vertical, and apex over the centre; comparing it with the equation at fig. 136, we see that it is the same as the locus of bending moments for a travelling uniform load of extent ns; that is, for the total load R spread uniformly over the actual extent of the load and one interval s more.

It is evident that this parabola gives too short an ordinate at all points except the junctions of fields. Still we have—

Cor.—Let R remain constant, and let n increase indefinitely while s decreases indefinitely; then the locus of the intersections D nearly coincides with $BD_1D_2 \ldots C$, and the equation to the locus is the same as for R spread uniformly over an extent only greater than the actual extent of the load by s an indefinitely small amount. In the limit, when n is infinite, the load is uniform and the result (fig. 136) follows from the present investigation.

Note.—In the general case, the locus of D is not a parabola nor any conic section. The general equation for any conic section may be written thus

$$x^2 + bxy + cy^2 + dx + ey + f = 0;$$

and if we suppose the locus to pass through the five points B, D_1, D_3, D_4, and C, example 96 and fig. 153, we find values for the five coefficients; and the equation becomes

$$x^2 + \tfrac{7}{930}\,xy + \tfrac{1}{7440}\,y^2 + \tfrac{532}{310}\,y - 441 = 0;$$

by trial we see that this locus does not pass through D_2; that is, the locus is not a conic section.

Examples.

169. A beam 76 feet span is subjected to a moving load R = 19 tons confined to the span, and concentrated on three wheels as under

Loads	7	4	8 tons = 19 tons.
Intervals	30		12 feet = 42 feet.
Loads distant from G	24	− 6	− 18 feet.

The 1st field extends from 38 to 10; 2nd field from 10 to − 6; 3rd field from − 6 to − 38.

Substituting in the general equation, we have

$${}_rM_x = \tfrac{1}{4}(38 + x - 2h_r)(c - x) + 2h_r\Sigma_1{}^{r-1}(W) - \Sigma_1{}^{r-1}(W \,.\, 2h).$$

${}_1M_x = \tfrac{1}{4}(14 + x)(38 - x)$, for values of x from 38 to 10.

${}_1M_{h1} = {}_1M_{12} = 169$, maximum for field 1.

${}_2M_x = \tfrac{1}{4}(44 + x)(38 - x) - 210$, for values of x from 8 to -6; the chord to be taken from 8 to 10.

${}_2M_{h2} = {}_2M_{-3} = 210\tfrac{1}{4}$, maximum for field 2.

${}_3M_x = \tfrac{1}{4}(56 + x)(38 - x) - 342$ for values of x from -6 to -38.

${}_3M_{h3} = {}_3M_{-9} = 210\tfrac{1}{4}$, maximum for field 3.

The maximum for whole span is equally ${}_2M_{-3}$ or ${}_3M_{-9}$.

170. For Ex. 164, find the equation to the bending moment at each point of span as b, a point of the load 8 feet to the left of G, comes over it (fig. 153).

To find D_2 the point of intersection on the junction of the fields commanded by W_2 and W_3 the weights on either side of b, suppose the load fixed with either of these weights over the point F_2, where $OF_2 = 11$, and calculate the bending moment there; or, substitute $x = 11$ into ${}_2M_x$ or ${}_3M_x$ of Ex. 164, and we find the height of D_2 to be 175; hence the coordinates of D_2 are $x = 11$, $y = 175$. The locus for the point b is the parabola whose apex A is on the vertical through S, where $OS = h = \frac{1}{2}Gb = 4$; this parabola passes through D_2, has its axis vertical, and its modulus is the load divided by the span; hence

$${}_bM_x = \frac{R}{2c}(c + x - 2h)(c - x) + C = (13 + x)(21 - x) + C;$$

since it passes through D_2, we have ${}_bM_{11} = 175$. Therefore

$$175 = (13 + 11)(21 - 11) + C, \quad \text{or} \quad C = -65;$$

and

$${}_bM_x = (13 + x)(21 - x) - 65,$$

for values of x from 12 to 0, since part of the load goes off the span when b occupies any position not comprised between these two limits.

171. Suppose the weights in example 164 and shown at fig. 153, to be the equivalent dead loads for the actual weights; combine with these an uniform load of $\frac{2}{3}$rds of a ton per foot of span, that is, a total load $U = 28$ tons, and find the equations to the maxima bending moments for the various fields.

Summing the moduli for R rolling and for U spread uniformly, we have

$$\frac{R}{2c} + \frac{U}{4c} = \frac{U + 2R}{4c} = \frac{28 + 84}{84} = \frac{4}{3},$$

the modulus of the parabolas for combined load.

The modulus for R rolling, divided by the sum of the moduli, is

$$\left(\frac{R}{2c}\right) \div \left(\frac{R}{2c} + \frac{U}{4c}\right) = \frac{2R}{U + 2R} = \frac{84}{28 + 84} = \frac{3}{4}.$$

Substituting into the general equation,

$${}_1M_x = \tfrac{4}{3}(21 + x - \tfrac{3}{4} \times 17)(21 - x) = \tfrac{4}{3}(8{\cdot}25 + x)(21 - x)$$

for values of x from 21 to 16.

$${}_2M_x = \tfrac{4}{3}(21 + x - \tfrac{3}{4} \times 12)(21 - x) + 12 \times 5 - 5 \times 17 = \tfrac{4}{3}(12 + x)(21 - x) - 25,$$

for values of x from 16 to 11.

$$\begin{aligned} {}_3M_x &= \tfrac{4}{3}(21 + x - \tfrac{3}{4} \times 4)(21 - x) + 4(5 + 5) - (5 \times 17 + 5 \times 12) \\ &= \tfrac{4}{3}(18 + x)(21 - x) - 105, \end{aligned}$$

for values of x from 8 to 0, the chord being taken for values from 11 to 8.

$$\begin{aligned} {}_4M_x &= \tfrac{4}{3}(21 + x + \tfrac{3}{4} \times 6)(21 - x) - 6(5 + 5 + 11) - (5 \times 17 + 5 \times 12 + 11 \times 4) \\ &= \tfrac{4}{3}(22{\cdot}5 + x)(21 - x) - 315, \end{aligned}$$

for values of x from -2 to -12, the chord being taken for values from 0 to -2.

$$\begin{aligned} {}_5M_x &= \tfrac{4}{3}(21 + x + \tfrac{3}{4} \times 13)(21 - x) - 13(5 + 5 + 11 + 12) \\ &\qquad\qquad - (5 \times 17 + 5 \times 12 + 11 \times 4 - 12 \times 6) \\ &= \tfrac{4}{3}(30{\cdot}75 + x)(21 - x) - 546, \text{ for values of } x \text{ from } -12 \text{ to } -21. \end{aligned}$$

The abscissæ of the apexes are

$\frac{3}{4} \times 8{\cdot}5 = 6{\cdot}375$; $\frac{3}{4} \times 6 = 4{\cdot}5$; $\frac{3}{4} \times 2 = 1{\cdot}5$, lies in field 3 ; $\frac{3}{4} \times (-3) = -2{\cdot}25$, lies in field 4 ; $\frac{3}{4} \times (-6{\cdot}5) = -4{\cdot}875$.

Here only two apexes lie in the corresponding fields ; substituting for these, we have

$${}_3M_{1\cdot5} = 402 \text{ maximum in field 3 ; and}$$

$${}_4M_{-2\cdot25} = 405{\cdot}75 \text{ maximum in field 4 ; and maximum for span ;}$$

the greatest bending moment occurs at $2\frac{1}{4}$ ft. to the right of the centre, when the fourth weight is over it. The joint centre of gravity is $2\frac{1}{4}$ ft. left of centre.

172. Suppose the weights in example 169 to be the equivalent dead loads for the actual weights, and combine an uniform load of $\frac{1}{2}$ ton per foot of span with the travelling load. Find the maximum bending moment at each point of span, for the load confined to the span.

The sum of the moduli of the parabolas for U and for R rolling is

$$\frac{U + 2R}{4c} = \frac{38 + 38}{152} = \tfrac{1}{2},$$

the modulus for the combined load.

The modulus of the parabola for R, divided by the sum of the moduli, is

$$\left(\frac{R}{2c}\right) \div \left(\frac{R}{2c} + \frac{U}{4c}\right) = \frac{2R}{U + 2R} = \frac{38}{38 + 38} = \tfrac{1}{2}.$$

In the general equation given for No. 169 replace $\frac{1}{4}$ by $\frac{1}{2}$, and in the trinomial factor replace $2h_r$ by h_r ; otherwise the expression remains the same ; thus

$${}_1M_x = \tfrac{1}{2}(38 + x - 12)(38 - x) = \tfrac{1}{2}(26 + x)(38 - x),$$

for values of x from 38 to 10.

$${}_1y_6 = \tfrac{1}{2}(32)^2 = 512,$$

height of apex of parabola 1, not in field 1.

$${}_2M_x = \tfrac{1}{2}(41 + x)(38 - x) - 210,$$

for values of x from 8 to -6 ; the chord to be taken from 8 to 10.

$${}_2M_{-1\cdot5} = \tfrac{1}{2}(39{\cdot}5)^2 - 210 = 570\tfrac{1}{8}, \text{ maximum in field 2.}$$

$${}_3M_x = \tfrac{1}{2}(47 + x)(38 - x) - 342, \text{ for values of } x \text{ from } -6 \text{ to } -38.$$

$${}_3y_{-4\cdot5} = \tfrac{1}{2}(42{\cdot}5)^2 - 342 = 561\tfrac{1}{8}, \text{ height of apex of parabola 3, not in field 3.}$$

The max. for span is $570\frac{1}{8}$ ft.-tons, and it occurs $1\frac{1}{2}$ ft. to the right of the centre when the second weight is over it. The centre of gravity is $1\frac{1}{2}$ ft. left of centre.

173. In a beam of 40 ft. span bearing a travelling load $W_1 = 2$, and $W_3 = 3$, tons fixed 20 feet apart, for the transit of the load find a maximum of 30 ft.-tons at the centre, and a maximum of 32 ft.-tons, four feet to the right of the centre. Supposing these weights to be the equivalent dead loads for the actual weights, and taking the weight of the beam, six tons uniformly distributed, into account, make the appropriate alteration upon the equations for the moving load only; since the distance apart of the wheels is greater than the shorter *field* of the span, the equations to the maximum bending moments for the transit are (fig. 148)

$$
\begin{aligned}
{}_1M_x &= \tfrac{1}{8}(8 + x)(20 - x),\\
{}_2M_x &= \tfrac{3}{40}(400 - x^2),\\
{}_3M_x &= \tfrac{1}{8}(12 - x)(20 + x);
\end{aligned}
$$

the first for values of x from between B and K or 20 to 10, the next from K to H or 10 to 0, and the last from H to C or 0 to -20.

Compounding the uniform load for the first and third equations, $R = 5$, and $U = 6$ tons; hence the new modulus is

$$\frac{U + 2R}{4c} = \frac{6 + 10}{80} = \frac{1}{5};$$

and the fraction

$$\frac{2R}{U + 2R} = \frac{6 + 10}{10} = \frac{5}{8}.$$

For centre equation $W_2 = 3$, and $U = 6$ tons; hence the new modulus is

$$\frac{U + 2W_2}{4c} = \frac{6 + 6}{80} = \frac{3}{20}.$$

Ans. ${}_1M_x = \tfrac{1}{5}(20 + x - 7{\cdot}5)(20 - x)$, for values of x from 20 to 10.
${}_2M'_x = \tfrac{3}{20}(400 - x^2)$, for values of x from 10 to 0.
${}_2M_x = \tfrac{1}{5}(20 - x - 5)(20 + x)$, for values of x from 0 to -20.
${}_1y_{3\cdot75} \doteq \tfrac{1}{5}(16{\cdot}25)^2 = 52{\cdot}81$, height of apex, not in field.
${}_2M'_0 = 60$, a maximum, at centre.
${}_2M_{-2\cdot5} = \tfrac{1}{5}(17{\cdot}5)^2 = 61{\cdot}25$, maximum in 2nd field, and for span.

In this example, where the distance between the wheels exactly equals the half-span, E coincides with h, fig. 148; hence A_2 will always be higher than E though an uniform load indefinitely great be compounded with it. When, however, the distance between the wheels is greater than the half-span, the diagram is as shown upon fig. 148; and if the E be higher than A_2, much more will it be higher when we add the ordinates of the parabola due to any uniform load; if E be lower than A_2, there is some uniform load which being added will make them equal in height; an uniform load greater than this being added will cause E to be higher than A_2.

174. A beam 20 feet span is subject to the transit of 5 weights, each 2 tons and fixed at intervals of 3 feet. Find the maximum bending moment at each point during the transit.

All the loads may be on the span at once, therefore $n = 5$; since the common interval is less than a fifth of the span, the locus is that for the whole load on the span (fig. 155). For the left half of span

$${}_1M_x = \frac{5w}{2c}(c + x - 2s)(c - x) = \tfrac{1}{2}(4 + x)(10 - x),$$

for values of x from 10 to 6.

$${}_2M_x = \tfrac{1}{2}(10 + x - 3)(10 - x) - 2 \times 3 = \tfrac{1}{2}(7 + x)(10 - x) - 6,$$

for values of x from 6 to 2.

$${}_3M_x = \tfrac{1}{2}(10 + x - 0)(10 - x) - 2 \times 6 - 2 \times 3 = \tfrac{1}{2}(10 - x) - 18,$$

for values of x from 2 to 0.

By symmetry the values for the right half of the span may be obtained, and the maximum for span during transit is ${}_3M_0 = 32$ ft.-tons.

175. A beam 20 feet span is subject to the transit of 5 weights, each 2 tons, and fixed at intervals of 3 feet. Find the maximum positive and negative shearing force at each foot of span during the transit. See example 174.

The slopes in fig. 157 are—1st link, $\frac{1}{10}$ vertical to 1 horizontal; 2nd, $\frac{2}{10}$ to 1; 3rd, $\frac{3}{10}$ to 1; 4th, $\frac{4}{10}$ to 1; and 5th $\frac{5}{10}$ to 1; hence upward ordinates to *Aa*..., foot by foot from left end are

1st link *Aa*... 0, ·1, ·2, ·3; 2nd link *aa*... ·3, ·5, ·7, ·9;
3rd ,, *aa*... ·9, 1·2, 1·5, 1·8; 4th ,, ,, ... 1·8, 2·2, 2·6, 3·0;
5th ,, ,, ... 3·0, 3·5, 4·0, 4·5, 5·0, 5·5, 6·0, 6·5, 7·0.

Hence the maxima are

$$\begin{cases} F_{10} = 7 \text{ and } 0, & F_9 = 6{\cdot}5 \text{ and } -{\cdot}1, \quad F_8 = 6 \text{ and } -{\cdot}2, \\ F_{-10} = 0 \text{ and } -7, & F_{-9} = {\cdot}1 \text{ and } -6{\cdot}5, \quad F_{-8} = {\cdot}2 \text{ and } -6, \end{cases}$$

$$\begin{cases} F_7 = 5{\cdot}5 \text{ and } -{\cdot}3, & F_6 = {\cdot}5 \text{ and } -{\cdot}5, \quad F_5 = 4{\cdot}5 \text{ and } -{\cdot}7, \\ F_{-7} = {\cdot}3 \text{ and } -5{\cdot}5, & F_{-6} = {\cdot}5 \text{ and } -5, \quad F_{-5} = {\cdot}7 \text{ and } -4{\cdot}5, \end{cases}$$

$$\begin{cases} F_4 = 4 \text{ and } -{\cdot}9, & F_3 = 3{\cdot}5 \text{ and } -1{\cdot}2, \quad F_2 = 3 \text{ and } -1{\cdot}5, \\ F_{-4} = {\cdot}9 \text{ and } -4, & F_{-3} = 1{\cdot}2 \text{ and } -3{\cdot}5, \quad F_{-2} = 1{\cdot}5 \text{ and } -3, \end{cases}$$

$$\begin{cases} F_1 = 2{\cdot}6 \text{ and } -1{\cdot}8, \\ F_{-1} = 1{\cdot}8 \text{ and } -2{\cdot}6, \end{cases} \quad \text{and } F_0 = 2{\cdot}2 \text{ and } -2{\cdot}2.$$

176. A beam 30 feet span is subject to the transit of 5 weights each 3 tons, and fixed at intervals of 7 feet. Find the maximum bending moment at each point.

The interval is not an aliquot part of the span, and when 5 weights (the whole load), *or* when only 4 weights, are on the span, no weight can overtake its field; the parts of the fields overtaken in each case correspond exactly with the parts not overtaken in the other.

Hence we have to find the locus $BD_1D_2 \ldots C$, when 5 weights are on the span, and also the locus $BD'_1D'_2 \ldots C$, when 4 weights are on, and take parts alternately of the two loci; the limits being found by laying off *a*, *b*, *c*, &c., *a'*, *b'*, *c'*, &c., at intervals of 7 feet from each end (see fig. 156). For left half of span

$${}_1M_x = \frac{5w}{2c}(c + x - 2s)(c - x) = \tfrac{1}{2}(1 + x)(15 - x),$$

for values of x from 15 to 13.

$${}_1M'_x = \frac{4w}{2c}(c + x - \tfrac{3}{2}s)(c - x) = \tfrac{2}{5}(4{\cdot}5 + x)(15 - x),$$

for values of x from 13 to 8.

$${}_2M_x = \frac{5w}{2c}(c + x - s)(c - x) - ws = \tfrac{1}{2}(8 + x)(15 - x) - 21,$$

for values of x from 8 to 6.

$${}_2M'_x = \frac{4w}{2c}(c + x - \tfrac{1}{2}s)(c - x) - ws = \tfrac{2}{5}(11{\cdot}5 + x)(15 - x) - 21,$$

for values of x from 6 to 1.

$${}_3M_x = \frac{5w}{2c}(c + x - 0)(c - x) - w \,.\, 2s - ws = \tfrac{1}{2}(15 + x)(15 - x) - 63,$$

for values of x from 1 to 0.

By symmetry the values for the right half of the span may be obtained. The two maxima are
$_3M_0 = 49\cdot5$ ft.-tons at the centre, and $_2M'_{1\cdot75} = 49\cdot255$ ft.-tons at a quarter interval from the centre; the former is the maximum for the transit.

Note.—The solution would be exactly the same although there were *many* weights, each 3 tons, and fixed at the same intervals, viz. 7 feet.

177. A beam 42 feet span is subject to the transit of a row of trunks 50 feet or more in length; the common interval between the wheels is 4 feet, and the weight on each wheel is 2 tons. Find the maximum bending moment for transit.

The greatest number of wheels that can be on the span at once is $n = 11$; and since $s = 4$, the distance between the wheels is not an aliquot part of the span, and the two loci for 11 and for 10 weights respectively intersect; the one has a maximum at the centre, and the other at 1 foot, a quarter interval, from the centre; so that it is only necessary to take the bending moments, one at the centre when a weight is over it, and another at 1 foot on either side of the centre when a weight is there; in each case the load being supposed to extend beyond the span on both sides. A weight at the centre gives

$$P = \frac{nw}{2}; \qquad M_0 = \frac{nw}{2} \,.\, c - ws\left(1 + 2 + 3 + \ldots + \frac{n-1}{2}\right)$$
$$= \frac{nw}{2}\,c - ws \,.\, \tfrac{1}{2}\,\frac{n-1}{2} \,.\, \frac{n+1}{2}$$
$$= \frac{w}{8}\,\{4nc - s(n^2 - 1)\} = 111 \text{ ft.-tons.}$$

When a weight is over the point 1 foot from the centre, another is just over the left point of support, and we may say that either 11 or 10 weights are on the span; that is, the two loci intersect at $x = 1$. The maximum for the 10 weights is the same as the ordinate at $x = 1$ to the locus for 11 weights, and its height is less than the above calculated value.

178. Find the locus of the bending moment at each point of the span (fig. 153), when a particular point of the wheel base of the locomotive is over it point by point. Numerical solution Ex. 170.

Let b be any point on the wheel base, or that base produced, and suppose the load standing as on fig. 153; let ordinates be drawn from the points where W_2 and W_3 (the weights on each side of b) stand to the corresponding parabolas 2 and 3, and let V and N the tops of those ordinates be joined by a straight line, which, as we have already seen, passes through D_2 the intersection of those parabolas. Let B be the point on that joining line on the same vertical with b, then the ordinate of B is the bending moment at the point over which b stands. Now by theorem, fig 95, when the load moves, the joining line, or double chord of the two parabolas turn about D_2; and since $\overline{vb}$ the horizontal projection of VB is constant, the point B traces out a parabola. This parabola passes through D_2, and is the same parabola as the others of the series; also by comparing figs. 95 and 153, it can be shown that its apex A lies to the left or right of O the centre, according as b lies to the left or right of G the centre of gravity, and at a distance given by the equation $\overline{OS} = \tfrac{1}{2}\overline{Gb}$.

Graphically.—Construct as much of the general solution as determines D_2, the junction of the parabolas corresponding to W_2 and W_3 the weights on the two sides of b. Lay off OS equal to half of $\overline{Gb}$, and draw a vertical through S; shift the parabolic segment till its apex is on that vertical, move the rollers till the curved edge passes through D_2, and draw the parabola whose apex is A. If b lies between the two end weights W_1 and W_2, the curved edge passes through D_1; if b is on the wheel base produced to the left, the curved edge passes through the extremity B. If the beam extends beyond the supports, as on fig. 144, the props being fixed and able to act upwards and downwards; then for the load in any position on the beam whatsoever, the locus BD_1D_2, &c., of fig. 153 gives the positive maxima bending moments, and the negative ordinates have the interpretation given at fig. 142.

179. In example 159, find the locus of the bending moment at each goint, as G the centre of gravity of the load comes over it (fig. 147). Data on model, fig. 149.

Put $x = 6$ in either equation, and we find the height of D to be 96 ; so that the coordinates of D are $x = 6$, $y = 96$. The parabola for G has its apex on the vertical through the centre O; its axis is vertical and it passes through D; the modulus is the load divided by the span; hence

$$_gM_x = \frac{R}{2c}(c + x - 0)(e - x) + C = \tfrac{1}{2}(324 - x^2) + C;$$

but $\quad _gM_6 = 96, \quad$ therefore $\quad 96 = \tfrac{1}{2}(324 - 36) + C, \quad$ or $\quad C = -48$;

therefore
$$_gM_x = \tfrac{1}{2}(324 - x^2) - 48,$$

for values of x from 14 to $-$ 10, that is for the whole load confined to the span.

If the load makes a transit, the above is still the locus for the central portion. For the left end portion, W_2 alone is on the span as G comes over each point; and for that portion the locus is the parabola for W_2 rolling; the apex is on the vertical through the point $x = 4$, since G is no longer the centre of gravity, but only a point lying 8 feet to the left of W_2 which is its own centre of gravity; the parabola passes through the left end, and the equation is

$$_gM_x = \tfrac{6}{36}(18 + x - 8)(18 - x), \text{ for values of } x \text{ from 18 to 14.}$$

Similarly $_gM''_x = \tfrac{12}{36}(18 - x - 4)(18 + x)$, for values of x from $-$ 10 to $-$ 18.

CHAPTER XIII.

COMBINED LIVE AND DEAD LOADS WITH APPROXIMATION BY MEANS OF AN EQUIVALENT UNIFORM LIVE LOAD.

WE already found that when the loads were all dead or fixed the joint construction of the shearing force diagram and the bending moment diagram recommended itself, the one aiding in the construction of the other. The shearing force diagram in itself, easy to draw, helped to construct the more difficult bending moment diagram.

For moving loads we continued to discuss the two diagrams together although they no longer had much in common, owing to the fact that the *position* of the load for a maximum bending moment and for a maximum shearing force at the same point was not the same position.

In the combination of live with dead loads to be dealt with in this Chapter matters are reversed, for the bending moment diagram is easily dealt with, while the shearing force diagram demands thorough investigation.

For practical purposes the *dead load* is to be reckoned as of uniform intensity per foot of span. The *live load* is also reckoned as uniform per foot of length, and as of sufficient length to cover the whole span.

The actual live load, no matter how it may be disposed, in positions concentrated on wheels regularly or irregularly spaced is to be replaced by an *equivalent uniform load.* That is, a uniform advancing load (at least as long as the span) which, in its transit, produces the same maximum bending moment at the central point of the span as the actual load produces at or near the centre. It is to be noted that although irregular load systems produce their maximum effect at a section a few feet from the centre, yet this is covered by the fact that in the practical design of girders the cross-sections at the centre and for a few feet on each side of it are identical. The equivalent load also produces the same shearing forces at the abutments as the actual load.

Again, it is necessary to augment the actual moving load to allow for the impulsive effect due to high velocities in a large proportion for short spans, and in a lesser proportion for longer spans. This we will afterwards illustrate by tables from a recent paper in the Transactions of the Institute of Civil Engineers.

The moving load when thus augmented and equivalated to a uniform intensity may now be called the *effective equivalent uniform live load,* but for brevity it will now be called simply the *live load.*

Seeing that both the *dead* and *live* loads are now uniform, the maximum bending moment will simply be a right parabolic segment standing on the span, for the *live load* must be put covering the span for maxima.

The bending moment diagram is now a parabola, the height of its vertex being *one-eighth* of the product of the *total load* and the span; where the *total load* means the sum of the dead load and of the equivalent live load covering the span. If u be the dead load per foot of span, and w the live load per foot, the equation to the bending moment diagram is

$$M_x = \frac{u + w}{2}(c^2 - x^2),$$

where $2c = l$ is the span, and x the distances of any cross-section from the centre. Now we shall investigate the important practical case of the

Shearing Force for a beam under a fixed uniform load together with an advancing load of uniform intensity (fig. 167).—Let u be the intensity of the fixed uniform load, and w the intensity of the advancing load in terms of its equivalent dead load; as in fig. 111, draw DE for the uniform dead load, and from DE as a sloping base plot the ordinates of the two parabolas (fig. 133) up and down respectively; the two loci FLE and DKG give the maximum and minimum, or the positive and negative maximum, shearing force at each point.

In fig. 133, p. 188, the parabola DC may be supposed to be drawn on the "Distorting Table" described in Chapter XI., p. 218. The table is then to be distorted till its base is at the slope DE (fig. 167) when we have FLE as the locus for the joint loads; this is still an arc of the same parabola with its vertex at A_2, for which see *Theorem A*, p. 131.

The ordinates to DKN, the portion of the curve which extends over the left half of span, are derived by taking the ordinates of the slope *OD* due to the dead load alone (fig. 111), and subtracting the ordinates of *BN* (fig. 133); or, what is the same thing, by adding the ordinates of the slope *OD* (fig. 111), and of the parabolic segment *JBN* (fig. 133), and then subtracting the constant quantity *ON*. When the ordinates of the slope and of the segment are added by the *Theorem A*, p. 131 the resulting locus is a portion of the *same* parabola as *BN*, but with its apex to the left of *B* at a distance

$$BS = a = \frac{BD}{2BJ} \cdot OB = \frac{\frac{1}{2}U}{2 \times \frac{1}{8}W} \cdot c = \frac{u}{w} \cdot 2c. \tag{1}$$

When the constant quantity *ON* is subtracted, the locus unaltered in form moves vertically downwards through that distance; the apex still remains to the left of *B* at the distance a; the curve passes through the point *D* whose height is $BD = uc$, and through the point *G* whose depth is $CG = (u + w)c$. Similarly *ELF* is symmetrical, left and right, up and down.

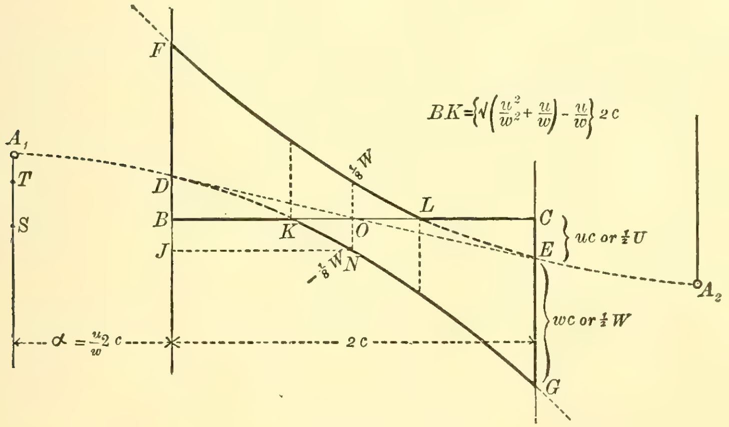


Fig. 167.

Taking the centre of span as origin, the equation to the parabola A_1DG is

$$y = -\frac{wc}{4} + \frac{2u + w}{2} x - \frac{w}{4c} x^2. \tag{2}$$

The point K is found by making $y = 0$, and finding the corresponding value of x; thus

$$OK = c - 2c\left(\sqrt{\frac{u^2}{w^2} + \frac{u}{w}} - \frac{u}{w}\right). \tag{3}$$

The ordinate to the apex A_1 is found by substituting $(c + a)$ for x in equation (2); and is

$$SA_1 = \left(\frac{u^2}{w} + u\right)c. \tag{4}$$

Graphical Solution (fig. 167).—Lay up BD and lay down CE, each equal to half the dead load. Then further lay down EG equal to half the live load. To construct as many points on the parabolic arc DKG as may be required, divide CB in any manner. In fig. 169 it is divided into twelve equal parts corresponding to the bays or panels in the girder. More correctly, however, it should have been divided by points over the *centres of the bays*, that is, it should have been divided into eleven equal central divisions and two half end divisions. Now divide EG *similarly* to the manner in which BC has been divided; then *vectors* drawn from D to the points of division of EG meet verticals drawn through the points of division of BC, and determine the corresponding points of the curve DKG. ELF is the same curve transferred by the dividers.

To the left of K the shearing force is always positive, to the right of L it is always negative, and between K and L there are both positive and negative maxima. The range at the centre is $\frac{1}{4}W$ as for W alone; if u be great compared to w, the range between K and L is nearly constant and equal to $\frac{1}{4}W$, since the apex is then far out and the portion of the parabola over KL is very flat.

The *critical points* K and L, between which the shearing stress sometimes produces distortion of the girder in one direction and at other times in the opposite direction, according to the position of the live load, are of great importance. Between these points the girder requires *counterbracing.* When the fig. 167 is drawn graphically, K and L are at once fixed; while, by calculation, the exact position of K is given by a modification of equation (3); thus

$$BK = 2c\left(\sqrt{\frac{u^2}{w^2} + \frac{u}{w}} - \frac{u}{w}\right). \tag{5}$$

This equation is so awkward and the position of K so important that we will determine its position by ascertaining the position of the live load which shall bring the common centre of gravity of the girder and the load on its left segment as near to the left abutment as possible.

The centre of gravity of the girder itself is at the middle. As the live load is pushed on from B, the common centre of gravity moves from the middle point O nearer and nearer the left abutment B for some time; then it begins to move back towards the centre O where it arrives when the whole girder is covered with the live load. For some segment such as BK loaded, the common centre of gravity of girder and load will be nearer the abutment B than for any other. This is the case when the common centre of gravity is at K the end of the loaded segment. For adding a load to the right of K will move the centre of gravity to the right of K; and removing a load to the left of K will have the like effect. To find K then it is only necessary to assume the shorter segment BK to be covered with the live load and at the same time the common centre of gravity to be at K. This is readily remembered; and can be applied at once to any numerical example.

That it leads to equation (5) is proved at once, for the supporting force at the right end can be calculated in two ways. First, we may suppose the dead load $w \,.\, 2c$ concentrated at the centre O, and the live load $w \,.\, \overline{BK}$ concentrated at the middle of BK, when the right reaction is

$$Q = uc + w \,.\, \overline{BK}\,\frac{\frac{1}{2}\overline{BK}}{2c} = uc + \frac{w}{4c}\,\overline{BK}^2.$$

Again, we may suppose the whole load $(u \,.\, 2c + w \,.\, \overline{BK})$ concentrated at K, when

$$Q = (2uc + w \,.\, BK)\,\frac{BK}{2c} = u\overline{BK} + \frac{w}{2c}\,\overline{BK}^2.$$

Equating these, and solving the quadratic equation, we have the same value for BK as that in equation (5).

Another way of considering the live load, that which Levy uses in his *Graphique Statique*, and which we follow in Chapter XXI., is to consider an *equivalent rolling load* on one wheel which will produce the same bending moment at the centre, and the same shearing force at the abutment as the maxima produced by the actual load. If R be the rolling load and U the dead load, then the bending moment diagram is a parabola, the height of the vertex being $(\frac{1}{4}R + \frac{1}{8}U)\,l$. By superposition of the diagrams, figs. 111 and 138, first distorting fig. 138, so that its base will coincide with the slope *DE*, we have the shearing force diagram, fig. 168.

Girder bridges are tested by passing back and forth over them two or three locomotives of the heaviest type and fully loaded, during which the deflections are observed. The effect is the more marked the shorter the span. With spans above 100 feet the effect is nearly the same as that calculated for a uniform load. That is the fact of the load being concentrated in parcels at the wheels may be overlooked almost. Compare figs. 155 and 156, and the formulæ referring to them.

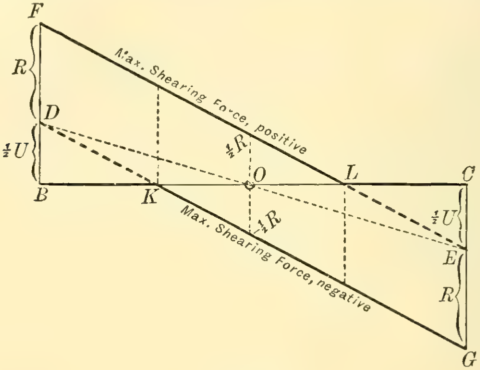


Fig. 168.

In 1890 the heaviest locomotive and tender used on the Caledonian Railway corresponded very nearly to the following—

Loads on wheels 14, 15, 15, 12, 13, 11 tons = 80 tons.
Intervals 7, 7, 9, 8, 6, 7, 6 feet = 50 feet.

Two such locomotives, the leading one running backwards, serve to test all girder bridges from the shortest spans up to a span of 100 feet. That wheel or set of wheels adjacent to each other which gives the greatest bending moment is taken for each span, and the greatest bending moment near the centre is calculated in this Chapter by the preceding rules or graphic constructions. Then the equivalent uniform load which would give a like maximum at the centre is calculated.

Thus on a span of 5 feet only, one pair of wheels could be on the bridge at one time, that is, the greatest load is the heaviest pair, namely, 15 tons of a rolling load. The equivalent uniform load is 30 tons spread or 6 tons per foot. For a span of 10 feet the 15 tons above the span and at its centre is the same as 30 tons spread or as 3 tons per foot; but when the 12 tons and 13 tons are both on the span, and placed so that the 13 tons is as far from the right abutment as their common centre of gravity is from the left abutment (see fig. 147, p. 203), in which reckon $W_1 = 12$ tons, $W_2 = 13$ tons, and $2h_1 + 2h_2 = 6$ feet; then the maximum bending moment under W_2 is to be calculated, and it will be found to be 4 per cent. greater than with the 15 tons alone at the centre.

In each case if w be the intensity of the equivalent uniform load, we must equate $\frac{1}{8}wl^2$ to the maximum bending moment for the severest position of the load.

The results are given in the following table*:—

Span in feet.	Maximum in foot-tons.	Equivalent uniform load in tons per foot.	Span in feet.	Maximum in foot-tons.	Equivalent uniform load in tons per foot.
5	$18\frac{3}{4}$	6·00	60	774	1·72
10	38	3·04	70	1010	1·65
20	104	2·08	80	1303	1·63
30	214	1·91	90	1633	1·62
40	372	1·86	100	2020	1·62
50	563	1·80			

Within the last ten years the weight of rolling stock has greatly increased, and results similar to the above, but on the most elaborate scale, are given in an excellent paper by Mr. W. B. Farr, called "Moving Loads on Railway Under Bridges," and published in the Transactions of the Institute of Civil Engineers, 1900, No. 3195. Mr. Farr has included the rolling stock of all the British and Belgian Railways. His argument too seems excellent, namely, that there is now no further possible increase of the weight on the wheels of rolling stock, so that increased weight of the rolling stock in future must be accomplished by increasing the number of wheels which would not affect the equivalent uniform load.

Another excellent feature of Mr. Farr's paper is that he gives a suitable augmentation of the equivalent uniform load on a *sliding scale* to allow for the impulse or impact of the load, so that his tables furnish at once reliable working values for the effective equivalent uniform load which are not likely to alter much in the immediate future.

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* These tabulated results were given by Dr. Thomson in a paper to the students of the Institute of Engineers and Shipbuilders of Scotland in the year 1890.

We quote here the summary of his results.

MAIN GIRDERS.

x = maximum uniform load per foot run (40 types of locomotives).

y = suggested per cent. increase for impact.

Spans.	x	y *	$x + y$
5·0 feet	7·60 tons	30·00 per cent.	9·88 tons
7·5 ,,	5·55 ,,	27·50 ,,	7·07 ,,
10·0 ,,	4·85 ,,	25·00 ,,	6·06 ,,
15·0 ,,	3·74 ,,	22·50 ,,	4·58 ,,
20·0 ,,	3·20 ,,	20·00 ,,	3·84 ,,
30·0 ,,	2·63 ,,	15·00 ,,	3·01 ,,
40·0 ,,	2·40 ,,	14·50 ,,	2·75 ,,
60·0 ,,	2·17 ,,	13·50 ,,	2·46 ,,
80·0 ,,	2·06 ,,	12·00 ,,	2·30 ,,
100·0 ,,	1·97 ,,	10·00 ,,	2·16 ,,

CROSS GIRDERS.

x = maximum concentrated loads in tons on each single line.

y = suggested per cent. allowance for impact, &c.

Distance apart.	x	y	$x + y$
3 feet	19·00 tons	50 per cent.	28·50 tons
5 ,,	19·00 ,,	45 ,,	27·55 ,,
7 ,,	19·00 ,,	40 ,,	26·60 ,,
8 ,,	21·50 ,,	35 ,,	29·02 ,,
9 ,,	23·50 ,,	30 ,,	30·55 ,,
10 ,,	25·11 ,,	25 ,,	31·39 ,,

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* We think Mr. Farr's addition too low, and suggest that y be increased throughout. See page 105; also see Fidler's *Bridge Construction*, page 241.

EXAMPLES.

180. A bridge 32 feet in span is subject to the transit of a 15 ton road roller; find the equivalent uniform load.

Here $R = 15$ tons and $l = 32$ feet, so that $M_0 = \frac{1}{4}Rl = 120$ ft.-tons. Equating this to $\frac{1}{8}wl^2$, we have *Ans.* $w = \frac{15}{16}$th of a ton per foot run.

181. A beam 36 feet in span is subject to the transit of a load on two wheels, viz. 12 tons on one and 6 tons on the other, spaced 12 feet apart. Find the equivalent uniform live load (see the model, fig. 149, p. 214).

$$ {}_1M_2 = \tfrac{18}{36} \times 16 \times 16 = 128 \text{ ft.-tons maximum.}$$

Put $$\tfrac{1}{8}wl^2 = \tfrac{1}{8}w \times 36 \times 36 = 128.$$

And $$w \text{ is a } \tfrac{64}{81}\text{th part of a ton.}$$

182. The beam 42 feet in span, shown on fig. 153, is subject to the transit of the locomotive shown on that figure. Find the equivalent uniform live load.

From the figure ${}_4M_{-3} = 261$ ft.-tons is the maximum bending moment. Equating this to $\frac{1}{8}w \times 42 \times 42$, we find w equal to $\frac{58}{49}$ tons.

183. A beam 30 feet span is subject to the transit of five weights, each 3 tons, fixed at intervals of 7 feet. Find the equivalent uniform live load.

As already calculated in Ex. 176, we have ${}_3M_0 = 49{\cdot}5$ ft.-tons. Equating $\frac{1}{8}wl^2$ to this, we find $w = {\cdot}44$ ton per foot.

184. An advancing load in length not less than the span, and of uniform intensity 3 tons per foot, passes over a beam 42 feet span. Find the maxima shearing forces positive and negative at the points $x = 21$, 14, 7, and 0.

			Positive tons.		Negative tons.		Range tons.
Ans.	F_{21}	..	63·0	..	0·0	..	63·0.
	F_{14}	..	43·75	..	1·75	..	45·5.
	F_7	..	28·0	..	7·0	..	35·0.
	F_0	..	15·75	..	15·75	..	31·5.

NOTE.—If the actual load be 3 tons per foot, these results are in terms of a moving load; if, however, we have increased the actual load to get its equivalent dead load, the results are in terms of a dead load. If the actual load were $1\frac{3}{4}$ tons per foot, the equivalent dead load would be about 3 tons per foot, and in this case the results above would be for the dead load equivalent to an actual advancing load of $1\frac{3}{4}$ tons per foot.

185. In the previous example, find the maximum value of the positive and negative shearing forces at the point $x = 7$, directly.

Let the segment to the right of the point be loaded; the amount of load will then be $28 \times 3 = 84$ tons; suppose this concentrated at the centre of the loaded segment and $+F_7 = P = (84 \div 4) \times 14 = 28$ tons; again let the segment to the left of the point be loaded, and $-F_7 = Q = (42 \div 42) \times 7 = 7$ tons.

186. The beam of example 184 is subject to an advancing dead load of 3 tons per foot, and an uniform dead load of 1 ton per foot of span. Find the maximum positive and negative value of the shearing force at intervals of 7 feet for the combined load, and the range at these points.

For the uniform load alone, the shearing forces at these points beginning at the left end are 21, 14, 7, 0, -7, -14, -21 tons. Taking the algebraic sums of these quantities, and the results given for example 184, we find that

$$\begin{aligned}
\textit{Ans.}\quad & F_{21} = 84, \; -F_{21} = 0; \quad F_{14} = 57\tfrac{3}{4}, \; -F_{14} = 0 \\
& F_7 = 35, \; -F_7 = 0; \quad F_0 = 15\tfrac{3}{4}, \; -F_0 = 15\tfrac{3}{4} \\
& F_{-7} = 0, \; -F_{-7} = 35; \quad F_{-14} = 0, \; -F_{-14} = 57\tfrac{3}{4} \\
& F_{-21} = 0, \; -F_{-21} = 84 \text{ tons.}
\end{aligned}$$

Between the two points $x = \pm 7$, we have both a positive and negative maximum; to the left of $x = 7$ the stress is always positive; to the right of $x = -7$ it is always negative.

The range of the shearing force at each of these points is respectively 84, $57\frac{3}{4}$, 35, $3\frac{1}{2}$, 35, $57\frac{3}{4}$, and 84 tons.

187. Find, for the previous example, the points between which the shearing force is sometimes positive and sometimes negative; and find the coordinates of the apex A_1. See fig. 167.

$$\textit{Ans.}\ OK = 21 - 42\left\{\sqrt{\left(\tfrac{1}{9} + \tfrac{1}{3}\right)} - \tfrac{1}{3}\right\} = 7 \text{ feet}; \ OL = -7 \text{ feet}.$$

For the apex A_1,

$$OS = x = c + \frac{u}{w}\,2c = 35 \text{ feet on horizontal scale},$$

$$SA_1 = y = \left(\frac{u^2}{w} + u\right)c = 28 \text{ tons on vertical scale}.$$

By using the coordinates of A_1, the graphical construction can be made with greater accuracy, more especially if the points A_1 and D are situated *near* each other.

188. A beam 24 feet span bears a load of 6 tons uniformly distributed, and is subject to a rolling load of 6 tons. Find the amounts and the range of the shearing forces at intervals of 2 feet.

Since the span is a little greater than 20 feet, the dead rolling load equivalent to the actual rolling load of 6 tons will be, say, 12 tons.

For the uniform load, F varies uniformly from 3 tons at the left end of span, to -3 tons at the right; for the rolling load, F varies uniformly from 12 tons to 0, positive from left to right, and negative from right to left; and we have for both loads

$$\begin{array}{llll} F_{12} = 15, & -F_{12} = 0; & F_{10} = 13{\cdot}5, & -F_{10} = 0; \\ F_8 = 12, & -F_8 = 0; & F_6 = 10{\cdot}5, & -F_6 = 1{\cdot}5; \\ F_4 = 9, & -F_4 = 3; & F_2 = 7{\cdot}5, & -F_2 = 4{\cdot}5; \\ F_0 = 6, & -F_0 = 6 \text{ tons}. & & \end{array}$$

The results for the right half of the span are similar to the above, the signs alone being changed.

The range at intervals of 2 feet for the left half of the span, beginning at the point of support, is 15, 13·5, 12, 12, 12, 12, and 12 tons.

189. A beam 24 feet span bears an uniform dead load of 100 lbs. per foot, and is subject to an advancing load as long as the span, and of intensity 400 lbs. per foot. Find the shearing forces, and their range, on the left half of span, at intervals of 2 feet. Find the *critical* point K, and the coordinates of the apex A_1. See fig. 169.

Since the span is a little greater than 20 feet, the dead advancing load equivalent to the actual advancing load of 400 lbs. per foot will be, say, 800 lbs. per foot.

Taking this value, we have

$$u = 100, \quad \text{and} \quad w = 800 \text{ lbs.}; \quad c = 12 \text{ feet}.$$

Substituting in equation (2), p. 255,

$$\begin{array}{llllll} F_{12} = 10800, & -F_{12} = 0; & F_{10} = 9067, & -F_{10} = 0; & F_8 = 7467, & -F_8 = 0; \\ F_6 = 6000, & -F_6 = 0; & F_4 = 4667, & -F_4 = 667; & F_2 = 3467, & -F_2 = 1467; \\ F_0 = 2400, & -F_0 = 2400 \text{ lbs.} & & & & \end{array}$$

The range at intervals of 2 feet beginning at left end of span is 10800, 9067, 7467, 6000, 5334, 4934, and 4800 lbs.

$$OK = 6 \text{ feet}; \qquad OS = 15 \text{ feet}; \qquad SA_1 = 1350 \text{ lbs}.$$

CHAPTER XIV.

RESISTANCE, IN GENERAL, TO BENDING AND SHEARING AT THE VARIOUS CROSS-SECTIONS OF FRAMED GIRDERS AND OF SOLID BEAMS.

FRAMED beams are built of pieces either freely jointed, or so slightly connected at their joints that they may be considered as freely jointed. If the load be applied at the joints, each piece is either a strut or a tie; and if the load be applied at intermediate points on, or distributed over a portion of, a piece, it (the load) is to be replaced by a pair of equivalent forces at the ends, evidently equal and opposite to the supporting forces of that piece looked upon as a short beam. Such a piece has two duties to perform, viz., to resist the bending moments and shearing forces as a small beam, and to act as a strut or tie in the built beam considered as loaded at the joints only; by making such pieces short enough, the duty they have to discharge as beams can be made so small compared to what they have to perform as struts or ties, that when designed to fulfil the latter duty they will also be able to fulfil the former; by making the pieces continuous at their joints, they are greatly strengthened for their duties as short beams.

If a cross-section cuts not more than three pieces, the unknown stresses are not more than three in number, and the three conditions of equilibrium (page 120) enable us to calculate the stress on each; or using the bending moment and shearing force diagrams, we can readily apply the conditions to as many sections as necessary, and so design the whole beam. The sections chosen for this purpose should lie just to one side of the joints for bending moment diagram, but for the shearing force diagram they should more correctly be at the centre of the bays, as the stress on a diagonal is constant and equal to the shearing force at centre of the bay multiplied by the secant of its slope. The variations of the shearing force for the half bay on each side of the centre is borne by the boom or stringer bridging across from joint to joint.

For any given system of loading a beam is said to be of uniform strength when the cross-section of each piece is such

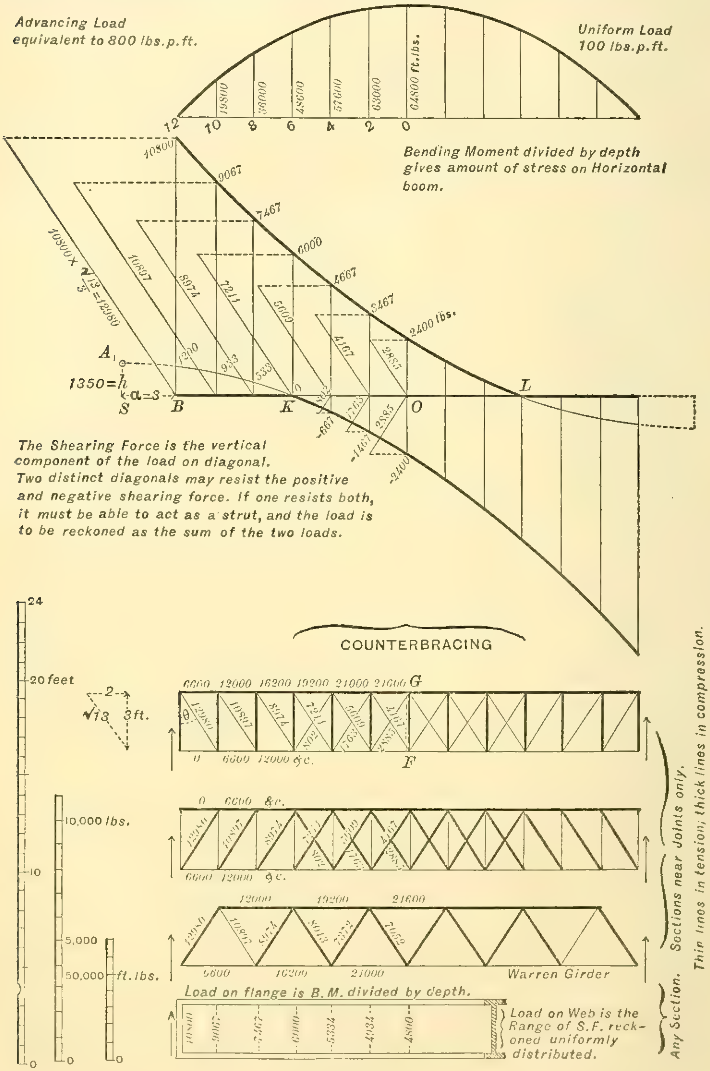

Advancing Load
equivalent to 800 lbs.p.ft.
Uniform Load
100 lbs.p.ft.
19800
36000
48600
57600
63000
64800 ft.lbs.
12
10
8
6
4
2
0
10800
9067
7467
6000
4667
3467
2400 lbs.
Bending Moment divided by depth
gives amount of stress on Horizontal
boom.
10800 × √13/3 = 12980
10897
8974
7211
5609
4167
2885
1200
933
533
0
302
763
2885
-667
-1467
-2400
A
1350=h
a=3
S
B
K
O
L
The Shearing Force is the vertical
component of the load on diagonal.
Two distinct diagonals may resist the positive
and negative shearing force. If one resists both,
it must be able to act as a strut, and the load is
to be reckoned as the sum of the two loads.
24
20 feet
10
0
2
√13
3 ft.
10,000 lbs.
5,000
0
50,000 ft.lbs.
0
COUNTERBRACING
6600 12000 16200 19200 21000 21600 G
0 6600 12000 &c.
F
0 6600 &c.
6600 12000 &c.
12000 19200 21600
6600 16200 21000
Warren Girder
Load on flange is B.M. divided by depth.
10800
9067
7467
6000
5334
4034
4800
Load on Web is the
Range of S.F. reckoned uniformly
distributed.
Sections near Joints only.
Any Section.
Thin lines in tension; thick lines in compression.

Fig. 169.

that the ratio of the ultimate or proof resistance of the material and the tension or compression which it has to bear is constant, an allowance being made, if necessary, for pieces which also act as small beams.

The term “flanged girder” is employed to denote all girders consisting of a web and one flange, or of two flanges connected together either by a continuous web or by open lattice work; in bridges, one at least of the flanges is usually straight, and also horizontal.

In figs. 84 and 85 the stress at A is horizontal and is denoted by p_a; if the flange is thin, as is often the case in iron bridges, this stress is sensibly constant; and since the intensity of the stress to which a piece is exposed should not exceed the strength of its material, we have

$$p_a \leqq f, \tag{1}$$

where f is the working or proof stress as may be desired; and if t = amount of stress on the horizontal flange, and S = the cross-sectional area of that flange, then

$$t \leqq S . f. \tag{2}$$

When there is only one set of triangles, as is shown in the lowest of the three systems of bracing (fig. 169), and also in the two upper systems if we neglect the counter-bracing, the amount of stress on the straight boom may be found thus:—Take a cross-section at a point, say F, just on either side of a joint in the boom opposite the straight boom (in fig. 169 both booms are straight); take moments round the point F, and since we neglect the counter-brace, the only member not passing through the point F is the upper boom; the product $t . h$ gives the moment where h is the depth of the beam at the point; this is equal to the bending moment, and we have by substituting the value of t given in equation (2)

$$t . h = S . f . h = M. \tag{3}$$

The quantity M is different at different points of the beam, and f is a constant quantity; if the above equation is to be fulfilled for every point, we make $S . h$ vary as M; in practice one of these two factors is generally kept constant, and the other is made to vary.

If we make h *constant*, then both booms are horizontal, and S varies as the bending moment; hence the bending moment diagram gives, upon a suitable scale, the area of the boom at each point. The vertical component of the stress on any diagonal is the amount of the shearing force at that end of the diagonal where the shearing force is greatest; hence if τ

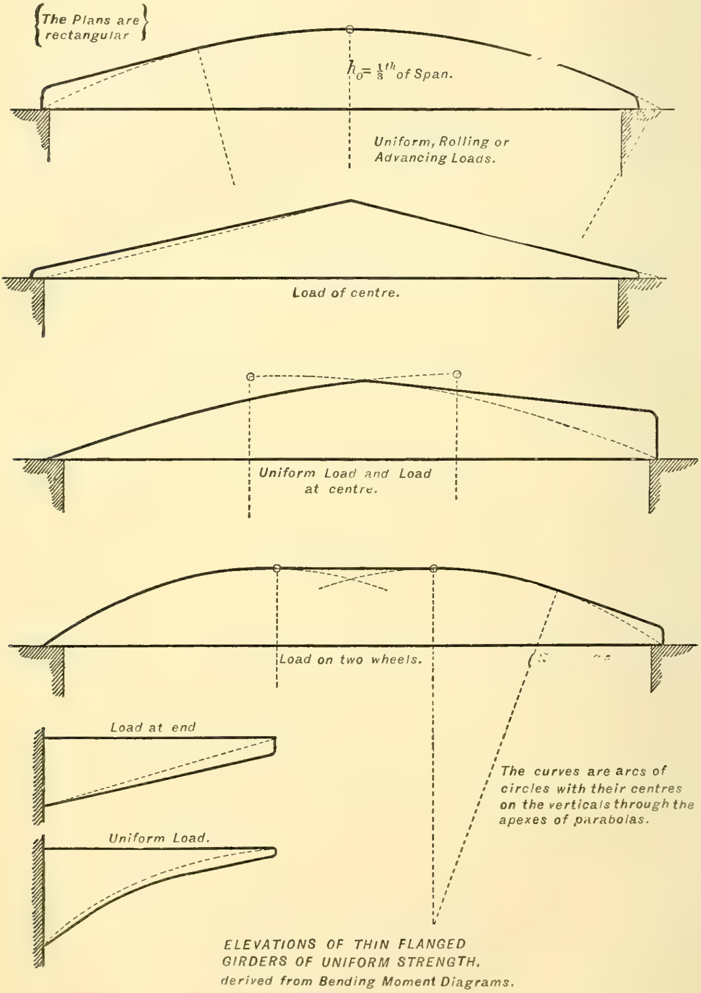

The Plans are rectangular
$h_0 = \frac{1}{8}^{th}$ of Span.
Uniform, Rolling or Advancing Loads.
Load of centre.
Uniform Load and Load at centre.
Load on two wheels.
Load at end
Uniform Load.
The curves are arcs of circles with their centres on the verticals through the apexes of parabolas.
ELEVATIONS OF THIN FLANGED GIRDERS OF UNIFORM STRENGTH.
derived from Bending Moment Diagrams.

Fig. 170.

be the stress on a diagonal, F the shearing force at the end of the diagonal where it is greatest and as given by the shearing force diagram, and θ the angle made by the diagonal with the vertical, then

$$\tau = F \sec \theta. \tag{4}$$

The depth h is chosen from $\frac{1}{8}$th to $\frac{1}{14}$th of the span to ensure stiffness; in fig. 169, h is taken at 3 feet, that is $\frac{1}{8}$th of span, and that figure shows how the stresses on the booms and on the diagonals are found. Between the points K and L, counterbracing is required; this is accomplished in the two upper girders by introducing *pairs* of diagonals between these points, both being ties or both struts as the case may be; the one diagonal resists the positive, the other resists the negative shearing force. In the Warren girder, the third in the figure, one diagonal resists both the positive and negative shearing force, the one being applied suddenly after the other; each diagonal should be designed to bear the stress due to the sum of these stresses. The lowest girder shown in the figure is one with thin flanges and a thin continuous web; part of the bending moment is resisted by the web and part of the shearing is resisted by the flanges, these parts, however, are small; practically the flanges are considered, as in the case of open beams, to resist the whole of the bending moment, and the web is considered to resist the whole of the shearing force; an approximate result is obtained if we consider that the shearing force is uniformly distributed over the cross-sectional area of the web.

If we make S *constant*, then h varies as M; and (fig. 170) the elevation of the beam will correspond with the bending moment diagram; h_0 the depth of the beam at the centre is, as in the previous case, to be taken sufficient to ensure stiffness. The curved boom will bear a share of the shearing force; this compounded with the stress on the horizontal boom at the same section will give the resultant stress on the curved boom. It is usual in practice to make the area of the curved and of the straight booms uniform, and to make the diagonals sufficiently strong to resist the whole of the shearing force as in the previous case. Where the curved member slopes considerably, as in the bowstring girder, it is made sufficiently strong to bear the whole shearing force, the diagonals being intended for another purpose, viz., to distribute partial loads in a sensibly uniform manner.

The theoretical elevations reduce to a height zero at the ends, and so give no material to resist the shearing force at the point where it is greatest; sufficient material is generally allowed at the ends either by making the span of the girder exceed the clear span, or by departing from the theoretical form along a tangent near the end. Further, whatever the curves may be, and, as we have seen, they are generally parabolas, they are usually replaced by circles which nearly coincide therewith ; when the figure passes from one curve to another, the passage is made along a tangent as will be seen on some of the figures. Approximate forms consisting entirely of straight lines enveloping the bending moment diagram, are sometimes adopted.

Moment of Resistance to Bending of Rectangular and Triangular Cross-Sections.

The moment of resistance to bending we have defined as the moment of the total stress upon the cross-section about any point in it; and this we have shown (figs. 83, 84, 85) to be equal to the couple which is the moment of the normal stress on the cross-section.

The stress (fig. 84) might be artificially produced by building on the portion $O'A$, columns of a material tending to gravitate towards the left; and on the portion $O'B$, columns gravitating towards the right. These columns standing on very small bases, being of uniform density, and of the proper height to produce the intensity at each point, will, if we suppose them to become one solid, form a wedge with a stepped or notched sloping surface; the more slender the columns are, the more accurately do they give the stress at each point, and the smaller are the notches on the wedge; hence *two right wedges* exactly represent the normal stress on the cross-section. Such a stress is called an *uniformly varying stress.* Taking the density of the wedges as unity, the height of one will be expressed by p_a, and the other by p_b ; the volumes of the wedges will give the two normal forces (fig. 86) the resultants of the thrusts and tensions respectively; these forces are equal, the volumes of the two wedges must therefore be equal, and the position of the neutral axis of the cross-section is thus determined. Further, each wedge, instead of distributing its weight over its base, may be supposed to stand on the point below its centre of gravity; this enables us to find the positions of the normal forces (fig. 86), and gives us the arm of the couple; if we multiply the volume of either wedge by this arm, we have M the moment of resistance to bending.

For a *rectangular cross-section*, the neutral axis is at the centre since the wedges are equal, and p_a equals p_b; and if the common volumes of the right wedges be represented by V, then

$$V = \tfrac{1}{2}p_a \times \tfrac{1}{2}bh = \tfrac{1}{4}p_a bh;$$

where b is the breadth and h is the depth of the beam, as shown in fig. 77. Each wedge stands on a rectangular base, so that the point on the cross-section below its centre of gravity is distant from O' by two-thirds of $O'A$, or $\tfrac{1}{3}h$; hence the arm of the couple is $\tfrac{2}{3}h$, and

$$\mathsf{M} = \tfrac{1}{4}p_a bh \times \tfrac{2}{3}h = \tfrac{1}{6}p_a bh^2.$$

By increasing the bending moment we can increase M till p_a becomes equal to f, the resistance of the material to direct tension or thrust, but no further; because if p_a becomes greater than f, the fibres at the skin will be injured; hence

$$\mathsf{M} = \tfrac{1}{6}fbh^2 \tag{1}$$

is the ultimate, proof, or working resistance to bending, according as f is the ultimate, proof, or working strength of the material. Since f is generally expressed in tons or lbs., per square *inch*, it is necessary to express b and h also in *inches*; in which case M will be in *inch*-tons or lbs.; on the other hand, M is generally expressed in *foot*-tons or lbs.; and it is well to observe that, if such be the case, M requires to be multiplied by 12 to reduce it to *inch*-tons or lbs. before equating to M.

For all cross-sections, if b and h are the dimensions of the circumscribing rectangle, the equation giving the moment of resistance is of the same form, but the numerical coefficient assumes values other than $\tfrac{1}{6}$, as will be proved hereafter. Rankine uses n for the value of this constant, which he calls the numerical coefficient of the moment of resistance to bending of any cross-section and we may put

$$\mathsf{M} = nfbh^2.$$

We shall now verify this for a triangular cross-section; an isosceles triangle is taken, but exactly the same result would be obtained for any triangle; the angle A, however, must not be too acute.

Triangular cross-section (figs. 171 and 172).—Assume that the neutral axis passes through O the centre of gravity of the triangle, and divide the part below into three areas as shown; put S and V with suffixes for the areas and volumes of the

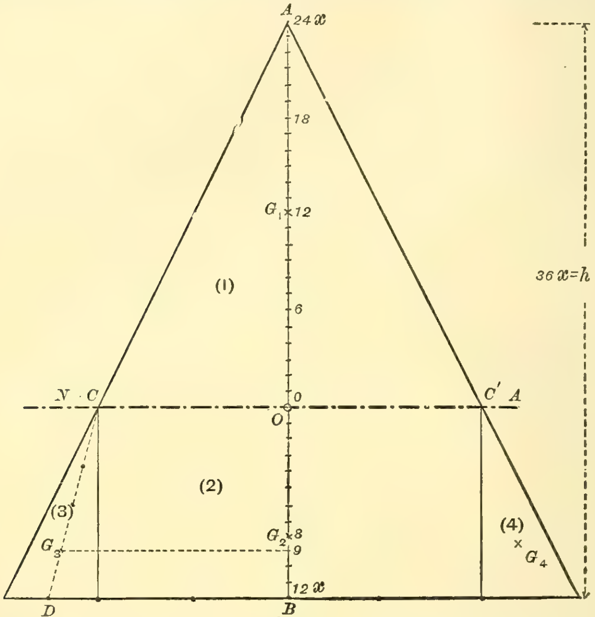


Fig. 171.

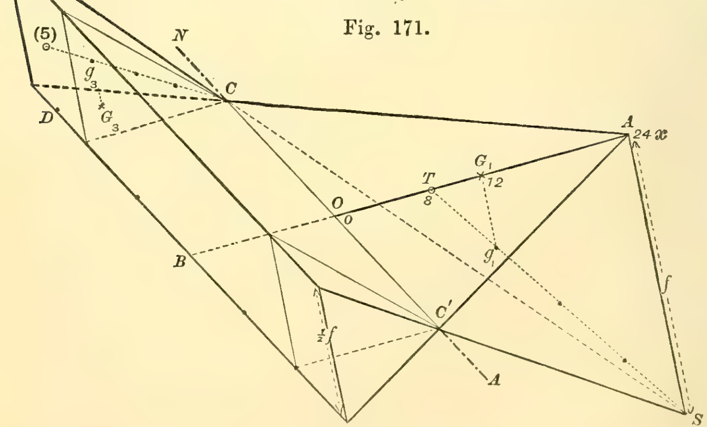


Fig. 172.

different parts, put l for the leverages about the neutral axis of their centres of gravity, and, for convenience, suppose $h = 36x$, and $b = 6y$.

V_1 is a pyramid on S_1 as a base, and let f be its height; V_2 is a right prism of height $\frac{1}{2}f$; V_3 is a pyramid on S_5 as base and of height $12x$. Now the centre of gravity of a pyramid is on the line joining the apex with the centre of base, and at three-quarters of the length of that line from the apex; and

$$S_1 = \tfrac{1}{2} . 24x . 4y = 48xy; \quad S_2 = 12x . 4y = 48xy; \quad S_5 = \frac{f}{2} y.$$

$$V_1 = \tfrac{1}{3} f S_1 = 16fxy; \qquad V_2 = \tfrac{1}{2} . \frac{f}{2} . S_2 = 12fxy;$$

$$V_3 = \tfrac{1}{3} . 12x . S_5 = 2fxy = V_4;$$

hence $V_1 = V_2 + V_3 + V_4$, and therefore the assumption as to the position of the neutral axis is correct.

Now

$$l_1 = 12x, \quad l_2 = 8x, \quad l_3 = l_4 = 9x;$$

and

$$\begin{aligned} \mathsf{M} &= V_1 l_1 + V_2 l_2 + V_3 l_3 + V_4 l_4 \\ &= (192 + 96 + 18 + 18) fyx^2 = 324 fyx^2. \end{aligned}$$

Substituting $y = \dfrac{b}{6}$, and $x^2 = \left(\dfrac{h}{36}\right)^2 = \dfrac{h^2}{1296}$,

we have

$$\mathsf{M} = \tfrac{1}{24} f b h^2, \tag{2}$$

so that for a triangular section, $n = \frac{1}{24}$.

Solid beams are sometimes made of *uniform section*; that is, at the point of maximum bending moment the section is made sufficient to resist the bending moment, and this section is adopted along the entire length; at every other section, therefore, the beam is too strong. This is frequently done with small timber beams cut out of one piece, because the material in excess, even if cut away, would be lost; the weight of this excess is very little since timber is light, and probably the cutting of it away would add to the expense of the beam.

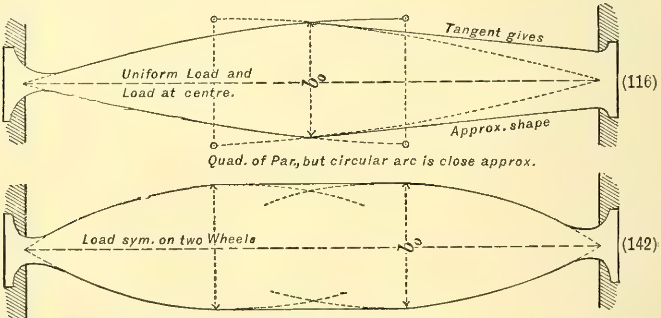
When the section of an uniform beam is designed to resist the maximum bending moment, it is generally many times more than sufficient to resist the maximum shearing force.

Solid rectangular beams are designed of uniform strength to economise material, and to reduce the weight when the material itself is heavy.

Rectangular beam of uniform strength and uniform depth.—In this case the breadth is varied so as to make the cross-section at every point just sufficient to resist the bending moment. It is evident that the elevation is a rectangle, while the plan will vary according to the load; the plan however is to be symmetrical about a centre line. The theoretical shape of the plan so designed has to be departed from near the ends, so as to make the end sections large enough to resist the shearing force; and some additional material is required at the ends of the beam to give it lateral stability; now

$$njbh^2 = M; \quad \text{therefore} \quad b = \frac{M}{nfh^2},$$

or b is proportional to M, since n, f, and h are constant quantities.



PLANS OF RECT. BEAMS OF UNIFORM STRENGTH & UNIFORM DEPTH SAME AS B. M. DIAGRAMS.

Fig. 173.

Hence the half plan is the bending moment diagram reduced so that its highest ordinate is equal to $\frac{1}{2}b_0$, the half breadth of the cross-section which can resist the maximum bending moment (see figs. 170 and 173). As has previously been explained, the parabolas being very flat are readily replaced by arcs of circles with their centres on the axes of the parabolas.

In designing such beams, then, the uniform depth is fixed as a fraction of the span from an eighth to a fourteenth to ensure the required degree of stiffness; b_0, the breadth of the cross-section where the bending moment is a maximum, is calculated to make that section sufficiently strong, exactly as in the preceding examples; then the bending moment diagram is reduced till its highest ordinate is $\frac{1}{2}b_0$, and drawn on both sides of a central line. All the curves may now be replaced by circular arcs, and all sudden changes of breadth bridged over by tangents; the curves are to be departed from, near the ends, to make the sections there strong enough to resist the shearing force. Otherwise, the breadths may be calculated at a number of sections and plotted, and a fair curve drawn through them, &c.

Rectangular beam of uniform strength and uniform breadth.—It is evident that the plan is a rectangle; and since $nfbh^2 = M$, we have h proportional to $\sqrt{M}$. Hence the elevation of the beam is obtained by degrading the bending moment diagram, so that the derived figure has its highest ordinate equal to h_0, the depth required to ensure stiffness; b is then to be made sufficient to resist the maximum bending moment (see *Theorems F*, *G*, and *H*, figs. 96, 97, 98, pages 134 and 135).

The figures in brackets refer to the corresponding bending moment diagrams.

Fig. 174 shows what the bending moment diagrams (107), (108), (111), (114) become when degraded by the preceding theorems. The elliptical elevation of a beam for an uniform load is readily struck from three centres, as shown in fig. 98; AD from a centre on AO and with a radius

$$r_2 = \frac{OB^2}{OA} = 60^2 \div 24 = 150;$$

BC from a centre K with radius

$$r_1 = \frac{OA^2}{OB} = 24^2 \div 60 = 10 \text{ nearly};$$

E is found by drawing OE from the first centre, and a circle about K with a radius

$$r_3 = h_0 - r_1 = 24 - 10 = 14;$$

further, if we choose we may retain the single circular arc AD, and depart along the tangent at D. For a beam with the load at the centre, the two parabolas may be replaced by their tangents; this gives an approximate form, and the beam will now consist of two straight portions tapering so that the depth at each side is $\frac{1}{2}h_0$, h_0 being the depth at the middle (see fig. 90); the area of the theoretical elevation is two-thirds, while that of the approximate elevation is three-quarters, of a rectangle of height h_0; these areas are as 8 to 9; hence the volume of the approximate form is only one-eighth in excess of the theoretical one, and the additional material is well placed to resist shearing; the approximate form is in some cases preferable, since it has the great advantage of straight boundaries. The same remarks apply to the elevation of a cantilever with the load at the end; on this principle spokes of wheels, when of uniform thickness, taper to half the depth from boss to tyre. More particularly in timber beams and cantilevers are the straight boundaries desirable. The material is light, and the extra wood at the ends holds the bolts and fastenings securely.

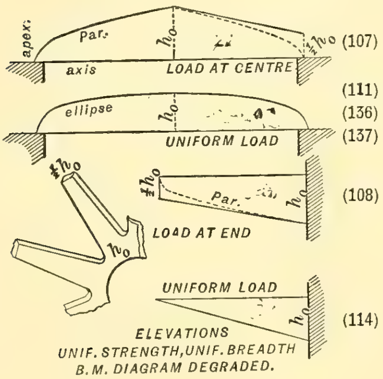


Fig. 174.

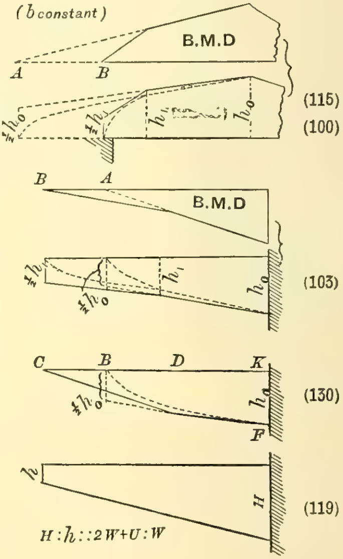


Fig. 175.

In fig. 175 are shown the effects of degrading such bending moment diagrams as (100), (103), (115), diagrams consisting of straight slopes. Producing the slopes to meet the base as at *A*, *B*, &c., the elevation for beams of uniform strength and uniform breadth will consist of a series of parabolas with their apexes at *A*, *B*, &c., and intersecting on the lines of action of the concentrated weights.

An approximate elevation is to be derived thus:—Lay up h_0 at the point of greatest bending moment, the proper fraction of the span to ensure stiffness, and calculate the breadth of that cross-section as in the preceding examples, so as to give the proper resistance to bending there; from the top of h_0, draw a tangent to the parabola whose apex is *A*, that is, draw the line which intercepts $\frac{1}{2}h_0$ on the vertical through A, and it will cut off h_1 on the line of nearest weight; from the top of h_1 draw a slope to cut off $\frac{1}{2}h_1$ on the vertical through B, the apex of second parabola and the point where the second slope of the bending moment diagram meets the base, this will be parallel to the tangent to that parabola, &c., &c.

It is evident then that these approximate elevations for concentrated loads consist of straight lines, each sloping at *half the rate* of the corresponding side of the bending moment diagram; from this fact they are readily drawn thus:—From the highest point in the bending moment diagram draw a line at half the slope of the adjacent side till it cuts the line of the weight nearest to that point; from the point thus found draw a line to cut the next weight at half the slope of the corresponding side of the bending moment diagram, &c.; lastly, reduce the ordinates so that the highest is h_0.

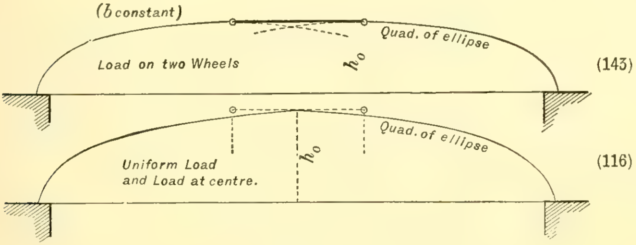


Fig. 176.

When fig. 130 is degraded, from K to D it will be a portion of the parabola whose apex is at B the centre of load, and from D to C a straight slope; the approximate form is a straight slope to D the end of load, and which, when continued, tapers to $\frac{1}{2}h_0$ at B; then from D a straight slope tapering to zero at the free end. On degrading the two parts of fig. 119 separately, that for U will be a taper from a at the fixed end to zero at the free end; while the approximate form for W will be a taper from b at the fixed end to $\frac{1}{2}b$ at the free end; hence the approximate elevation for the combined load is a taper from $(a + b)$ at the fixed end to $\frac{1}{2}b$ at the free end; substituting for a and b their values, we have $H : h :: 2W + U : W$.

Fig. 176 shows (116) and (143) degraded, the quadrants of parabolas becoming quadrants of ellipses.

In those bending moment diagrams which consist of a series of intersecting parabolic segments all of one modulus, the degraded figures will consist of a series of intersecting semi-ellipses on the same base line, intersecting on the same verticals, and having their heights proportional to their bases; this follows from theorem at fig. 98, or at once from the fact that the heights of the parabolic segments were proportional to the squares of their bases; in some figures it will be necessary to bridge over gaps at junctions of pairs of ellipses. The following is an easy method (fig. 177) of degrading these diagrams, and as an example we will take fig. 147; from S_1 with radius S_1B, describe the circular arc BA_1D; from S_2 with radius S_2D, draw arc DA_2C; draw mn the common tangent to the two circles, and $BmnC$ is the degraded figure; it only remains now to reduce the ordinates of m, A_1, n, A_2, so that the highest is h_0, when mn will still be a straight line and a common tangent to mB and nC, which will be arcs of ellipses whose centres are S_1 and S_2.

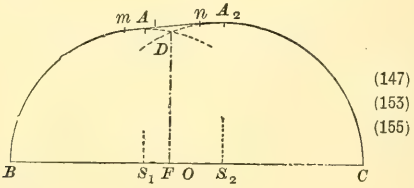


Fig. 177.

The same method applies to fig. 153, and in drawing the intermediate elliptic arcs, they will be struck from one centre as they will not extend past the portion AD in fig. 98; the end arcs will be struck from three centres, as also shown in fig. 98. In fig. 177, we may depart along tangents to the circles near B and C; in which case, when we reduce the ordinates, *all* the elliptic arcs may be struck from one centre, and the elevation will consist of straight lines and circular arcs; the radii of the circular arcs being calculated like r_2 in fig. 98, and their centres lying in the axes of the parabolas in the bending moment diagram.

The same construction degrades fig. 120, and here no bridging over is required; thus from D_1 with radius D_1B describe a circular arc BE; from D_2 with radius D_2E describe arc EF, &c., &c., and reduce the ordinates so that the highest is h_0.

When fig. 126 is degraded, BAE becomes an elliptic arc, and EC a parabola with its apex at C and its axis horizontal; the two curves have a common tangent at E; the parabola EC will be replaced by the tangent at E, which will cut off on the vertical at C one-half of FE; the ordinates of the figure are readily reduced. Otherwise, from D with radius DB describe a circular arc BAE; draw a tangent from E to meet the vertical through C, and reduce the ordinates.

In the same way fig. 124 can be degraded.

In degrading such figures as (128), (148), (156), it is to be remembered that the parabolas have not a common modulus. For instance, in degrading (148) each of the parabolas 1 and 2 is to be replaced by a semi-ellipse, whose height is proportional to the product of the base of the corresponding parabolic quadrant into the square root of its modulus; while BEC is to be replaced by a semi-ellipse, whose height is proportional to the product of its base into the square root of *its* modulus. Now the common modulus of 1 and 2 is $\frac{R}{2}$, while that of BEC is $\frac{W_2}{2c}$; hence for each semi-ellipse we have

$$\text{height of No. 1} : \text{height of No. 2} : \text{height of ellipse } BEC :: R \,.\, BS_1 : R \,.\, CS_2 : W_2 \,.\, OB.$$

Rectangular beam of uniform strength and similar cross-section.—In this case both b and h vary, but they bear to each other a constant ratio; it is evident that the plan and elevation will have the same form. The plan is always to be symmetrical about a centre line; the elevation may either have one straight boundary, or be symmetrical about a centre line.

Since $\qquad b \propto h, \quad \text{then} \quad bh^2 \propto b^3, \quad \text{or} \quad h^3;$

now $\qquad nfbh^2 = M, \quad bh^2 \propto M;$

therefore $\qquad b \text{ and } h \propto \sqrt[3]{M}.$

That is, both plan and elevation are derived by drawing a locus whose ordinates are proportional to the cube roots of those of the bending moment diagram.

For bending moment diagrams with straight slopes, that is, for concentrated loads, the degraded figures were parabolas with axes horizontal, &c.; in the same way, when the new figure is made with its ordinates proportional to the cube roots of the ordinates of the straight slopes, it becomes what is called a cubic parabola; a property of this curve is that the tangent cuts off *two-thirds* of the ordinate upon the vertical through the apex, instead of *one-half* as in the case of the common parabola. Hence figs. 174 and 175 give approximate elevations and half plans of beams of uniform strength and similar cross-sections, if we use $\frac{2}{3}h_0$ instead of $\frac{1}{2}h_0$ in making the construction; or if we draw the tapers at two-thirds of the slopes of the bending moment diagram instead of one-half. Observe that in this case there is a double taper in planes at right angles to each other.

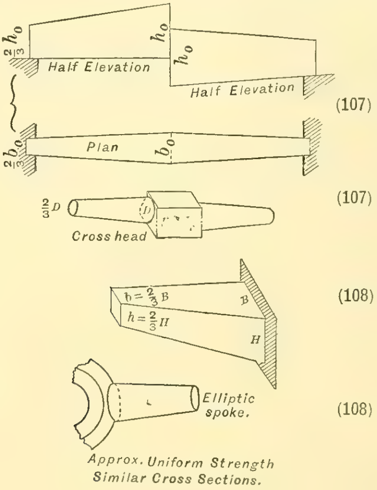


Fig. 178.

On this principle the crosshead of a piston-rod may have a conical taper, so that the diameter at each end may be two-thirds of the diameter at the centre; and the spokes of wheels may have a conical taper from the boss to two-thirds (linear dimensions) at the tyre.

Examples.

190. Find the working moment of resistance to bending of a rectangular section, 10 inches deep and 3 inches broad, the working strength of the material being 4 tons per square inch.

$$\mathrm{M} = nfbh^2 = \tfrac{1}{6} \,.\, 4 \,.\, 3 \,.\, 100 = 200 \text{ inch-tons.}$$

Find the same for an isosceles cross-section inscribed in the above rectangle, and with the base horizontal.

Ans. $\mathsf{M} = nfbh^2 = \frac{1}{24} \,.\, 4 \,.\, 3 \,.\, 100 = 50$ inch-tons.

191. Find suitable dimensions for a cast-iron beam 20 feet span, of uniform and rectangular cross-section, and subject to a load of 10 tons at the centre.

Taking $h = 20$ inches, a twelfth of the span, to ensure stiffness; $f = 2$ tons per square inch; and $\mathsf{M} = M_0 = mWl$, the maximum bending moment.

Ans. $\frac{1}{6} \,.\, 2 \,.\, b \,.\, 20^2 = \frac{1}{4} \,.\, 10 \,.\, (12 \times 20)$; $\therefore\ b = 4{\cdot}5$ inches.

192. If the breadth be taken at 6 inches, what depth would give sufficient strength?

Ans. $\frac{1}{6} \,.\, 2 \,.\, 6 \,.\, h^2 = \frac{1}{4} \,.\, 10 \,.\, (12 \times 20)$; $\therefore\ h = 17{\cdot}3$ inches.

193. If the load is uniformly distributed, and the cross-section is a triangle whose base is horizontal and 8 inches broad, find the height of the triangle.

Ans. $\frac{1}{24} \,.\, 2 \,.\, 8 \,.\, h^2 = \frac{1}{8} \,.\, 10 \,.\, (12 \times 20)$; $\therefore\ h = 21{\cdot}2$ inches.

194. Find the greatest cross-section for a wrought-iron beam of rectangular section and 15 feet span, to bear a load of 20 tons uniformly distributed, together with a load of 5 tons at the centre. Take $f = 4$ tons per square inch, and $h = 15$ inches to give sufficient stiffness.

Ans. $\frac{1}{6} \,.\, 4 \,.\, b \,.\, 15^2 = (\frac{1}{8} \,.\, 20 \,.\, 15 + \frac{1}{4} \,.\, 5 \,.\, 15) \times 12$; $\therefore\ b = 4{\cdot}5$ inches.

195. Taking the depth one-twelfth of the span, and $f = 4$ tons per square inch, find the breadth for a wrought-iron beam of rectangular section, to resist the maximum bending moment in Ex. 160.

Ans. $\frac{1}{6} \times 4 \times b \times 36^2 = 112\frac{1}{2} \times 12$; $\therefore\ b = 1{\cdot}56$ inches.

196. Design a rectangular cantilever 10 feet long of approximately uniform strength and of uniform breadth, of timber whose working strength is 1 ton per square inch; the load is 2 tons at the free end, and $h_0 = 15$ inches, an eighth of the length.

Ans. $nfbh_0{}^2 = m \,.\, W \,.\, l$; $\frac{1}{6} \,.\, 1 \,.\, b \,.\, 15^2 = (1 \,.\, 2 \,.\, 10)\ 12$, gives $b = 6{\cdot}4$ inches for uniform breadth; the depth tapers from 15 inches at fixed end to 7·5 inches at free end.

197. If an additional uniform load of 4 tons be added, and h_0 be still retained 15 inches, find the value now of b, and of h at the free end for Ex. 196.

Ans. At the fixed end the bending moment will be double its former amount, so that b will be doubled; that is, $b = 12{\cdot}8$ inches, and $h_0 : h :: 2W + U : W :: 15 : h :: 2 \times 2 + 4 : 2$; $\therefore\ h_{10} = 3{\cdot}75$ ins.

198. A wooden cantilever 12 feet long bears 3 tons uniformly distributed on the half next the free end. Design an approximate elevation, supposing the breadth to be uniform, and $f = 1$ ton per square inch; take depth at fixed end as 18 inches.

Ans. $M_0 = 27$ foot-tons $= 324$ inch-tons; $\frac{1}{6} \,.\, 1 \,.\, b \,.\, 18^2 = 324$; $\therefore\ b = 6$ inches constant.

Between the fixed end and the end of the load next thereto, the elevation will taper at the rate of 1 inch per foot, and thence to zero at the free end; that is, $h_0 = 18$, $h_6 = 12$, and $h_{12} = 0$ inches.

199. Design a cantilever for Ex. 196, supposing its section to be a square, taking dimensions to the nearest whole number in inches.

> *Ans.* Put $b_0 = h_0 =$ side of square at fixed end, and equating $nfb^3 = m \cdot W \cdot l$, gives $b_0 = h_0 = 11{\cdot}3$, say 12 inches. The side of the square at the free end is $b_{12} = h_{12} = 8$ inches.

200. Design a beam whose span is 20 feet, of timber whose strength is 1 ton per square inch; the beam to be of uniform breadth and approximately uniform strength; the loads are 4, 3, and 4 tons at points 5, 12, and 16 feet respectively from the left end.

> *Ans.* $P = 5$, $Q = 6$ tons; in the bending moment diagram, beginning at the left end, we have the slopes as follows:—

	Vert.		Horiz.	
$s_1 = P \quad . \quad . \quad .$	$= 5$	to	1	up to right,
$s_2 = P - W_1 \quad . \quad .$	$= 1$	to	1	up to right,
$s_3 = P - W_1 - W_2 \quad .$	$= 2$	to	1	down to right,
$s_4 = P - W_1 - W_2 - W_3$	$= 6$	to	1	down to right;

hence $M_{12} = 32$ foot-tons maximum.

In the degraded figure let $h_{12} = 32$ parts; from the top of this ordinate the slope down to the left will be half of the corresponding slope in the bending moment diagram; that is, ·5 vertical to 1 horizontal, and it extends over 7 feet; hence $h_5 = h_{12} - {\cdot}5 \times 7 = 28{\cdot}5$ parts. The next slope from this down to left is at the rate 2·5 to 1, and extends over 5 feet; so that $h_0 = h_5 - 2{\cdot}5 \times 5 = 16$ parts. Beginning again at top of h_{12} the slope down to right is at the rate 1, and extends over 4 feet, so that $h_{16} = h_{12} - 1 \times 4 = 28$ parts. The next slope down to right is at the rate 3 and extends over 4 feet; so that $h_{20} = h_{16} - 3 \times 4 = 16$ parts; thus

$$h_1 : h_5 : h_{12} : h_{16} : h_{20} : : 16 : 28{\cdot}5 : 32 : 28 : 16.$$

Putting $h_{12} = 20$ inches a twelfth of the span for stiffness, and reducing the others in the proportion of 20 to 32, we have on the elevation

$$h_1 = 10, \quad h_5 = 18, \quad h_{12} = 20, \quad h_{16} = 17{\cdot}5, \quad h_{20} = 10 \text{ inches.}$$

To find the uniform breadth put

$$\tfrac{1}{6} \times 1 \times b \times 20^2 = 384 \text{ inch-tons}; \quad \text{and} \quad \therefore\ b = 5{\cdot}76 \text{ inches.}$$

AREA, GEOMETRICAL MOMENT, AND MOMENT OF INERTIA.

The functions of a plane surface which we require for our investigations regarding the moment of resistance to bending of a cross-section in general, are the area, geometrical moment, and moment of inertia.

The *area* of a rectangle is the product of its two adjacent sides; the area of any other surface is the sum of all the elementary rectangles into which it may be divided. We take it for granted that the area of the triangle, the circle, the ellipse, and parabolic quadrant are respectively half the product of the height into the base, π into radius squared, π into the product of the semi-major and semi-minor diameters, and two-thirds of the product of the circumscribed rectangle.

The area of any figure is quite independent of the position of the axis to which the figure is referred; and the word *area* is conveniently used as a name for a finite plane surface.

Definition.—The *geometrical moment* of a surface about any line in its plane as axis, is the sum of the products of each elemental area into its leverage or perpendicular distance from that axis; the leverages which lie to one side of the axis being reckoned positive, and those to the other side negative.

Each elemental area is to be so small that the distance from the axis to every point in it is sensibly the same. If the surface were a plate of unit thickness and unit density, it is evident that the statical moment of the plate would be exactly the geometrical moment of the surface. Suppose the axis, for instance, to be a knife edge upon which the plate rests, the weights of the portions on different sides tend to cause the plate to rotate in opposite directions, and their statical moments are of different sign; the definition shows the geometrical moment of these portions of the surface also to be of opposite sign.

It will be convenient for us always to choose a *horizontal* axis; and if we consider leverages *up* to be positive, then, when the axis is below the area, the geometrical moment is positive; when above it, negative; when the axis cuts the area, it will be positive or negative according as the axis is near one or other edge; and for one position of the axis, cutting the area, the positive and negative products will destroy each other, and the geometrical moment will be zero.

An axis about which the geometrical moment of an area is zero passes through a point called the *geometrical centre* of the area. From this it appears that the geometrical centre of an area corresponds with the centre of gravity of a thin plate of uniform thickness and of that area; and for this reason the geometrical centre of an area is often called its centre of gravity.

Theorem.—The geometrical moment of a surface about any axis in its plane is equal to the area multiplied by the distance of the geometrical centre from the axis (fig. 179). Suppose G the geometrical centre of the surface to be known; through G draw OO' parallel to the axis AA; let s and s' be a pair of elemental areas, one on each side of OO', such that the sum of their geometrical moments is zero; that is, $s' . a = s . b$. It is evident that the whole area can be divided into such pairs from the definition of the geometrical centre. The geometrical moment of s' about AA is $s'(d + a)$, that of s is $s(d - b)$, and their joint moment is $(s' + s)d + (s'a - sb) = (s' + s)d$, since the second term is zero. In the same way the moment of each pair is their sum multiplied by d; hence the geometrical moment of the area about AA is the sum of all the pairs into d, that is, the area multiplied by the distance of the geometrical centre from the axis.

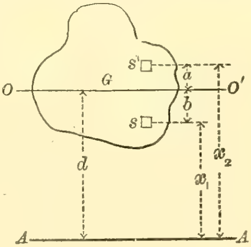


Fig. 179.

Cor.—The geometrical moment of an area which can be divided into simpler figures whose geometrical centres are known, may be found by multiplying the area of each such figure by the distance of the axis from its geometrical centre and summing algebraically.

Theorem.—The geometrical moment of an area about an axis in its plane is expressed by the number which denotes the volume of that portion of a right-angled isosceles wedge whose sloping side passes through the axis, and which stands on the area as base.

Let $AEKFA'$ (fig. 180) be the wedge, and AF its sloping side passing through the axis AA'; the angle at A is 45° and that at E is 90°; let $BCDE$ be the area, then the geometrical moment of $BCDE$ relatively to the axis AA' is represented by the volume of $DBCKL$.

If s be an elemental portion of the area $BCDE$, its geometrical moment about AA' is s multiplied by its distance from AA'; but the column of the wedge standing on s is sensibly a parallelepiped whose height is the same as the distance of s from AA', and the volume of that column expresses the geometrical moment of s; hence the volume of $DBCKL$, a portion of the isosceles wedge, is the geometrical moment of $BCDE$ about AA'.

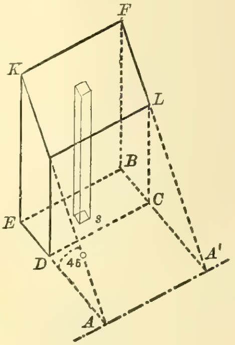


Fig. 180.

When the axis cuts the area, the plane sloping at 45° will form a wedge *above* one portion and *below* the other; by considering these of different signs, their algebraic sum is still the geometrical moment. For example, if we wish to find the geometrical moment of the triangle (fig. 172) about the axis NA; let $f = \frac{2}{3}h$, then the plane will slope at 45°; in the figure, the volume on one side of the axis is equal to the volume on the other, that is, the geometrical moment is zero; the axis NA must pass therefore through the geometrical centre.

To find an axis passing through the geometrical centre of a plane area then, it is only necessary to draw a plane sloping at 45° which will cut off an equal volume of the wedge above and below; the intersection of this plane with the area will be the axis required.

By similar triangles a plane through that axis at *any* slope will cut off equal volumes above and below; the wedges which represent the normal stress (fig. 84) require to be equal, hence *the neutral axis of a cross-section passes through the geometrical centre* (*or centre of gravity*) *of the cross-section.*

The distance of the neutral axis from the furthest away skin is, in each cross-section, a definite fraction of h the depth of the circumscribing rectangle; for instance, for a triangular cross-section (figs. 171, 172)

$$OA = \tfrac{2}{3}h.$$

Rankine expresses this generally thus

$$OA = m'h$$

where *m' is the fraction which the distance from the neutral axis to the farthest away skin is of the depth*; for all cross-sections, symmetrical above and below, as a rectangle, ellipse, hollow rectangle, &c., $m' = \frac{1}{2}$.

Definition.—The *moment of inertia* of a surface, about a line in its plane as axis, is the sum of the products of each elemental area into the square of its distance from the axis.

Whether the horizontal axis intersects, or is below or above the area, the moment of inertia will be positive; for, though the leverage be negative, the square of that quantity is always positive; it is also impossible that the sum can ever be zero. When the horizontal axis is at a great distance below, the moment of inertia is very great, since the leverage of each element is great; as the axis approaches, the moment of inertia decreases; when the axis has passed above the area and recedes, the moment of inertia again increases; hence for one position of the horizontal axis, the moment of inertia was less than when that axis was in any other position.

Theorem.—The moment of inertia of a surface about any axis in its plane equals that about a parallel axis through its geometrical centre, together with the product of the area into the square of the deviation of the axis from the centre.

Let s and s' (fig. 179) be a pair of elemental areas, the sum of whose geometrical moments about OO' is zero; and let x_1 and x_2 be their leverages about AA respectively. The sum of their moments of inertia about AA is

$$\begin{aligned} s'x_2^2 + sx_1^2 &= s'(d+a)^2 + s(d-b)^2 \\ &= (s'+s)d^2 + 2(s'a - sb)d + (s'a^2 + sb^2) \\ &= (s'+s)d^2 + (s'a^2 + sb^2), \quad \text{since} \quad (s'a - sb) = 0. \end{aligned}$$

Summing the left side for all pairs we have the moment of inertia of the area about AA. The first term on the right side is the area of each pair into the square of the deviation; and the sum of these for all pairs is the area into the square of the deviation; the second term on the right side is the moment of inertia of s and s' about OO', the sum will be the moment of inertia of the whole area about OO'. If I_A and I_O represent the moments of inertia round the axes A and O respectively, then

$$I_A = S \,.\, d^2 + I_O,$$

where S is the area of the figure.

Cor.—For any set of parallel axes, the moment of inertia about that axis which passes through the geometrical centre is a minimum; and those axes which give minima values for the moment of inertia intersect at a point.

When we speak of the moment of inertia of a body without specifying with regard to the axis, it is to be understood that the axis is horizontal and passes through the geometrical centre. The point of intersection for those axes which give minima values for the moments of inertia is called the *centre of inertia* of the area; this point coincides with the geometrical centre of the area, and with the centre of gravity of a thin plate of that area. Hence we may say shortly—The neutral axis of a cross-section of a beam passes through the centre of the section; this centre being called the geometrical centre, the centre of inertia, or the centre of gravity.

Theorem.—The moment of inertia of a plane area about an axis in its plane is expressed by the number which denotes the statical moment of that portion of a right-angled isosceles wedge of unit density whose sloping side passes through the axis, and which stands on the area as base.

To obtain the moment of inertia of the elemental area s (fig. 180), we multiply its area by the square of its distance from AA'; but the height of the column standing on s is equal to that distance. Multiplying s by that equivalent we have the volume of the column, and multiplying again by the leverage we have the statical moment of that column about AA'. Hence, summing for all elemental areas, we have the moment of inertia of $BCDE$ about AA' equal to the statical moment of $DBCKL$, a portion of the isosceles wedge, standing on $BCDE$, reckoning its density unity. Now $DBCKL$, instead of extending all over its base, may be supposed to stand on the point of $BCDE$ directly below its centre of gravity; the moment of inertia is thus readily found, if we know the volume of the isosceles wedge and the position of its centre of gravity.

We generally wish to find I_O, since we can readily find I_A from it by the theorem above. Now, for I_O, one portion of the wedge stands below and one above, and on taking statical moments these must be supposed to gravitate in opposite directions.

Cor.—If the plane area be divided into parts, the moment of inertia for the whole will be the sum of the moments for the parts; thus, for fig. 171, making $f = \frac{2}{3}h$ so as to make the wedges isosceles, we have

$$I_O = 16fxy \,.\, 12x + 12fxy \,.\, 8x + 4fxy \,.\, 9x = \tfrac{1}{36}bh^3,$$

for a triangle about an axis parallel to b, and passing through the *centre.*

For every cross-section, I_O will be of the same form, a constant multiplied by the breadth and multiplied by the cube of the depth of the circumscribing rectangle. Rankine puts generally $I_O = n'bh^3$ where n' is the numerical coefficient of the moment of inertia of the cross-section about its neutral axis, the other factors being the breadth and cube of the depth of the circumscribing rectangle.

If through the neutral axis of a cross-section we draw a plane sloping at 45°, it will form two isosceles wedges, or two portions, one on each side of the plane; the sum of the products got by multiplying the volume of each by the distance of the point *under* its centre of gravity from the neutral axis gives I_O for the cross-section; for this purpose we may take each portion as a whole or subdivide it into a number of parts if such is more convenient.

For a rectangular wedge $AEKFBA'$ (fig. 180) let S = area of base $AEBA'$, f = height EK, V = volume, x_0 = distance from AA' to the point which is *under* the centre of gravity, then

$$V = \tfrac{1}{2}S \,.\, f\,; \qquad x_0 = \tfrac{2}{3}\overline{AE}.$$

For an isosceles-triangular wedge $ACC'S$ (fig. 172), let S = area of base ACC', f = height $\overline{AS}$, x_0 = distance from CC' of the point which is *under* the centre of gravity, then

$$V = \tfrac{1}{3}S \,.\, f\,; \qquad x_0 = \tfrac{1}{2}\overline{OA}.$$

If *any* sloping plane be drawn through the neutral axis, it will cut off two wedges; and since the volumes of all such wedges are proportional to their heights, we have

$$V' : V :: f' : f :: f' : m'h,$$

where V' is the volume corresponding to f' the new value of EK in fig. 180, and of AS in fig. 172; and $m'h$ is the height of the isosceles wedge, that is the distance of the skin from the neutral axis. Since the leverages are the same as before, the statical moments of the wedges are also in the above ratio.

CHAPTER XV.

CROSS-SECTIONS: THEIR RESISTANCE TO BENDING AND SHEARING, AND DISTRIBUTION OF STRESS THEREON.

RESISTANCE TO BENDING OF CROSS-SECTIONS.

WE see then that I_O the moment of inertia of the cross-section about its neutral axis is represented by the statical moment of the isosceles wedges made by a plane sloping at 45° and passing through the neutral axis; that the highest point of these wedges is $y_{a \text{ or } b} = m'h$, the distance from the neutral axis to the further skin; while **M**, the moment of resistance to bending, is represented by the statical moment of the wedges made by a plane passing through the neutral axis, and sloping so that the height of the highest point of these wedges is $p_{a \text{ or } b} = f$, the stress on the skin A or B, whichever is further from the neutral axis; hence

$$\mathsf{M} = \frac{\text{normal stress on skin furthest from neutral axis}}{\text{distance of skin from neutral axis}} \times I_O$$

$$= \frac{p_a}{m'h} I_O = \frac{p_b}{m'h} I_O = \frac{f}{m'h} I_O$$

is the proof, or working moment of resistance according as f is the proof, or working strength of the material supposed to be the same for both skins, that is, for thrust and tension.

Rectangular Section (fig. 181).—Let $BCDE$ be a rectangle. Its moment of inertia about DC one side is found thus:—The height of the isosceles wedge is zero at C and BC at B; the average height is therefore $\frac{1}{2}BC$; its volume is $\frac{1}{2}EB \cdot BC^2$, and since the point below the centre of gravity of the wedge is two-thirds of BC from DC, we have

$$I_{DC} = \tfrac{1}{2}EB \cdot BC^2 \times \tfrac{2}{3}BC = \tfrac{1}{3}EB \times BC^3$$

as the moment of inertia of a rectangle about a side.

Now, if BE' be a rectangular cross-section of which DC is the neutral axis, then its moment of inertia about that axis will be double the above; and thus the moment of inertia of a rectangle about an axis through the centre and parallel to a side is

$$I_O = 2 \times \tfrac{1}{3} \,.\, EB \,.\, BC^3 = \tfrac{1}{12}bh^3.$$

In this case $m' = \frac{1}{2}$, so that

$$\mathsf{M} = \frac{f}{m'h} I_O = \frac{f}{\frac{1}{2}h} \times \tfrac{1}{12}bh^3 = \tfrac{1}{6}fbh^2.$$

Thus, for a rectangle, $m' = \frac{1}{2}$, $n' = \frac{1}{12}$, and $n = \frac{1}{6}$.

Hollow Rectangular Section.—For a hollow rectangular section symmetrical above and below the neutral axis, that is, when the whole rectangle and rectangle removed have their centres on the same horizontal axis, the moment of inertia is the difference of the moments of the two rectangles.

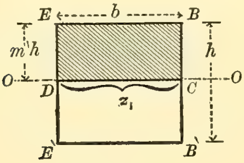


Fig. 181.

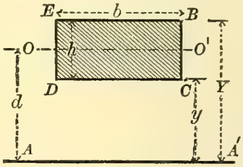


Fig. 182.

Let H, h, and B, b be the outer and inner dimensions respectively, so that the area is $BH - bh$; then

$$I_O = \tfrac{1}{12}BH^3 - \tfrac{1}{12}bh^3 = \tfrac{1}{12}\left(1 - \frac{bh^3}{BH^3}\right)BH^3;$$

$$\mathsf{M} = \frac{f}{m'H} I_O = \tfrac{1}{6}\left(1 - \frac{bh^3}{BH^3}\right)fBH^2.$$

Hence, for a hollow rectangle,

$$m' = \tfrac{1}{2}, \quad n' = \tfrac{1}{12}\left(1 - \frac{bh^3}{BH^3}\right), \quad \text{and} \quad n = \tfrac{1}{6}\left(1 - \frac{bh^3}{BH^3}\right);$$

and for the dimensions given in fig. 183, viz.,

$$H = 30, \quad B = 10, \quad h = 24, \quad b = 6 \text{ inches},$$

we obtain

$$I_O = 15588, \quad \text{and} \quad \mathsf{M} = 1039{\cdot}2f \text{ inch-lbs., if } f \text{ be in lbs.}$$

Tabular Method.—The following is a convenient form for expressing the area, geometrical moment, and moment of inertia of a rectangle:—Let $BCDE$ (fig. 182) be a rectangle, and AA' an axis parallel to DC, and let Y and y be the distances of its sides EB and DC from AA'. The area is

$$S = b(Y - y), \tag{1}$$

the breadth into the difference of the ordinates. For EL, part of the isosceles wedge (fig. 180) $EK = Y$, $CL = y$; and the geometrical moment of the rectangle $BCDE$ about AA' is

$$\begin{aligned} G_A &= \text{volume of part of isosceles wedge} \\ &= b(Y - y) \times \tfrac{1}{2}(Y + y) = \tfrac{1}{2} . b(Y^2 - y^2), \end{aligned} \tag{2}$$

one-half the breadth into the difference of the squares of the ordinates of the sides parallel to the axis. By theorem on

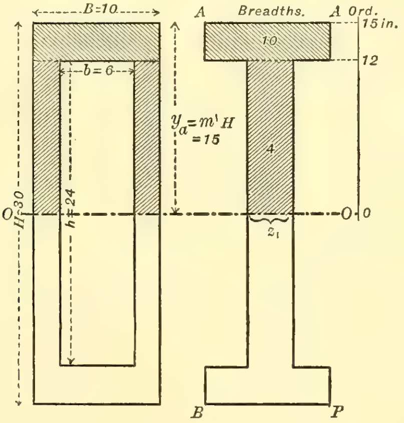


Fig. 183.

page 283 we have the moment of inertia of the rectangle $BCDE$ (fig. 182) about AA'

$$\begin{aligned} I_A &= I_O + Sd^2 = \tfrac{1}{12} EB . BC^3 + EB . BC \left(\frac{Y + y}{2}\right)^2 \\ &= \tfrac{1}{12} b(Y - y)^3 + b(Y - y) \frac{(Y + y)^2}{4} = \tfrac{1}{3} . b . (Y^3 - y^3), \end{aligned} \tag{3}$$

one-third of the breadth into the difference of the cubes of the ordinates of the sides parallel to the axis.

To apply this to the case of a hollow rectangle and a symmetrical double-T section, which give similar results (fig. 183). Choose OO the neutral axis in the centre from symmetry; and if only the upper half of the section be considered, it consists of two rectangles, the ordinates to the edges being 0, 12, 15; for one rectangle $Y = 15$, $y = 12$; for the other $Y = 12$, and $y = 0$; and for each (equation (3)), $I_O = \frac{1}{3}b(Y^3 - y^3)$.

b	Y	Y^3	$Y^3 - y^3$	$\frac{1}{3}b(Y^3 - y^3)$
	15	3375		
10			1647	5490
	12	1728		
4			1728	2304
	0	0		
—	—	—	—	$\frac{1}{2}I_O = 7794$

This tabular method applies to any section made up of rectangles and which is symmetrical above and below the neutral axis. As we require G' the geometrical moment of the semi-section relatively to the neutral axis when we come to resistance to shearing, it is convenient in making the table to find G'; thus, for the symmetrical section, one-half of which is shown in fig. 184, we have the following results:—

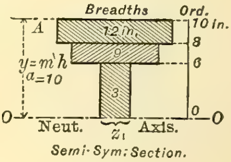


Fig. 184.

b	Y	Y^2	$Y^2 - y^2$	$\frac{1}{2}b(Y^2 - y^2)$	Y^3	$Y^3 - y^3$	$\frac{1}{3}b(Y^3 - y^3)$
	10	100			1000		
12			36	216		488	1952
	8	64			512		
9			28	126		296	888
	6	36			216		
3			36	54		216	216
	0	0			0		
—	—	—	—	$G' = 396$	—	—	$\frac{1}{2}I_O = 3056$

Where G' is for semi-section only.

Suppose the working strength $f = 4$ tons per square inch, then the distance from *OO* the neutral axis to the skin is 10 inches; hence

$$\mathsf{M} = \frac{f}{m'h} I_0 = \tfrac{4}{10} \times 6112 = 2444{\cdot}8 \text{ inch-tons.}$$

The use of G' will be shown at the proper place.

Symmetrical sections are suitable for materials for which the strengths to resist thrust and tension are equal, as p_a and p_b become f simultaneously (fig. 84). For materials whose strengths are unequal, f_a is put for the greatest value of p_a and f_b for that of p_b; for wrought iron, for instance, the working resistance to thrust is $f_a = 4$ tons per square inch, while to tension it is $f_b = 5$ tons per sq. inch. For symmetrical sections in such materials the value of f employed must obviously be the smaller. Because of this property of materials, cross-sections are made unsymmetrical above and below.

Unsymmetrical Section.—Let the unsymmetrical section (fig. 185) be given, and let $f_a = 4$, and $f_b = 5$ tons per square inch. In order to determine its resistance to bending we find I_0; we will also find the area of the section, and G' the geometrical moment of the portion of the section lying to one side of the neutral axis whose position is not yet known.

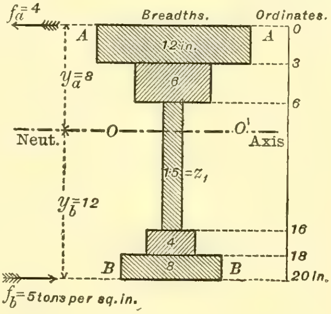


Fig. 185.

Choose *any* line as axis, say, the upper skin; the diagram and table show the breadths, and ordinates laid down from this line.

b	Y	$Y-y$	*Area.*	Y^2	Y^2-y^2	$\frac{1}{2}b(Y^2-y^2)$	Y^3	Y^3-y^3	$\frac{1}{3}b(Y^3-y^3)$
	0			0					
12		3	36		9	54		27	108
	3			9			2		
6		3	18		27	81		189	378
	6			36			216		
1·5		10	15		220	165		3880	1940
	16			256			4096		
4		2	8		68	136		1736	2315
	18			324			5832		
8		2	16		76	304		2168	5781
	20			400			8000		
—	—	—	$S = 93$	—	—	$G_A = 740$	—	—	$I_A = 10522$

Now $y_a = \frac{G_A}{S} = \frac{740}{93} = 8$ inches (by theorem at fig. 179);

$\therefore \quad y_b = 20 - 8 = 12$ inches.

Again, $I_0 = I_A - Sy_a^2 = 10522 - 93 \times (\frac{740}{93})^2 = 4634$.

The neutral axis divides the depth almost exactly as 2 to 3; and if we had $f_a : f_b :: 2 : 3$, then both skins would come to their working stress simultaneously, and we might obtain **M** by multiplying I_0 either by $\frac{f_a}{y_a}$ or by $\frac{f_b}{y_b}$; but since $f_a : f_b :: 4 : 5$, it is evident that both skins cannot come to their working strength at once. In this example, when the skin B comes to $f_b = 5$ tons, the skin A will be at $\frac{2}{3} f_b$ or $3\frac{1}{3}$ tons per square inch, and so is not at its full strength 4, yet the stresses are now at their greatest; on the other hand, the skin A cannot come to its strength $f_a = 4$, because then the skin B would be at $\frac{3}{2} \times 4$, or 6 tons, and so be overtaxed; hence **M** is to be obtained by multiplying I_0 by the ratio $\frac{f_b}{y_b}$, not by $\frac{f_a}{y_a}$ which would give too much. Of the two ratios

$$\frac{f_a}{y_a} = \frac{4}{8} = \cdot 5, \quad \text{and} \quad \frac{f_b}{y_b} = \frac{5}{12} = \cdot 42,$$

select the less, and

$$\mathsf{M} = \frac{f_b}{y_b} \times I_0 = \cdot 42 \times 4634 = 1931 \text{ inch tons.}$$

Taking the neutral axis now as origin, the geometrical moment for the part of the section lying above or below, is $G'_0 = 296$; a quantity to be used in calculating the resistance to shearing.

b	Y	Y^2	$Y^2 - y^2$	$\frac{1}{2}b(Y^2 - y^2)$
	0	0		
1·5			64	48
	8	64		
4			36	72
	10	100		
8			44	176
	12	144		
—	—	—	—	$G'_0 = 296$

Graphical Solution (fig. 186 : for the same data as fig. 185).—Replace the areas of the rectangles by the forces (1), (2), (3), (4), (5) acting at their centres of gravity, the amount of each force being the same as the number of units in the area corresponding; draw the first link polygon as in fig. 101, and L, the intersection of the end links, gives the centre of the forces and therefore the centre of gravity and neutral axis of the cross-section. The intercepts on OO are the geometrical moments of the areas respectively, that is, the product of each area into its leverage about OO ; the scale is found by subdividing the scale for areas by 10, since 10 on the scale for dimensions was taken as the polar distance.

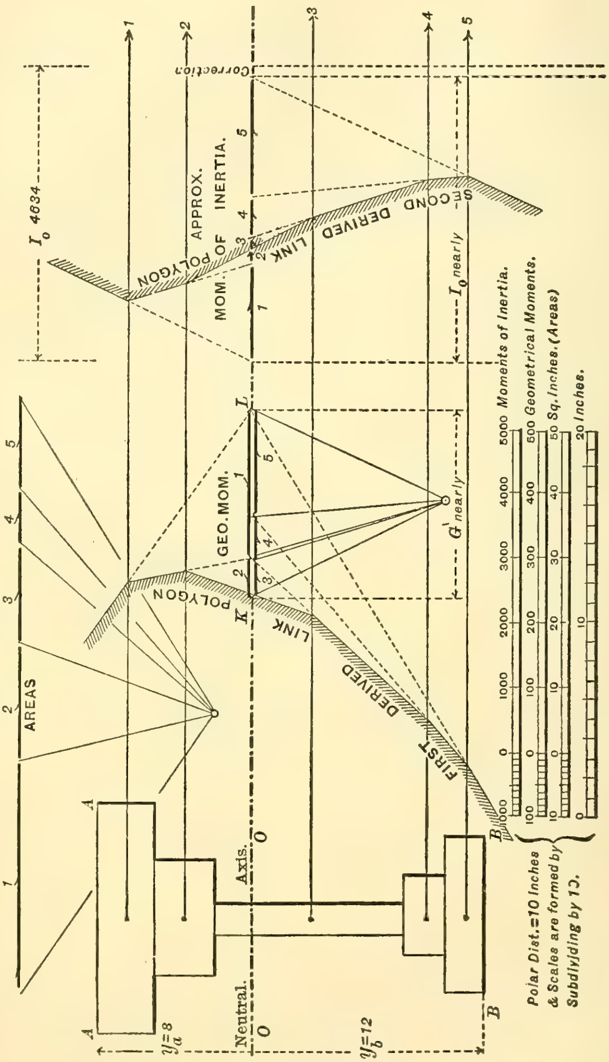
Considering these intercepts as magnitudes of forces acting in the same lines as before, and drawing a second link polygon, its intercepts on OO are the products of these forces each into its leverage about OO; or these intercepts are the areas each into the square of its leverage about OO; the scale is derived from the previous one by again subdividing by the polar distance, 10 on the scale for dimensions. The proof is given at fig. 101.

The sum of the intercepts made by the second link polygon is nearly I_0; being deficient by one-twelfth of the sum of the products of the breadth of each rectangle into the cube of its depth.

Also KL the geometrical moment of areas (1) and (2) is nearly $G^{\backprime}_0$, the deficiency being that of the rectangle lying between the area (2) and the neutral axis; this rectangle, however, is small and has a short leverage.

There is also shown in fig. 187 a link polygon drawn for the three areas (1), (2), ($3^{\backprime}$), which constitute the portion of the section lying above the neutral axis; the sum of the intercepts is $G^{\backprime}_0$.

Corrections.—The manner of correcting is shown in figs. 186 and 187. Take the lines of action of the areas first along their (say) *upper* edges, and construct the first link polygon; treating the intercepts on OO as forces acting along the *under* edges of the rectangles, construct a second link polygon; then to the sum of the intercepts made by the second link polygon in fig. 186, add, as a correction, one-third of its excess above the sum of the intercepts made by the second link polygon in fig. 187.



I_o 4634
Correction
1
2
3
4
5
APPROX.
MOM. OF INERTIA.
POLYGON
SECOND DERIVED LINK POLYGON
I_o nearly
5000 Moments of Inertia.
500 Geometrical Moments.
50 Sq. Inches. (Areas)
20 Inches.
4000
3000
2000
1000
0
1000
400
300
200
100
0
100
40
30
20
10
0
10
10
0
L
K
GEO. MOM.
G' nearly
AREAS
FIRST DERIVED LINK POLYGON
A
A
B
B
O
O
Axis.
Neutral.
$y_a=8$
$y_b=12$
Polar Dist.=10 Inches & Scales are formed by Subdividing by 10.

Fig. 186.

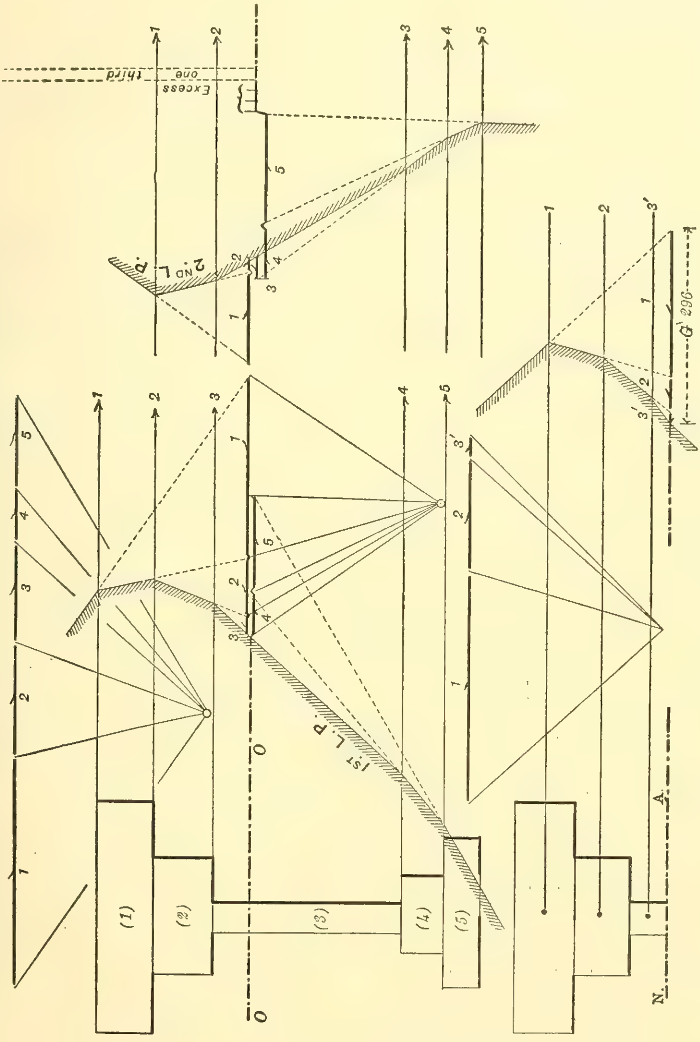


Fig. 187.

The proof of the correction on I_0, which is exact, is shown thus:—For any rectangle

$$
\begin{aligned}
I_A &= \tfrac{1}{3}b(Y^3 - y^3) = \tfrac{1}{3}b(Y - y)(Y^2 + Yy + y^2) \\
&= \tfrac{1}{3}bh\{(Y + y)^2 - Yy\} = S\left\{\left(\frac{Y + y}{2}\right)^2 + \tfrac{1}{3}\left(\frac{Y + y}{2}\right)^2 - \frac{Yy}{3}\right\} \\
&= S\{d^2 + \tfrac{1}{3}(d^2 - Yy)\} = Sd^2 + \tfrac{1}{3}(Sd^2 - SYy);
\end{aligned}
$$

where S is the area of the rectangle under consideration. The smaller h becomes, the more nearly does the value of Yy approach d^2, and the smaller is the correction. The greater the number of rectangles into which the cross-section is divided, the more nearly accurate will be the approximation given by fig. 186 alone.

In the example, the correction on I_0 is about 4 per cent.; in the rectangle $EBB'E'$ (fig. 181), the areas of the two halves each into the square of its leverage about OO is less than I_0 by one-twelfth, or about 8 per cent.

Any cross-section can be blocked out into rectangles, and I_0 and G_0 easily calculated for it by the tabular method, if we have a table of squares and cubes. If the cross-section has a very irregular outline, it may require to be blocked into a great many rectangles, and the construction (fig. 186) will probably give a sufficiently close approximation.

DEFINITION.—A cross-section for which the neutral axis divides the depth in the same ratio as the strengths of a given material to resist tension and thrust is called a *cross-section of uniform strength* for that material.

The cross-section (fig. 185 for instance) would be of uniform strength for a beam of a material whose resistance to tension and thrust is as 3 to 2, AA being the compressed skin, and BB the stretched skin; for a cantilever of the same material it would be turned upside down.

Triangular cross-section and sections which can be divided into triangles.—They are of no practical importance but lead up to others which are.

Triangular Section.—On making $f = \frac{2}{3}h$, the wedges (figs. 171 and 172) become isosceles, and we have by substituting in the expressions given thereat

$$G' = \text{vol. of wedge on triangular portion above } NA = \tfrac{4}{81}bh^2.$$

and $$I_0 = \tfrac{1}{36}bh^3.$$

In the following way, I_0 may be derived from the rectangle (fig. 188). The moment of inertia of the shaded triangle about the central axis CC is half that of the rectangle; for the triangle, then

$$I_C = \tfrac{1}{2}\left(\tfrac{1}{12}bh^3\right) = \tfrac{1}{24}bh^3.$$

$$I_0 = I_C - Sd^2 = \tfrac{1}{24}bh^3 - \frac{bh}{2} \times \left(\frac{h}{6}\right)^2 = \tfrac{1}{36}bh^3.$$

$$\mathrm{M} = \frac{f}{\frac{2}{3}h} I_0 = \tfrac{1}{24}fbh^2.$$

Hence for a triangular section

$$m' = \tfrac{2}{3}, \quad n' = \tfrac{1}{36}, \quad n = \tfrac{1}{24}.$$

Hexagonal Section.—As an example of a built figure, we will take a hexagon about a diameter as axis (fig. 189).

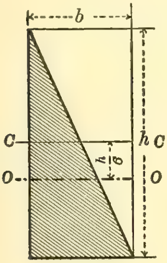


Fig. 188.

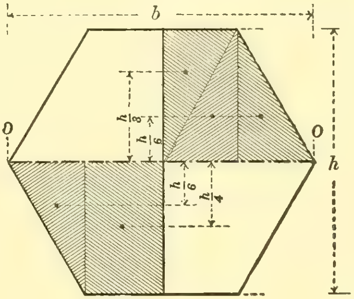


Fig. 189.

Taking a quadrant, we can divide it into three equal triangles; the breadth of each is $\frac{1}{4}b$, and the height $\frac{1}{2}h$; hence the area of each is $\frac{1}{16}bh$, and the moment of inertia of each about its own neutral axis is $\frac{1}{36} \times \frac{1}{4}b \times \frac{1}{8}h^3 = \frac{1}{1152}bh^3$. If I_0 be the moment of inertia of the hexagon about OO, we have for a quadrant

$\frac{1}{4}I_0$ = moment of each of the three triangles about its own neutral axis, together with the area of each into the square of the distance of its centre from OO,

$$= 3 \times \frac{bh^3}{1152} + \frac{bh}{16}\left\{\left(\frac{h}{6}\right)^2 + \left(\frac{h}{6}\right)^2 + \left(\frac{h}{3}\right)^2\right\} = \tfrac{5}{384}bh^3;$$

$$\therefore \quad I_0 = \tfrac{5}{96}bh^3.$$

Or, taking another quadrant divided into a rectangle and a triangle

$$\left.\begin{aligned} \tfrac{1}{4}I_0 = \tfrac{1}{12}\left(\frac{b}{4}\right)\left(\frac{h}{2}\right)^3 + \frac{bh}{8}\left(\frac{h}{4}\right)^2 &\text{ for rectangle} \\ + \tfrac{1}{36}\left(\frac{b}{4}\right)\left(\frac{h}{2}\right)^3 + \frac{bh}{16}\left(\frac{h}{6}\right)^2 &\text{ for triangle} \end{aligned}\right\} = \tfrac{5}{384}bh^3;$$

$$\therefore \quad I_0 = \tfrac{5}{96}bh^3.$$

Dividing the semi-hexagon above the axis into any set of convenient figures, and multiplying the area of each by the deviation of its centre of gravity, we have

$$G_0 = \tfrac{1}{12}bh^2.$$

Also
$$\mathsf{M} = \frac{f}{m'h}\,I_0 = \frac{f}{\frac{1}{2}h} \times \tfrac{5}{96}bh^3 = \tfrac{5}{48}fbh^2.$$

Hence for a hexagonal section about a diameter as axis,

$$m' = \tfrac{1}{2}, \quad n' = \tfrac{5}{96} = \cdot 05208, \quad n = \tfrac{5}{48} = \cdot 10417.$$

Rhomboidal Section.—As another example, we will take a section in the form of a rhombus, with diagonal as neutral axis (fig. 190).

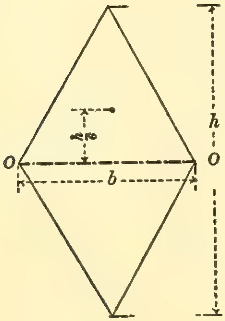


Fig. 190.

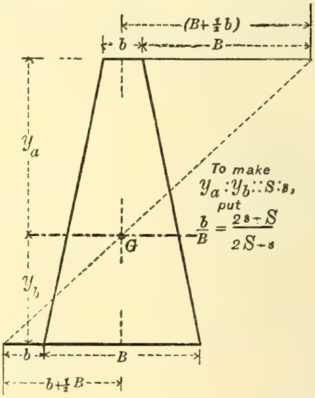


Fig. 191.

For upper half section, the area is $\frac{bh}{4}$, the moment of inertia about its own neutral axis is $\frac{1}{36}b\left(\frac{h}{2}\right)^3 = \frac{bh^3}{288}$, and the deviation of its centre of gravity is $\frac{h}{6}$; hence

$$\tfrac{1}{2}I_0 = \frac{bh^3}{288} + \left(\frac{bh}{4}\right)\left(\frac{h}{6}\right)^2 = \tfrac{1}{96}bh^3;$$

therefore
$$I_0 = \tfrac{1}{48}bh^3, \quad \text{and} \quad \mathsf{M} = \frac{f}{\frac{1}{2}h}\,I_0 = \tfrac{1}{24}fbh^2.$$

$$G' = \left(\frac{bh}{4}\right)\left(\frac{h}{6}\right) = \tfrac{1}{24}bh^2.$$

Hence for a rhomboidal section with diagonal as neutral axis,

$$m' = \tfrac{1}{2}, \quad n' = \tfrac{1}{48}, \quad n = \tfrac{1}{24}.$$

Square Section.—A square section lying with its diagonal horizontal is a particular case of this.

In order to compare the strengths of a square section when lying with a side horizontal, and when lying with a diagonal horizontal; let a be the side of the square, then $d = a\sqrt{2}$ is the diagonal; for b and h substitute a in the one case, and $a\sqrt{2}$ in the other; then

$$\mathsf{M} = \tfrac{1}{6}fa^3;\ \mathsf{M}' = \tfrac{1}{24}f(a\sqrt{2})^3 = \frac{\sqrt{2}}{12}fa^3;$$

so that $\mathsf{M} : \mathsf{M}' :: \sqrt{2} : 1$, being stronger when the side is horizontal.

Trapezoidal Section (fig. 191).—Observe that a triangular section is of *uniform strength* for a material whose strengths to resist tension and thrust are in the ratio 1 : 2, or 2 : 1. It is evident, then, that for a material, the ratio of whose strengths is between 1 and 2, a cross-section of uniform strength can be made by selecting the proper frustum of a triangle; that is, a trapezoid with the parallel sides horizontal. Suppose the strengths are as $S : s$, not greater than 2 : 1; then to find the ratio of the parallel sides B and b (fig. 191); a well-known construction for finding G is to lay off B in a continuation of b, and b in a continuation of B, one in each direction; the line joining the extremities cuts the medial line at G; hence

$$B + \frac{b}{2} : b + \frac{B}{2} :: y_a : y_b :: S : s;$$

therefore

$$\frac{b}{B} = \frac{2s - S}{2S - s}.$$

For instance, if the strengths are as 3 : 2, then the parallel sides ought to be in the ratio

$$\frac{b}{B} = \frac{2 \times 2 - 3}{2 \times 3 - 2} = \tfrac{1}{4}.$$

when the neutral axis will divide the depth as 3 : 2.

Though such sections are thus far economical, still they are not the most economical, as too much of the material lies near the centre of the section where it cannot act effectively in resisting bending.

Circular Section.—In the circle (fig. 192) inscribe a hexagon with a diameter horizontal; for the hexagon

$$b = d, \quad h = d \,.\, \frac{\sqrt{3}}{2}, \quad \text{and} \quad I_0 = \tfrac{5}{96}(d)\left(d\,\frac{\sqrt{3}}{2}\right)^3 = \cdot 0338d^4,$$

an approximation on the small side to that for the circle about a diameter.

Again, circumscribe a hexagon with a diameter horizontal; for this hexagon

$$h = d, \quad b = d \,.\, \frac{2}{\sqrt{3}} = \frac{2d}{3}\sqrt{3}, \quad \text{and} \quad I_0 = \tfrac{5}{96}\left(\frac{2d}{3}\sqrt{3}\right)(d)^3 = \cdot 0602d^4,$$

an approximation on the large side.

Taking the average of these, we have for the circle

$$I_0 = {\cdot}047d^4,$$

where d is the diameter of the circle, and the breadth and depth of the circumscribing rectangle.

It will be seen that $n' = {\cdot}047$ is a close approximation, being correct to 2 decimal places.

In the same way an approximation to G'_0 may be found.

The exact value of I_0 is found thus:—From O the centre of the circle (fig. 193) draw three axes OX, OY diameters at right angles, and OZ normal to the plane of the paper. Let I, J, K be the moments of inertia of the circle about these axes respectively, and let s be any small elementary area. Now $\overline{OA}$, $\overline{OB}$, and $\overline{OS}$ are its leverages about the three axes respectively, and by definition

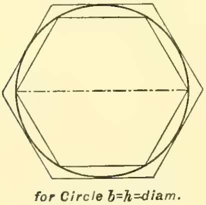


Fig. 192.

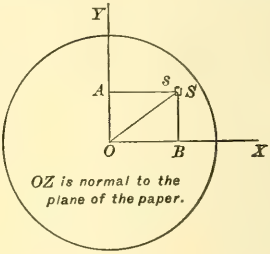


Fig. 193.

$$I = (s \times \overline{OA^2}) \text{ summed for each element,}$$

$$J = (s \times \overline{OB^2}) \quad ,, \quad ,, \quad ,,$$

$$K = (s \times \overline{OS^2}) \quad ,, \quad ,, \quad ,,$$

but, by Euclid I. 47, $\overline{OS^2} = \overline{OA^2} + \overline{OB^2}$ for each element, hence $K = I + J$; and further, since I and J are equal, each being the moment of inertia about a diameter, $K = 2I$, so that if we find K, the value of I is at once obtained.

On the element s build a column of material of unit density, and whose height is $\overline{OS}$; suppose this column to gravitate *tangentially*, that is, at right angles to $\overline{OS}$, then its statical moment about OZ, that is its volume into $\overline{OS}$, gives the moment of inertia of the element about OZ; all these columns will form a solid standing on the circle as base, whose height at the circumference is r, and whose upper surface is a conical surface, apex at O, and sloping at 45° to the plane of the circle. The volume of this solid is that of a cylinder of height r standing on the circle, minus a cone of equal height and base; its volume is therefore two-thirds of the cylinder, viz.

$$V = \tfrac{2}{3} \times \pi r^2 \times r = \tfrac{2}{3}\pi r^3,$$

every particle gravitating tangentially and in the same direction. If we cut this solid (fig. 194) into slices by planes at right angles to the plane of the circle, and passing through consecutive radii, the slice between two adjacent planes will be a pyramid with its apex at O, and having its base on the cylindrical surface; one dimension of the base, therefore, is r, and the other an arc of the circle. Now by taking the slices thin enough, the base of each pyramid becomes in the limit a rectangle, and the points in the circle below the centres of gravity of these pyramids will form a circle of radius $\tfrac{3}{4}r$; the whole weight may be supposed to be applied by a cord on a pulley of radius $\tfrac{3}{4}r$, and hence

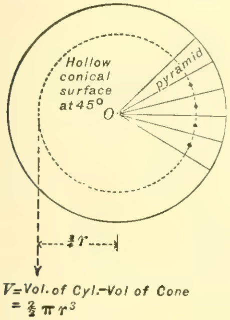

Hollow conical surface at 45°
pyramid
O
¾r
V=Vol. of Cyl.–Vol of Cone = ⅔πr³

Fig. 194.

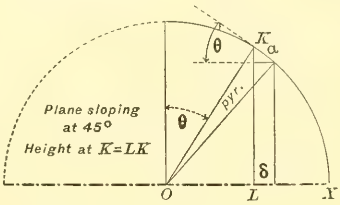


Fig. 195.

$$K = \text{volume of solid} \times \text{radius of pulley} = \tfrac{2}{3}\pi r^3 \times \tfrac{3}{4}r = \tfrac{1}{2}\pi r^4.$$

$$\therefore \quad I_0 = \tfrac{1}{4}\pi r^4 = \frac{\pi}{64}d^4, \quad \text{or} \quad \frac{\pi}{64}bh^3;$$

$$\mathsf{M} = \frac{f}{\frac{1}{2}h}I_0 = \frac{\pi}{32}fd^3, \quad \text{or} \quad \frac{\pi}{32}fbh^2.$$

Hence for a circular section,

$$m' = \tfrac{1}{2}, \quad n' = \frac{\pi}{64} = \cdot 049, \quad n = \frac{\pi}{32} = \cdot 098.$$

To find G'_0:—Suppose a wedge standing on the quadrant (fig. 195) formed by a 45° sloping plane passing through OX;

the geometrical moment of a semicircle about a diameter will be twice this volume. Cut the wedge into slices by planes at right angles to the plane of the circle, and passing through consecutive radii. Each slice is a pyramid with apex at O, and of height r; its base is part of a cylindrical surface, and the upper edge of the base is sloping. Let a be the short arc as shown in the figure, and δ its projection on OX; then in the limit when the slices are thin, the base becomes a plane rectangle whose dimensions are a and the height of the sloping plane at K; but height of $K = LK = r\cos\theta$; hence

$$\text{volume of pyramid} = \tfrac{1}{3} \times r \times a \,.\, r\cos\theta = \tfrac{1}{3}r^2 \,.\, \delta,$$

since $$\delta = a\cos\theta.$$

For each pyramid $\frac{1}{3}r^2$ is constant; so that the volume of wedge on the quadrant is $\frac{1}{3}r^2$ multiplied by the sum of the quantities δ; this sum is r whether the slices be thick or indefinitely thin; hence the

$$\text{volume of wedge} = \tfrac{1}{3}r^3;$$

and doubling, we have for semicircle

$$G'_0 = \tfrac{2}{3}r^3.$$

If we divide G'_0 by the area of the semicircle, we obtain the distance from the centre of the circle to the centre of gravity of the semicircle

$$\bar{y} = \frac{4r}{3\pi}.$$

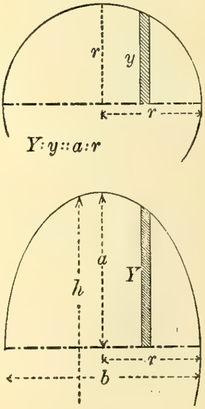


Fig. 196.

Elliptic Section.—The ellipse is immediately derived from the circle thus:—Let the ellipse (fig. 196) have the same minor diameter as the circle above it, so that $b = 2r$; all the vertical dimensions of the circle are, however, to be altered in the constant ratio $a : r$, since the vertical radius of the circle is altered to a. Let the circle be divided into rectangular elements, each with an edge lying on the neutral axis; when the circle is changed into the ellipse, each element remains of the same breadth b' as before, but its vertical dimension is changed from y to Y, where $Y : y :: a : r$.

For each element of circle and ellipse respectively

$$I = \tfrac{1}{3}b'(y^3 - 0^3), \quad \text{and} \quad I = \tfrac{1}{3}b'(Y^3 - 0^3).$$

For each element, and therefore for the whole figure, the moment of inertia has changed in the ratio

$$Y^3 : y^3 = a^3 : r^3.$$

For circle

$$I_0 = \frac{\pi}{4} r^4 ;$$

hence, for ellipse,

$$I_0 = \frac{a^3}{r^3}\left(\frac{\pi}{4} . r^4\right) = \frac{\pi}{4} ra^3 = \frac{\pi}{64} bh^3,$$

putting $\tfrac{1}{2}b$ for r, and $\tfrac{1}{2}h$ for a,

$$\mathsf{M} = \frac{f}{\tfrac{1}{2}h} I_0 = \frac{\pi}{32} fbh^2.$$

Hence, for elliptic section,

$$m' = \tfrac{1}{2}, \quad n' = \frac{\pi}{64}, \quad n = \frac{\pi}{32}.$$

For a semi-ellipse about a diameter as axis, we have

$$G_0 = \tfrac{2}{3}ra^2 ; \quad \text{and} \quad \bar{y} = \frac{4a}{3\pi}.$$

Hollow Circular Section.—For a hollow circle or ellipse the reduction is the same as for the hollow rectangle only $\frac{\pi}{32}$ replaces $\frac{1}{6}$; hence

$$\mathsf{M} = \frac{\pi}{32}\left(1 - \frac{bh^3}{BH^3}\right) fBH^2.$$

RESISTANCE OF CROSS-SECTIONS TO SHEARING, AND DISTRIBUTION OF SHEARING STRESS ON A CROSS-SECTION.

In fig. 197 let AB be the cross-section shown in figs. 82, 83, 84, 85, and 86, and let $A'B'$ be another section lying at a small distance δx to the left thereof; in figs. 82 and 83, we assume AB to be in a position such that P is greater than $W_1 + W_2$, that is, AB lies to the left of the section of maximum bending moment; hence the bending moment on $A'B'$ will be less than that on AB.

Consider the *horizontal* equilibrium of $AHH'A'$ (fig. 197) part of the slice of the beam between these sections, and bounded below by the horizontal face HH' a portion of the plane CD (fig. 82). There is no stress on the free surface. On AH the horizontal stress is indicated by arrows p_a, &c.; and on $A'H'$ by the shorter arrows p'_a, &c. (see fig. 84), shorter because the bending moment on $A'B'$ is less; in so far as these affect the *horizontal* equilibrium of $AHH'A'$, they may be replaced by arrows $(p_a - p'_a)$, &c., on the face AH alone. For *horizontal* equilibrium, there must be a tangential stress q acting towards the right on the remaining face $H'H$; and since $H'H$ is very small, the stress on it will be of constant intensity q; therefore

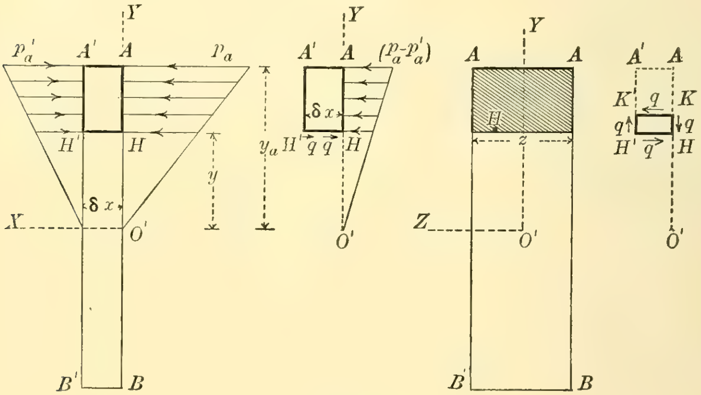


Fig. 197.

$q \times$ area of plane $H'H$ = sum of the arrows $(p_a - p'_a)$, &c., on plane AH;

or $q \,.\, z \,.\, \delta x$ = volume of frustum of wedge standing on HAA the shaded part of section as base, and of height $(p_a - p'_a)$.

Now if G stands for the geometrical moment of the shaded part of the cross-section relatively to the neutral axis, that is, for the volume of the frustum standing on HAA of the isosceles wedge made by a plane sloping at 45° and passing through the neutral axis, then the frustum of the wedge above is the same fraction of G, that $(p_a - p'_a)$ its height is of y_a the height of the isosceles wedge, and

$$q \,.\, z \,.\, \delta x = \frac{p_a - p'_a}{y_a} G.$$

Now on the cross-section AB, the bending moment equals the moment of resistance, that is

$$M = \mathsf{M} = \frac{p_a}{y_a} I_0 ; \quad \therefore \; \frac{p_a}{y_a} = \frac{M}{I_0}. \tag{1}$$

For the sections AB and $A'B'$, y_a and the moments of inertia are the same, because in the limit when δx is indefinitely small the sections coincide; so that if M' be the bending moment at $A'B'$, then

$$M' = \frac{p'_a}{y_a} I_0 ; \quad \therefore \; \frac{p'_a}{y_a} = \frac{M'}{I_0}, \quad \text{and} \quad \frac{p_a - p'_a}{y_a} = \frac{M - M'}{I_0} = \frac{\delta M}{I_0},$$

where δM stands for the small difference of the bending moments on the sections AB and $A'B'$ at the small distance δx apart; therefore

$$q \,.\, z \,.\, \delta x = \frac{\delta M}{I_0} G, \quad \text{or} \quad q = \frac{\left(\dfrac{\delta M}{\delta x}\right)}{I_0} \cdot \frac{G}{z}.$$

In the limit when δx becomes indefinitely small, so does δM; but their ratio $\dfrac{\delta M}{\delta x}$ becomes F the shearing force at the section AB, by theorem, page 139, and therefore

$$q = \frac{F}{I_0} \cdot \frac{G}{z}; \tag{2}$$

since H' now coincides with H, we are warranted in assuming q constant over $H'H$.

Now q is the intensity of the shearing or tangential stress at the point H in the horizontal plane CD (fig. 82); but it is *also* the intensity of the shearing stress at the point H in the vertical plane AB (see fig. 32); hence (fig. 85) the intensity of the shearing stress at H, any point of the cross-section AB, is

$$q = \frac{F}{I_0} \cdot \frac{G}{z}; \qquad (2)$$

where F is the shearing force on the cross-section; I_0 is the moment of inertia of the cross-section about the neutral axis; G is the geometrical moment of the portion of the cross-section beyond the point, about the neutral axis; and z is the breadth of the cross-section at the point.

On the cross-section AB, it will be seen that q acts downwards at *any* point H; if we choose K at an indefinitely small distance *above* H, the tangential stress on $K'K$ will also be q since it is indefinitely near $H'H$; and on $K'K$ the *lower* surface of $AKK'A'$, q will act to the right just as on $H'H$, but on $K'K$ the *upper* surface of the small prism $KHH'K'$, q acts to the left. The horizontal stresses q form a couple tending to turn the small prism in the left-handed direction; hence for equilibrium, the vertical stress q on the faces KH and $K'H'$ tend to turn it in the opposite direction; so that q on the face KH acts downwards, an assumption made in fig. 85, which is now proved.

On the other hand, if P were less than $W_1 + W_2$, &c., then the bending moment on $A'B'$ would be the greater; q on face $H'H$ would act towards the left, the arrows q all round the prism would be reversed, and so q on the face HK would act upwards; that is, q at *any* point H of the cross-section would act upwards.

Observe that if q be evaluated for two different points of a cross-section of a beam loaded in any manner, F and I_0 will be the same in both, and the two values of q will therefore be to each other directly as the geometrical moments of the parts of the section beyond the points, and inversely as the breadths at the points respectively. But for any given form of cross-section, as rectangular, circular, elliptical, &c., the ratio of the geometrical moments of the portions beyond two points definitely situated in the section with respect to each other, is the same whether the section be large or small; whether, for instance, it be a large circle or a small one; so also is the ratio of the breadths; and hence the distribution of the shearing stress on a cross-section, or the manner in which q varies from point to point in the section, depends *only* upon the *form* of the cross-section

It is instructive to know how the shearing stress is distributed on cross-sections of various forms employed in practice ; and it is of the greatest practical importance to know *where* the intensity is a maximum, the amount of that intensity, and the ratio of its maximum and average values. This ratio is an abstract quantity, and depends only upon the form of the cross-section.

In the graphical solutions it is inconvenient to draw the tangential arrows q as in fig. 85, since they interfere with each other; we will therefore draw them at right angles to AB, when their extremities will give a locus which specifies the distribution. For such a locus the origin will always be at O the neutral axis, the abscissæ y are measured on the vertical axis, and the ordinates q are measured on the horizontal axis; and at any point H whose abscissa y is given, the ordinate q gives the intensity of the shearing stress.

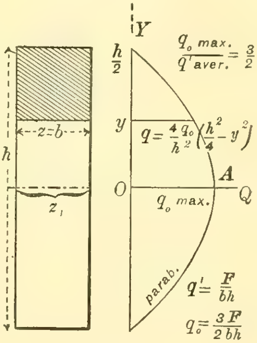

Y
$\frac{h}{2}$
$\frac{q_o \, max.}{q' \, aver.} = \frac{3}{2}$
$z = b$
h
y
$q = \frac{4 q_o}{h^2}\left(\frac{h^2}{4} - y^2\right)$
A
O
Q
$q_o \, max.$
z_1
parab.
$q' = \frac{F}{bh}$
$q_o = \frac{3F}{2bh}$

Fig. 198.

Rectangular cross-section (fig. 198).—Since $z = b$ = constant, q will vary as G the geometrical moment of the shaded portion of the section, that is as the isosceles (frustum) wedge standing on that portion; now this wedge will be greatest when $y = 0$, for then the shaded portion will be the whole isosceles wedge on the semi-section. If y be negative, we have more than the semi-section shaded, but the portion lying below the neutral axis gives a negative geometrical moment, and q is again less than q_0; hence q is a maximum at the neutral axis, it has equal values for equal values of y above and below O, and is zero at each skin. The maximum value is thus

$$q_0 = \frac{F}{I_0} \cdot \frac{G_0}{b} = \frac{F}{\frac{1}{12}bh^3} \; \frac{\frac{1}{2}b\left\{\left(\frac{h}{2}\right)^2 - 0^2\right\}}{b} = \frac{3F}{2bh}.$$

Graphical Solution.—Lay off $\overline{OA} = q_0$; and if we take A as origin, $q_0 - q$ the ordinate of any point will be proportional to the volume of the isosceles wedge on the semi-section minus that on the shaded part, that is to the isosceles wedge on the part of the section from O to y; but the breadth being constant, the volume of that wedge is proportional to $\overline{Oy}^2$; so that from

A as origin, the ordinate of any point on the curve is proportional to y^2, and the curve is a parabola. The modulus of the parabola is

$$\frac{q_0}{(\frac{1}{2}h)^2} = \frac{4q_0}{h^2},$$

and the equation, with O as origin, is

$$q = \frac{4q_0}{h^2}\left(\frac{h^2}{4} - y^2\right).$$

Let $q'_{\text{aver.}}$ represent the average value of the intensity of the shearing force on the cross-section, then

$$q'_{\text{aver.}} = \frac{\text{shearing force}}{\text{area}} = \frac{F}{bh};$$

and we have for the ratio of the maximum and average intensity

$$\frac{q_{0\,\text{max.}}}{q'_{\text{aver.}}} = \frac{3}{2}.$$

*Hollow rectangular cross-section, or symmetrical double-*T *section* (fig. 199).—For values of y from $\frac{1}{2}H$ to $\frac{1}{2}h$, q varies as G

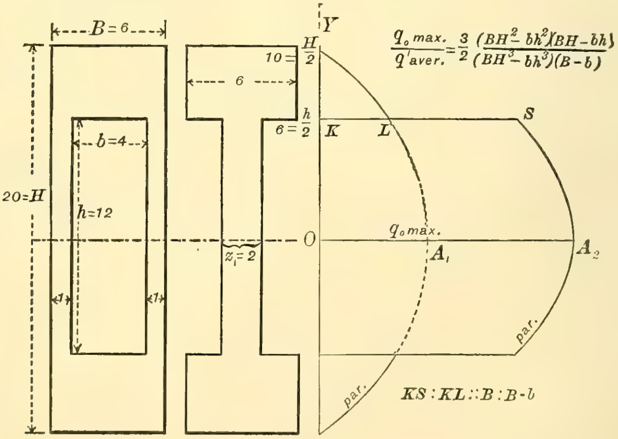


Fig. 199.

exactly as in the previous case, so that the locus from Y to L is a parabola whose apex is A_1; the modulus of this parabola is greater than that for the solid rectangle since the constant divisor I_0 is now less. For values of y between $\frac{1}{2}h$ and 0, consider the effect upon each of the items that make up q. If the hollow were filled up, SA_2 would be part of the parabola in fig. 198; I_0 is now less, however, and allowing for that fact alone SA_2 would coincide with LA_1. The removal of the centre decreases z from B to z_1; this increases each ordinate in the same proportion, so that SA_2 is still a parabola. Lastly, for values of y between $\frac{1}{2}h$ and 0, the removal of the centre alters the value of G from what it would be for the solid figure, by the geometrical moment of the part of the hollow beyond y; now the geometrical moment of that part of the hollow is $\frac{1}{2}b\{(\frac{1}{2}h)^2 - y^2\}$; this leaves the equation consisting of a term in y^2 and a constant term, so that SA_2 is finally a parabola with apex at A_2, but with a modulus different from that of LA_1; and hence $q_0 = OA_2$ is the maximum.

For semi-section,

$$G'_0 = \tfrac{1}{8}(BH^2 - bh^2);$$

for total section,

$$I_0 = \tfrac{1}{12}(BH^3 - bh^3).$$

Area

$$S = (BH - bh), \quad \text{and} \quad z_1 = (B - b)$$

is the breadth at the neutral axis;

$$\therefore \quad q_0 = \frac{F}{I_0} \cdot \frac{G'_0}{z_1} = \frac{3F}{2} \cdot \frac{BH^2 - bh^2}{(BH^3 - bh^3)(B - b)};$$

$$q'_{\text{aver.}} = \frac{F}{S} = \frac{F}{(BH - bh)};$$

$$\therefore \quad \frac{q_{0\text{max.}}}{q'_{\text{aver.}}} = \frac{3(BH^2 - bh^2)(BH - bh)}{2(BH^3 - bh^3)(B - b)}.$$

For the dimensions given in fig. 199,

$$q_{0\text{max.}} = \cdot 033F,$$

$$\frac{q_{0\text{max.}}}{q'_{\text{aver.}}} = 2\cdot 4, \quad \text{and} \quad KS : KL :: 6 : 2.$$

The locus (fig. 199) gives the shearing stress on the horizontal layers as well as on the cross-section. Now, as you pass from the horizontal layer K cutting the web to another cutting the flange, there is a sudden change of intensity from

KS to KL; this change cannot be supposed to take place altogether at K, as the free overhanging surface of the section at that point does not bear any share of the shearing at all, and for a small distance just above K the portion overhanging does not bear its proper share; in the vicinity of K, therefore, the stress is not constant on the horizontal on the cross-section. The consequence of this is to introduce shearing stress on the *vertical* plane through K. In order that the intensity of the stress should change from KL to KS absolutely at K, there would require to be an infinite amount of shearing force on the horizontal plane at K; and since the intensity changes from KL to KS in passing through a *small* vertical distance at K, there must be a *great* amount of shearing force on the horizontal plane at K. Hence sections of this shape very readily give way by shearing at K; cast-iron sections being specially liable to do so under the shearing force developed by irregularities in cooling. Re-entrant angles, as that at K, are to be rounded off in castings and rolled plates, and filled in with angle irons in built sections; this allows the breadth to change *gradually* from B to $(B - b)$, and the intensity of shearing stress to change *gradually* from KL to KS.

It follows as a corollary from fig. 199 that, for symmetrical sections made up of rectangles with breadths diminishing towards the centre, q_0 is the maximum.

Approximate method.—For a cross-section, such as is shown in fig. 199, the web bears the greater share of the shearing stress; and, moreover, the stress is nearly uniform in its distribution. A close approximation to the resistance to shearing for such a cross-section will therefore be obtained by multiplying the area of the web into $q_0 = f$, f being the shearing strength of the material. This is equivalent to considering that the web bears *all* the shearing stress uniformly distributed over it; or that the central parabola, in such a diagram as fig. 199, is replaced by a rectangle of height h and breadth q_0, and that the upper and lower portions of the diagram are left out of consideration. The area of the web required for a double-T section is readily found by the converse of the above, and is given by the equation

$$S = \frac{F}{f},$$

where S is the area of web in square inches; F is the amount of the greatest shearing force in lbs. at the section; and f is the resistance of the material to shearing in lbs. per square inch.

Circular cross-section (figs. 200 and 201).—On the shaded sector (fig. 200) suppose an isosceles wedge standing, made by a plane sloping at 45° and intersecting the horizontal radius, and cut up into pyramids as in fig. 195. In this case the sum of the small quantities δ is $OC = r \sin \theta$, and the volume of the wedge is $\frac{1}{3}r^2 \times r \sin \theta$; for the sector, the geometrical moment about OX, $G = \frac{1}{3}r^3 \sin \theta$; and deducting for triangle ODB, we have for the geometrical moment of ABD about OX as axis

$$G = \tfrac{1}{3}r^3 \sin \theta - (\tfrac{1}{2} . r \sin \theta . r \cos \theta) \times \tfrac{2}{3}r \cos \theta = \frac{r^3}{3} \sin^3\theta.$$

Hence for the shaded part of the circle in fig. 201

$$G = \tfrac{2}{3}r^3 \sin^3\theta ; \quad z = 2r \sin \theta ;$$

and $$\frac{G_0}{z} = \frac{r^2}{3} \sin^2\theta,$$

where θ is half the angle subtended at centre.

But $$y = r \cos \theta ; \quad \therefore \quad \cos \theta = \frac{y}{r} ;$$

$$\sin^2\theta = \frac{r^2 - y^2}{r^2}, \quad \text{and} \quad \frac{G_0}{z} = \tfrac{1}{3}(r^2 - y^2) ;$$

so that $q \propto \frac{1}{3}(r^2 - y^2)$, and the locus is a parabola with its axis on OQ.

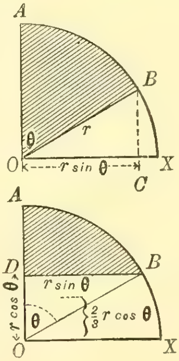


Fig. 200.

Hence q_0 is the maximum; and since

$$G'_0 = \frac{2}{3} r^3 ; \quad I_0 = \frac{\pi}{4} r^4 ; \quad S = \pi r^2 ; \quad \text{and} \quad z_1 = 2r ;$$

$$\therefore \quad q_{0\text{max.}} = \frac{FG'_0}{I_0 z} = \frac{4F}{3\pi r^2} ; \quad q'_{\text{aver.}} = \frac{F}{\pi r^2} ; \quad \frac{q_{0\text{max.}}}{q'_{\text{aver.}}} = \frac{4}{3} .$$

Elliptical section (fig. 201).—Let the ellipse shown in the figure be derived from the circle by altering in a constant ratio the breadth z of the circle at any point, to z' the breadth of the ellipse at the point corresponding, that is, let $z = nz'$.

Suppose the shaded part of the circle to be divided by vertical lines into elementary rectangles, and let the ellipse be correspondingly divided; by considering each of these elementary rectangles, it is easily seen that I, G, S, and z' for the ellipse are derived from the corresponding quantities for the circle by multiplying by n; for the circle we had

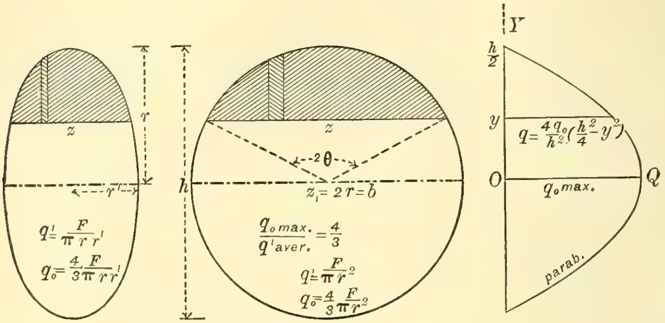

r
z
r'
$q' = \frac{F}{\pi r r'}$
$q_0 = \frac{4}{3}\frac{F}{\pi r r'}$
h
z
2θ
$z_1 = 2r = b$
$\frac{q_0 \text{max.}}{q' \text{aver.}} = \frac{4}{3}$
$q' = \frac{F}{\pi r^2}$
$q_0 = \frac{4}{3}\frac{F}{\pi r^2}$
Y
$\frac{h}{2}$
y
$q = \frac{4q_0}{h^2}\left(\frac{h^2}{4} - y^2\right)$
O
Q
$q_0$max.
parab.

Fig. 201.

$$q = \frac{F}{I}\,\frac{G}{z};$$

so that for the ellipse

$$q = \frac{F}{nI}\,\frac{nG}{nz} = \frac{F}{I}\,\frac{G}{z'}; \quad q_0 = \frac{4}{3}\,\frac{F}{\pi r r'};$$

$$q'_{\text{aver.}} = \frac{F}{\pi r r'}; \qquad \frac{q_{0\text{max.}}}{q'_{\text{aver.}}} = \frac{4}{3},$$

the same as for the circle.

Symmetrical section of three rectangles.—When the middle rectangle is the narrowest, q_0 is the maximum as we saw in the previous case.

In fig. 202 the middle rectangle is the broadest; and in this case, the intensity of the shearing force has two maxima values, one at $y = 0$, the other at $y = \frac{h}{2}$.

For the whole section, $S = bH + (B - b)\,h$;

$$I_0 = \frac{bH^3 + (B - b)\,h^3}{12}; \text{ and } q'_{\text{aver.}} = \frac{F}{bH + (B - b)\,h}.$$

For the middle portion of the section, the maximum value of q is

$$q_0 = \frac{F}{I_0}\,\frac{G_0}{B} = \frac{3F}{2B}\,\frac{bH^2 + (B-b)\,h^2}{bH^3 + (B-b)\,h^3}.$$

For the portion of the cross-section beyond K,

$$G = \frac{b}{8}\,(H^2 - h^2);$$

and the maximum value of q is

$$q_{\frac{h}{2}} = \frac{3F}{2}\,\frac{H^2 - h^2}{bH^3 + (B-b)\,h^3}.$$

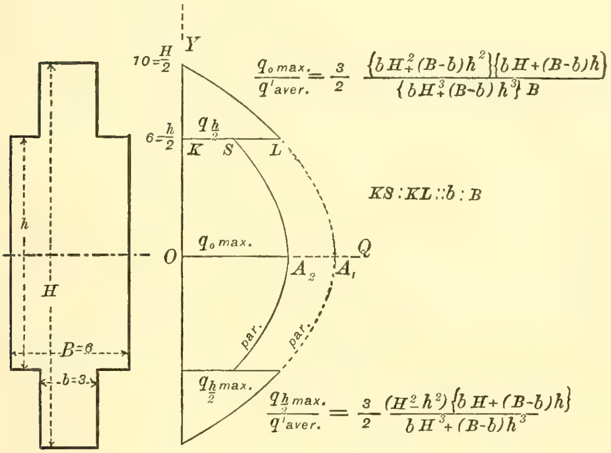

Y
10=H/2
q₀ max./q′aver. = 3/2 {bH²+(B-b)h²}{bH+(B-b)h} / {bH³+(B-b)h³}B
6=h/2
q h/2
K
S
L
KS : KL :: b : B
h
O
q₀ max.
Q
A₂
A₁
H
par.
par.
B=6
b=3
q h/2 max.
q h/2 max./q′aver. = 3/2 (H²-h²){bH+(B-b)h} / bH³+(B-b)h³

Fig. 202.

To find the ratio between the maximum and the average values of q, we have the two equations

$$\frac{q_0\text{max.}}{q'\text{aver.}} = \frac{3}{2}\,\frac{\{bH^2 + (B-b)\,h^2\}\,\{bH + (B-b)\,h\}}{B\{bH^3 + (B-b)\,h^3\}};$$

and

$$\frac{q_{\frac{h}{2}}\text{max.}}{q'\text{aver.}} = \frac{3}{2}\,\frac{(H^2 - h^2)\,\{bH + (B-b)\,h\}}{bH^3 + (B-b)\,h^3};$$

in any example the greater of these two ratios is to be used.

In the numerical example shown in fig. 202.

$$q_0 = \cdot 014F, \quad \text{and} \quad q_{\frac{h}{2}} = \cdot 013F;$$

and

$$\frac{q_0\text{max.}}{q'\text{aver.}} = 1\cdot 3; \quad \frac{q_{\frac{h}{2}}\text{max.}}{q'\text{aver.}} = 1\cdot 25.$$

Triangular section (fig. 203).—Let y, which includes its sign, be the ordinate of the base of any portion such as is shaded in the diagram; then for that portion

$$G_0 = \tfrac{1}{2}\left(\frac{2h}{3} - y\right) z \left\{ y + \tfrac{1}{3}\left(\frac{2h}{3} - y\right)\right\} = \frac{z}{27}(2h - 3y)(h + 3y).$$

Now $$q = \frac{F}{I_0}\frac{G_0}{z}, \text{ and since } \frac{F}{I_0} \text{ is constant,}$$

$$q \propto \frac{G_0}{z}, \text{ or } q \propto (2h - 3y)(h + 3y).$$

The sum of these factors is constant, and their product is therefore greatest when they are equal; that is when

$$(2h - 3y) = (h + 3y), \quad \text{or} \quad y = \frac{h}{6};$$

hence q is a maximum at middle of depth h, and the locus is a parabola with its axis horizontal.

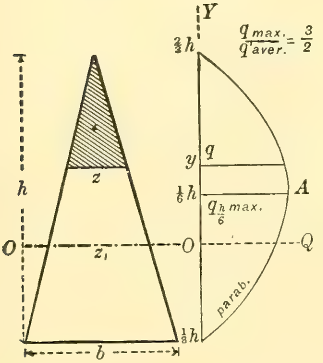


Fig. 203.

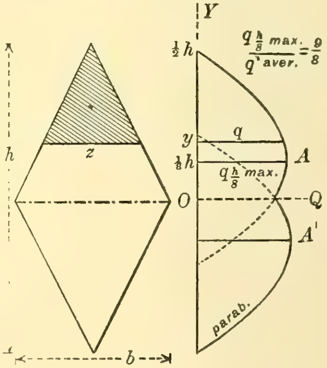


Fig. 204.

For the portion above this central point

$$G_0 = \tfrac{1}{2}\frac{h}{2}\cdot\frac{b}{2} \times \left(\frac{h}{6} + \tfrac{1}{3}\frac{h}{2}\right) = \frac{bh^2}{24}, \quad \text{and} \quad z = \frac{b}{2}.$$

For the whole section $I_0 = \frac{1}{36}bh^3$; hence

$$q_{\frac{h}{6}}\,\text{max.} = \frac{FG_0}{I_0\,.\,z} = \frac{F\,.\,\dfrac{bh^2}{24}}{\dfrac{bh^3}{36} \times \dfrac{b}{2}} = \frac{3F}{bh};$$

$$q\ \text{aver.} = \frac{F}{S} = \frac{F}{\frac{1}{2}bh} = \frac{2F}{bh}; \quad \therefore \quad \frac{q_{\frac{h}{6}}\,\text{max.}}{q'\ \text{aver.}} = \frac{3}{2}.$$

The equation to the parabola is easily found.

Rhomboidal section (fig. 204).—For a portion such as is shaded, in the upper half of section, that is for positive values of y,

$$G_0 = \tfrac{1}{2}\left(\frac{h}{2} - y\right)z\left\{y + \tfrac{1}{3}\left(\frac{h}{2} - y\right)\right\} = \frac{z}{24}\left\{h^2 + 8y\left(\frac{h}{4} - y\right)\right\}.$$

Now $$q \propto \frac{G}{z} \propto h^2 + 8y\left(\frac{h}{4} - y\right),$$

so that the locus is a parabola with its axis horizontal. The sum of the factors $y\left(\frac{h}{4} - y\right)$ is constant, and the product is greatest when these are equal, so that when $y = \frac{h}{8}$, q is a maximum. The locus for the upper half is a parabola whose axis is $\frac{1}{8}h$ above the neutral axis, and for the under half a similar parabola symmetrical below.

For that part of the section above $y = \frac{h}{8}$;

$$G'_0 = \tfrac{1}{2}\cdot\frac{3h}{8}\cdot\frac{3b}{4} \times \left\{\frac{h}{8} + \tfrac{1}{3}\frac{3h}{8}\right\} = \tfrac{9}{256}bh^2; \quad \text{and} \quad z = \frac{3b}{4}.$$

For the cross-section

$$\tfrac{1}{2}I_0 = \tfrac{1}{36}b\left(\frac{h}{2}\right)^3 + \tfrac{1}{2}b\frac{h}{2} \times \left(\tfrac{1}{3}\cdot\frac{h}{2}\right)^2 = \frac{bh^3}{96}; \quad \therefore I_0 = \frac{bh^3}{48};$$

$$\therefore \quad q_{\frac{h}{8}}\text{ max.} = \frac{FG_0}{I_0 \, . \, z} = \frac{9}{4}\frac{F}{bh}; \quad q'\text{aver.} = \frac{F}{\frac{1}{2}bh} = \frac{2F}{bh}; \quad \therefore \frac{q_{\frac{h}{8}}\text{ max.}}{q'\text{ aver.}} = \frac{9}{8}.$$

Regular hexagonal section, diameter vertical (fig. 205).—If we suppose the sloping sides above and below produced to meet the neutral axis, we have a form like fig. 204; and for values of y from $\frac{1}{4}h$ to $\frac{1}{2}h$, everything so far as regards q is the same as if those sides were produced, except the constant I_0 which is less; hence for those values of y, the locus is a parabola with its apex A_1 on the horizontal $\frac{1}{8}h$

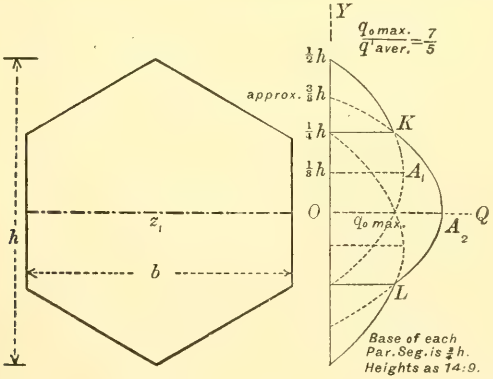


Fig. 205.

above the centre as in the previous case. For values of y from 0 to $\frac{1}{4}h$, suppose the vertical sides produced and the rectangle completed; the four triangles which require to be added on to complete that rectangle increase the constant I_0, but so far as that affects q the locus is still a parabola with its apex on the neutral axis; they also increase G_0 for every value of y, augmenting both the constant term and the term in y^2 in the expression for q; the locus, however, is still a parabola with its axis coinciding with the neutral axis, but the modulus is altered.

Now we know that this parabola intersects the other pair at K and L, points beyond their apexes, hence OA_2 is greater than ordinate of A_1; and $q_0 = OA_2$ is the maximum. From Ex. 213, p. 320, we have

$$G'_0 = \tfrac{7}{96}bh^2; \quad I_0 = \tfrac{5}{128}bh^3; \quad z_1 = b; \quad \text{and} \quad S = \frac{3bh}{4};$$

$$\therefore \quad q_0\,\text{max.} = \frac{FG'_0}{I_0 z} = \frac{28F}{15bh}; \quad q'\text{aver.} = \frac{F}{\frac{3}{4}bh} = \frac{4}{3}\frac{F}{bh}; \quad \therefore \quad \frac{q_0\,\text{max.}}{q'\text{aver.}} = \frac{7}{5}.$$

Regular hexagonal section with diameter horizontal (fig. 206).—Let z be the horizontal line at any distance y from the neutral axis. Suppose the two shaded triangles joined together; they will form an equilateral triangle, for which

$$S = \tfrac{1}{2}\left(\frac{h}{2} - y\right)\frac{1}{\sqrt{3}}\left(\frac{h}{2} - y\right) = \frac{1}{4\sqrt{3}}(h - 2y)^2;$$

$$G_0 = \frac{1}{4\sqrt{3}}(h - 2y)^2\left\{y + \tfrac{1}{3}\left(\frac{h}{2} - y\right)\right\}$$

$$= \frac{1}{24\sqrt{3}}(h - 2y)^2(h + 4y);$$

$$z = \frac{h}{\sqrt{3}} + \frac{2}{\sqrt{3}}\left(\frac{h}{2} - y\right) = \frac{2}{\sqrt{3}}(h - y).$$

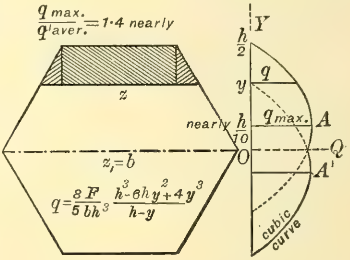

$\frac{q\,\text{max.}}{q'\text{aver.}} = 1{\cdot}4$ nearly
Y
$\frac{h}{2}$
q
y
z
nearly $\frac{h}{10}$
q max.
A
Q
O
$z_1 = b$
A'
$q = \frac{8F}{5bh^3}\,\frac{h^3 - 6hy^2 + 4y^3}{h - y}$
cubic curve

Fig. 206.

For shaded rectangle

$$G_0 = \tfrac{1}{2}\cdot\frac{h}{\sqrt{3}}\left\{\left(\frac{h}{2}\right)^2 - y^2\right\} = \frac{h}{8\sqrt{3}}(h^2 - 4y^2).$$

For the total shaded part, summing and reducing

$$G_0 = \frac{1}{6\sqrt{3}}(h - 2y)(h^2 + 2hy - 2y^2);$$

$$\frac{G_0}{z} = \tfrac{1}{12}\frac{(h - 2y)(h^2 + 2hy - 2y^2)}{h - y};$$

and since $\dfrac{F}{I_0}$ is a constant quantity, we have

$$q \propto \frac{h^3 - 6hy^2 + 4y^3}{h - y}$$

a cubic curve; the maximum value occurs when $y = \dfrac{h}{10}$ nearly; and evaluating for $q_{\frac{h}{10}}\,\text{max.}$ and $q'\text{aver.}$, we find the ratio is $\dfrac{q_{\frac{h}{10}}\,\text{max.}}{q'\text{aver.}} = 1{\cdot}26$.

When several points for this equation are plotted, they give, as shown in the figure, a pair of curves resembling parabolas.

This case shows that the locus giving values of q is not necessarily a parabola.

EXAMPLES.

201. Find the working moment of resistance to bending, and the working resistance to shearing of the section, of which the upper half is shown in fig. 207.

In wrought-iron built sections, the piercing of holes for rivets reduces the effective area to resist tension but not to resist thrust; the area as thus reduced is to the original area in the ratio of about 4 : 5, which is the same as the ratio of the strengths; hence it is usual to make such built sections symmetrical above and below, to consider the working resistance to tension and thrust as each equal to 4 tons per sq. inch, and then to neglect the fact that the holes diminish the effective sectional area; the working resistance to shearing may also be taken at 4 tons per square inch.

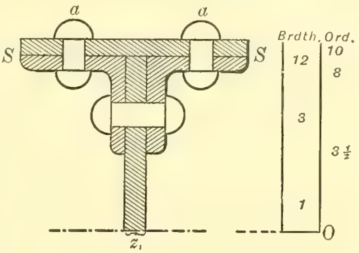


Fig. 207.

b	*Ord.*	*Dif.*	*Dif.* $\times b$	*Ord.*^{2}	*Dif.*	*Dif.* $\times \frac{b}{2}$	*Ord.*^{3}	*Dif.*	*Dif.* $\times \frac{b}{3}$
	10			100			1000		
12		2	24		36	216		488	1952
	8			64			512		
3		4·5	13·5		52	78		469	469
	3·5			12·3			43		
1		3·5	3·5		12	6		43	14
	0			0			0		
$41 \times 2 = 82 = S.$				$G'_0 = 300.$			$2435 \times 2 = 4870 = I_0.$		

$$\mathsf{M} = \frac{f}{\frac{1}{2}h} I_0 = \tfrac{4}{10} \times 4870 = 1948 \text{ inch-tons.}$$

$$q_0 = \frac{\mathsf{F} G'_0}{I_0 z_1}; \quad \therefore \quad \mathsf{F} = \frac{q_0 I_0 z_1}{G'_0} = \frac{4 \times 4870 \times 1}{300} = 65 \text{ tons.}$$

202. If the rivets *a*, *a*, fig. 207, be pitched at 4 inches apart, find the diameter necessary for each rivet.

For part of section beyond *SS*, $G_0 = \frac{1}{2} \times 12\,(10^2 - 9^2) = 114$;

$$\therefore \quad q_0 = \frac{FG_0}{I_0 z} = \frac{67 \times 114}{4870 \times 12} = {\cdot}13 \text{ tons per square inch}$$

is the intensity of the shearing stress on the horizontal plane *SS* at the cross-section, and it will be sensibly constant on *SS* for a few inches on either side of cross-section. There is one rivet for each 24 sq. inches of *SS*; hence a rivet has to bear ·13 × 24 = 3·12 tons of shearing force. If the rivet be very tight, this shearing force would be uniformly distributed on its section, and the area required would be found by dividing by $f = 4$. If the rivet be not perfectly tight, there will be a bending moment on it, and we must consider the shearing stress distributed as in fig. 201; in which case

$$q_0 = 4; \quad \therefore \quad q' = \tfrac{3}{4} \times 4 = 3 \text{ tons per square inch average intensity.}$$

$$\therefore \qquad \text{area} = \tfrac{3\cdot12}{3} = 1\cdot04, \text{ which gives a diameter } d = 1\cdot2 \text{ inches.}$$

Taking 1·2 inches as the diameter of the rivets, the six holes reduce the area of the cross-section by 16·8 sq. inches, almost exactly a fifth as we supposed it would do.

This is on the supposition that the section bears the full shearing force that it can resist; at other cross-sections where the shearing force is less, the rivets might either be made smaller in diameter or be more widely pitched.

203. Find the resistance to shearing of a cross-formed section: width of each pair of wings 5 inches, thickness of metal 1·5 inches; $f = 4$ tons per sq. inch (fig. 208); see also fig. 202.

For semi-section

$$G'_0 = \tfrac{1}{2} \times 1\cdot5\,(2\cdot5^2 - \cdot75^2) + \tfrac{1}{2} \times 5\,(\cdot75^2 - 0^2) = 5\cdot7;$$

and $\dfrac{G'_0}{z_1} = \dfrac{5\cdot7}{5} = 1\cdot14$ is a maximum.

$$\tfrac{1}{2}I_0 = \tfrac{1}{3} \times 1\cdot5\,(2\cdot5^3 - \cdot75^3) + \tfrac{1}{3} \times 5\,(\cdot75^3 - 0^3);$$

therefore $I_0 = 16\cdot6$.

For portion beyond K

$$G_0 = \tfrac{1}{2} \times 1\cdot5\,(2\cdot5^2 - \cdot75^2) = 4\cdot27;$$

and $\dfrac{G_0}{z} = \dfrac{4\cdot27}{1\cdot5} = 2\cdot84$ is therefore the maximum.

$$q_{\cdot75} = \frac{\mathsf{F}}{I_0}\,\frac{G_0}{z}; \quad 4 = \frac{\mathsf{F}}{16\cdot6} \times 2\cdot84; \quad \therefore \ \mathsf{F} = 23\cdot4 \text{ tons.}$$

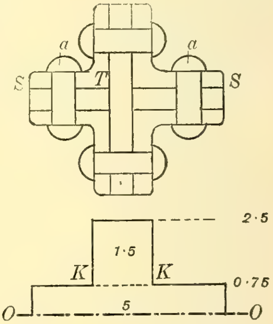


Fig. 208.

204. If the section (fig. 208) be built of half-inch plates as shown, and fixed with rivets a, a, one-inch diameter, find what should be the pitch near the cross-section which is under $\frac{1}{3}$ of the full working shearing force F, if $f = 5$ tons per square inch.

For the part beyond SS, $G_0 = \frac{1}{2} \times 1\cdot5\,(2\cdot5^2 - \cdot75^2) + \frac{1}{2} \times 5\,(\cdot75^2 - \cdot25^2) = 5\cdot5$;

therefore $$q_{\cdot25} = \frac{\frac{1}{3}\mathsf{F}}{{}_0}\,\frac{G_0}{z} = \frac{\frac{1}{3} \times 23\cdot4 \times 5\cdot5}{16\cdot6 \times 5} = \cdot5 \text{ tons per square inch}$$

is the intensity of the shearing stress on the horizontal plane SS.

If the working strength of the rivets be 5 tons per square inch, the average resistance to shearing will be $\frac{4}{5} \times 5 = 4$ tons per square inch nearly; hence $\cdot5 \div 4$, that is $\frac{1}{8}$th of the area along ST, and normal to the paper, must be pierced with holes. But $ST = 2$ inches; so that for every four inches measured on ST normal to the paper, there should be one square inch pierced; that is, rivets about one inch in diameter should be pitched four inches apart.

205. Find the moment of resistance to bending of the symmetrical section, the upper half of which is shown in fig 209; the material is wrought-iron, for which the weaker working strength is $f_a = 4$ tons per square inch.

Consider the two triangles as one of breadth 8 inches; for a triangle about its own neutral axis, $I = \frac{1}{36}bh^3 = 48$; hence for semi-section

$$\tfrac{1}{2}I_0 = (48 + 24 \times 8^2) \text{ for triangle} + \tfrac{1}{3} \times 2\,(10^3 - 0^3) \text{ for rectangle};$$

therefore $I_0 = 4501$; and $\mathsf{M} = \frac{f}{\frac{1}{2}h} I_0 = \frac{4}{10} \times 4501 = 1800$ inch-tons.

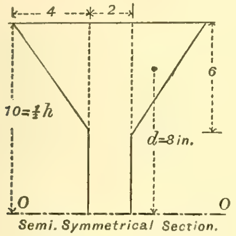


Fig. 209.

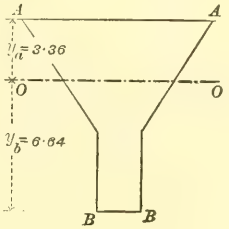


Fig. 210.

206. For the cross-section shown in fig. 210, and which is the same as fig. 209, find the working value of M, if the working strengths of the material be $f_a = 4$, and $f_b = 5$ tons per square inch.

Choose BB as an axis of reference; then

$$S = 24 + 20 = 44; \quad G_B = (24 \times 8) + \tfrac{1}{2} \times 2\,(10^2 - 0^2) = 292.$$

$$I_B = \tfrac{1}{2}I_0 \text{ of Ex. 205 } = 2251.$$

$$y_b = \frac{G_B}{S} = \frac{292}{44} = 6{\cdot}64, \quad \text{and} \quad y_a = 10 - 6{\cdot}64 = 3{\cdot}36;$$

$$I_0 = I_B - S \,.\, y_b^2 = 315; \quad \frac{f_a}{y_a} = \frac{4}{3{\cdot}36} = 1{\cdot}19, \quad \text{and} \quad \frac{f_b}{y_b} = \frac{5}{6{\cdot}64} = {\cdot}752;$$

taking the smaller value, we have

$$\mathsf{M} = {\cdot}752 \times I_0 = 237 \text{ inch-tons.}$$

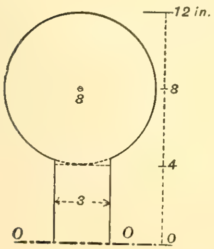


Fig. 211.

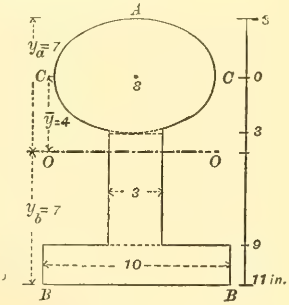


Fig. 212.

207. Find M for the section, the upper half of which is shown in fig. 211; the material is cast iron, for which $f_b = 2$ tons per square inch.

For the circle about its own neutral axis,

$$I = \frac{\pi}{64} bh^3 = \cdot049 \times 8^4 = 201 \text{ ; } \quad \text{and} \quad S = 50\cdot26.$$

For the cross-section,

$$\tfrac{1}{2} I_0 = (201 + 50\cdot26 \times 8^2) + \tfrac{1}{3} \times 3\,(4^3 - 0^3) \text{ ;}$$

$$\therefore \quad I_0 = 6964 \text{ ; } \quad \text{and} \quad \mathsf{M} = \frac{f_b}{\frac{1}{2}h} I_0 = \tfrac{2}{12} \times 6964 = 1161 \text{ inch-tons.}$$

208. Find M for the section shown in fig. 212, the working strengths f_a and f_b being 4 and 5 tons per square inch respectively.

It is convenient to choose CC the diameter of the ellipse as the axis of reference, and we have

$$\begin{array}{lccccc} & \text{Ellipse.} & & \text{Middle rect.} & & \text{Lower rect.} \\ S = & \pi \times 4 \times 3 & + & 3 \times 6 & + & 10 \times 2 = 75\cdot7 \text{ ;} \\ G_C = & 0 & + & \tfrac{1}{2} \times 3\,(9^2 - 3^2) & + & \tfrac{1}{2} \times 10\,(11^2 - 9^2) = 308 \text{ ;} \\ I_C = & \frac{\pi}{64} \times 8 \times 6^3 & + & \tfrac{1}{3} \times 3\,(9^3 - 3^3) & + & \tfrac{1}{3} \times 10\,(11^3 - 9^3) = 2793 \text{ ;} \end{array}$$

$$\bar{y} = \frac{G_C}{S} = \frac{308}{75\cdot7} = 4 \text{ inches sensibly ; so that } y_a = 7, \text{ and } y_b = 7 \text{ inches ;}$$

$$I_0 = I_C - S\bar{y}^2 = 1582.$$

The neutral axis being sensibly in the middle, take $f_b = 4$ the smaller strength, and $\mathsf{M} = \frac{4}{7} I_0 = 904$ inch-tons.

209. Find the resistance to bending for the section shown in fig. 213; the dimensions are in inches, and the strengths of the material are $f_a = 4$ tons (thrust), and $f_b = 5$ tons (tension) per square inch.

Choose CC the diameter of the ellipse as an axis of reference; and for semi-ellipse

$$S = 23\cdot6 \text{ ; } \qquad G_C = -30 \text{ ; } \qquad I_C = 53.$$

For semicircle about its diameter, $I = 245$, and distance from centre to g its centre of gravity is $2\cdot12$ inches; hence about its own neutral axis, that is an axis through g, we have $I_0 = I - S\,(2\cdot12)^2 = 69$; and for semicircle

$$S = 39\cdot3 \text{ ; } \qquad G_C = 398 \text{ ; } \qquad I_C = 4093.$$

For the two triangles, reckoned as one about its own neutral axis $I_0 = 42\cdot7$; hence for double triangle

$$S = 12 \text{ ; } \qquad G_C = 64 \text{ ; } \qquad I_C = 384.$$

For rectangle,

$$S = 24 \text{ ; } \qquad G_C = 96 \text{ ; } \qquad I_C = 512.$$

For total section,

$$S = 98\cdot9 \text{ ; } \quad G_C = 528 \text{ ; } \quad I_C = 5042 \text{ ;}$$

$$\bar{y} = \frac{G_C}{S} = \frac{528}{98\cdot9} = 5\cdot3 \text{ ;}$$

hence $y_b = 8{\cdot}3$ ins., and $y_a = 7{\cdot}7$ ins., and $I_0 = I_C - S\bar{y}^2 = 2250$;

$$\frac{f_a}{y_a} = \frac{4}{7{\cdot}7} = {\cdot}519, \quad \text{and} \quad \frac{f_b}{y_b} = \frac{5}{8{\cdot}3} = {\cdot}602 \; ;$$

taking the smaller value, we have

$$\mathsf{M} = {\cdot}519 I_0 = 1167 \text{ inch-tons.}$$

As shown in the figure, the section is lying in a position suitable for a cantilever ; if intended for a beam, it should be turned upside down ; should it be kept as it stands, however, then for a beam we take the smaller of the two ratios, $\frac{5}{7{\cdot}7}$ and $\frac{4}{8{\cdot}3}$, viz. ·482 ; multiplying I_0 by this quantity gives a value of M, which is less than the above.

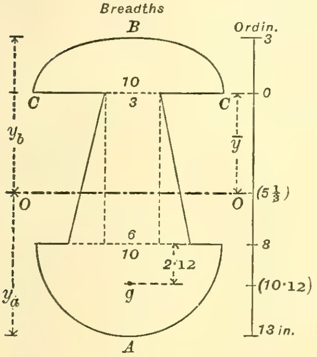


Fig. 213.

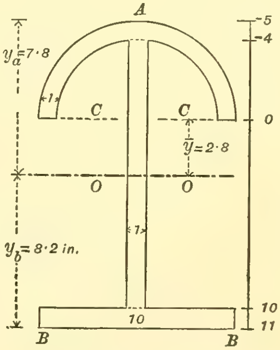


Fig. 214.

210. Find the resistance to bending of the wrought-iron section, fig. 214 ; the dimensions are in inches, and the metal everywhere is 1 inch thick.

Choose CC the diameter of the hollow semicircle as an axis of reference ; then for

	S	G	I_C
Hollow semicircle, ...	14	− 41	144
Rectangle above CC, ...	4	− 8	21
First rectangle below CC, ...	10	50	333
Second rectangle below CC, ...	10	105	1103
Whole section, ...	38	106	1601

For whole section

$$y = \frac{G_C}{S} = \frac{106}{38} = 2\cdot 8\,; \quad \therefore \quad y_a = 7\cdot 8, \quad y_b = 8\cdot 2\,;$$

$$I_0 = I_C - Sy^2 = 1601 - 38 \times 2\cdot 8^2 = 1305.$$

$$\frac{f_a}{y_a} = \frac{4}{7\cdot 8} = \cdot 512, \quad \text{and} \quad \frac{f_b}{y_b} = \frac{5}{8\cdot 2} = \cdot 609\,;$$

$$\therefore \quad \mathsf{M} = \cdot 512 I_0 = 668 \text{ inch-tons.}$$

211. For a wheel, design an elliptical spoke of approximately uniform strength, of a material whose smaller strength is 2 tons per square inch ; length of spoke 3 feet, load at end due to a force applied to the circumference of wheel $\frac{1}{20}$th of a ton, and at each cross-section the depth is to the breadth in the ratio 3 : 1.

$$M_{\text{max.}} = Wl = \tfrac{3}{20} \text{ ft.-tons} = 1\cdot 8 \text{ inch-tons}\,;$$

$$\mathsf{M} = nfbh^2 = \frac{\pi}{32} \times 2 \times b\,(3b)^2 = 1\cdot 76b^3 \text{ inch-tons.}$$

Equating these values of M and M, we have

$$1\cdot 76b^3 = 1\cdot 8\,; \quad \text{therefore} \quad b^3 = 1.$$

Therefore $b = 1$, and $h = 3$ inches, are the diameters of the elliptic section at boss, while two-thirds of these are the diameters at tyre (see fig. 178).

212. Find the thickness of a cast-iron pipe whose external diameter is 2 feet, that it may have a working moment of resistance to bending of 800 inch-tons ; the smaller working strength of cast-iron is 2 tons per square inch.

Put $D = 24$ inches, and let d be the inside diameter ; then

$$I_0 = \frac{\pi}{64}(D^4 - d^4), \quad \text{and} \quad \mathsf{M} = \frac{f}{\frac{1}{2}D} I_0\,;$$

therefore $\quad 800 = \cdot 00818\,(D^4 - d^4), \quad$ or $\quad (D^4 - d^4) = 97800.$

Ans. $d = 22$ inches ; and for the thickness of the metal, we have $t = 1$ inch.

213. Find the relative strengths of a regular hexagonal section to resist bending when lying with a diameter horizontal, and when lying with a diameter vertical. Find also G'_0 for semi-section in the latter case.

For the diameter horizontal, page 296, we had $\mathsf{M} = \frac{5}{48} fbh^2$; if a be put for the side of the hexagon, then $h = a\sqrt{3}$, and $b = 2a$; this gives

$$\mathsf{M} = \tfrac{5}{48} f \,.\, 2a \,.\, (a\sqrt{3})^2 = \tfrac{5}{8} fa^3.$$

When the diameter is vertical, the half-section consists of a rectangle and a triangle ; the rectangle is of breadth b, one edge is against the neutral axis, and the other is at a distance $\frac{1}{4}h$ therefrom ; the triangle is of breadth b, its height is $\frac{1}{4}h$, and its centre of gravity is $\frac{1}{3}h$ distant from the neutral axis ; hence

$$\tfrac{1}{2}I_0 = \tfrac{1}{3}b\left\{\left(\frac{h}{4}\right)^3 - 0^3\right\} + \left\{\tfrac{1}{36}b\left(\frac{h}{4}\right)^3 + \left(\tfrac{1}{2}\,.\,b\,.\,\frac{h}{4}\right)\left(\frac{h}{3}\right)^2\right\};$$

therefore $I_0 = \frac{5}{128}bh^3$ and $\mathsf{M} = \dfrac{f}{\frac{1}{2}h}I_0 = \frac{5}{64}fbh^2.$

In this case $h = 2a$, and $b = a\sqrt{3}$; so that

$$\mathsf{M} = \tfrac{5}{64}f\,.\,(a\sqrt{3})\,(2a^2) = \frac{5\sqrt{3}}{16}fa^3.$$

Ans. The strengths are as $2 : \sqrt{3}$, being greater when the diameter is horizontal; and

$$G'_0 = \tfrac{1}{2}b\left\{\left(\frac{h}{4}\right)^2 - 0^2\right\} + \left(\tfrac{1}{2}\,.\,b\,.\,\frac{h}{4}\right)\left(\frac{h}{3}\right) = \tfrac{7}{96}bh^2.$$

214. The section of a beam is a rectangle 2 inches broad by 6 inches deep ; the material is wrought-iron whose working resistance to shearing is $f = 5$ tons per square inch. Find the working resistance to shearing of the cross-section.

F will be such that q_0 the maximum intensity shall attain to $f = 5$ tons per square inch ;

$$\therefore \quad q_0 = 5\text{; but } q'_{\text{aver.}} = \tfrac{2}{3}q_0 = \tfrac{10}{3}\text{ tons.}$$

$$\mathsf{F} = \text{average intensity} \times \text{area} = q'\,.\,bh = \tfrac{10}{3} \times 12 = 40\text{ tons.}$$

215. Compare the shearing force and the resistance to shearing of the cross-section at the fixed end of the cantilever of Ex. 196, taking the working strength of the timber to resist shearing as $\frac{1}{10}$ of a ton per square inch.

$$F = 2\text{ tons; } \quad q_0 = \tfrac{1}{10}\text{ ton per sq. inch ; } \quad \therefore\ q' = \tfrac{2}{3} \times \tfrac{1}{10} = \tfrac{1}{15}\text{ ton per sq. inch.}$$

$$\mathsf{F} = q' \times \text{area} = \tfrac{1}{15} \times 6{\cdot}4 \times 15 = 6{\cdot}4\text{ tons.}$$

So that in providing for the bending moment at the fixed end, there is three times the necessary strength to resist shearing; but near the free end F is still 2 tons, while F will only be half as much as before, namely 3·2 tons.

216. Find the resistance to shearing of the cross-section, fig. 185, taking $f = 4$ tons per square inch, q_0 is the maximum value.

From the tabular form or graphical solution we had

$$I_0 = 4634, \qquad G'_0 = 296, \quad \text{and} \quad z_1 = 1{\cdot}5\text{ ;}$$

but $$q_0 = \frac{\mathsf{F}G'_0}{I_0 z_1}, \quad \text{or} \quad 4 = \frac{\mathsf{F} \times 296}{4634 \times 1{\cdot}5}\text{ ;} \quad \therefore \quad \mathsf{F} = 94\text{ tons.}$$

217. Find the resistance to shearing of the section shown in fig. 184, taking f the resistance of the material to shearing at 4 tons per square inch.

$$q_0 = \frac{\mathsf{F}G'_0}{I_0 z_1}\text{ ;} \qquad 4 = \frac{\mathsf{F} \times 396}{6112 \times 3}\text{ ;} \qquad \therefore \quad \mathsf{F} = 185\text{ tons.}$$

218. In section (fig. 184), find the ratio of the maximum and average intensity of the shearing stress.

$$q_0 = \frac{FG'_0}{I_0 z_1}, \quad \text{and} \quad q' = \frac{F}{S};$$

therefore $$\frac{q_0 \text{ max.}}{q' \text{ aver.}} = \frac{SG'_0}{I_0 z_1} = \frac{120 \times 396}{6112 \times 3} = 2{\cdot}6.$$

219. Find F for fig. 199, taking $f = 4$ tons per square inch; and find the ratio of the maximum and average intensity of the shearing force.

$$\tfrac{1}{2} I_0 = \tfrac{1}{3} \times 6\,(10^3 - 6^3) + \tfrac{1}{3} \times 2\,(6^3 - 0^3); \quad \therefore \quad I_0 = 3424;$$

$$G'_0 = \tfrac{1}{2} \times 6\,(10^2 - 6^2) + \tfrac{1}{2} \times 2\,(6^2 - 0^2) = 228.$$

$$\tfrac{1}{2} S = 6\,(10 - 6) + 2\,(6 - 0); \quad \therefore \quad S = 72.$$

$z_1 = 2$, and we wish to have $q_0 = 4$; but $q_0 = \dfrac{\mathsf{F} G'_0}{I_0 z_1}$; $\therefore$ F = 120 tons.

$$q' \text{aver.} = \frac{\mathsf{F}}{72} = \tfrac{5}{3} \text{ tons}; \quad \text{and} \quad \frac{q_0 \text{ max.}}{q' \text{ aver.}} = \frac{4}{\frac{5}{3}} = 2{\cdot}4.$$

220. Find the ratio of the maximum and the average intensity of the shearing stress for fig. 212.

Take the under half which consists entirely of rectangles, and find $G'_0 = 157{\cdot}5$.

ow $I_0 = 1582$, $S = 75{\cdot}7$, and $z_1 = 3$.

Ans. $\dfrac{q_0 \text{ max.}}{q' \text{aver.}} = 2{\cdot}5.$

221. A triangular section, as shown in fig. 203, is 12 inches deep, and the resistance to shearing is $f = 4$ tons per square inch. Find what the breadth should e in order to make F = 100 tons.

$$q_0 = 4; \quad \therefore \; q' = \tfrac{2}{3} \times 4 = \tfrac{8}{3}; \quad S = \tfrac{1}{2} bh = 6b; \quad \mathsf{F} = 100.$$

$$\mathsf{F} = q'S, \quad \text{or} \quad 100 = \tfrac{8}{3} \times 6b; \quad \therefore \quad b = 6{\cdot}25 \text{ inches}.$$

222. Compare the resistance to shearing for the cross-section shown in fig. 183, as obtained by the exact and approximate formulæ respectively, taking $f = 5$ tons per square inch.

$$\mathsf{F} = \frac{I_0 z_1 q_0}{G'} = \frac{15588 \times 4 \times 5}{693} = 450 \text{ tons (exact)}.$$

$$\mathsf{F} = fS = 5 \times 96 = 480 \text{ tons (approx.)}.$$

223. Suppose the web of the cross-section shown in fig. 184 to extend to the under side of the upper plate, take $f = 4$ tons per square inch, and find approximately the resistance to shearing. Compare the approximate result with that obtained for Ex. 217.

$$\mathsf{F} = 4 \times 48 = 192 \text{ tons}.$$

224. Suppose the web of the cross-section shown in fig. 185 to extend to the inner faces of the outer plates, and find the approximate resistance to shearing, supposing $f = 4$ tons per square inch; compare this with the correct result of Ex. 216.

$$\mathsf{F} = 4 \times 15 \times 1{\cdot}5 = 90 \text{ tons (approx.)}.$$

225. The depth of a girder is 20 inches, the upper and lower flanges are each 1 inch in thickness, and the shearing force to be resisted is 65 tons. Find the thickness required for the web, supposing $f = 4$ tons per square inch.

$$S = \tfrac{65}{4} = 16\tfrac{1}{4} \text{ square inches};$$

the depth of the web is 18 inches, so that the thickness required is a little over nine-tenths of an inch.

The resistance to shearing of the section shown in fig. 207 is (Ex. 201) 65 tons by the exact formula, and the web is 1 inch thick.

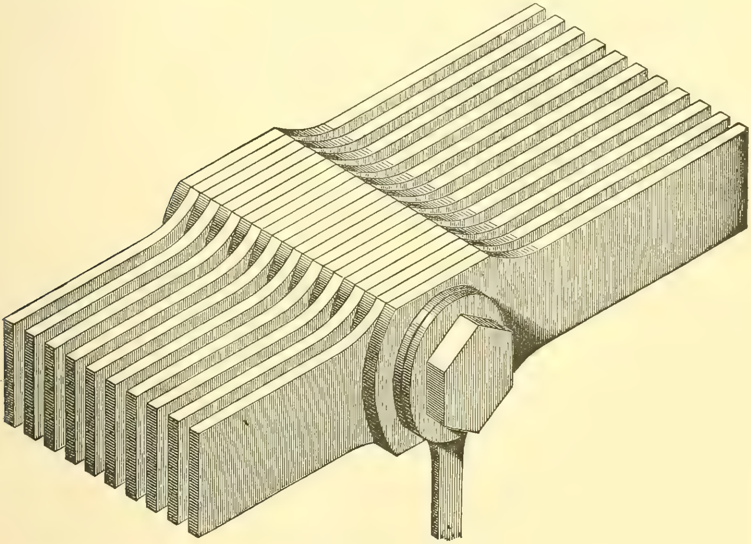


Fig. 215.

226. In the chain of a suspension bridge ten flat links dove-tail with eleven alternately, and a cylindrical pin passes through the eyes. If the pull on the chain be 500 tons, find the area of the pin, supposing that $f = 6$ tons per square inch.

As the pin would require to shear at 20 sections simultaneously, $500 \div 20 = 25$ tons is the shearing force on the section,

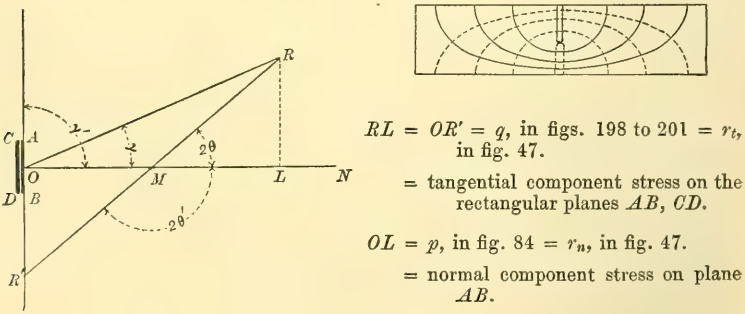
$$q_0 = f = 6, \quad \text{and} \quad q' = \tfrac{3}{4} q_0 = 4{\cdot}5\,; \quad \mathsf{F} = q'.\,S, \quad \text{or} \quad 25 = 4{\cdot}5\,S\,;$$

therefore area of pin = 5·6 square inches, or diameter = 2·7 inches.

CHAPTER XVI.

STRESS AT AN INTERNAL POINT OF A BEAM.

RETURNING to fig. 82, p. 118, we have now found the intensity and obliquity of the stresses at the point *H*, on *AB* and *CD* the pair of rectangular planes through it, viz.:—On *CD* the total stress tangential and given by q in such diagrams as fig. 198, and on *AB* the total stress of intensity r at obliquity γ (fig. 216) given in terms of its normal component p, and its tangential component q on such diagrams as fig. 198. The planes of principal stress at *H* are to be found as on fig. 143; making that construction, and noting that γ' is a right angle, we have (fig. 216)



$RL = OR' = q$, in figs. 198 to 201 $= r_t$, in fig. 47.

= tangential component stress on the rectangular planes *AB*, *CD*.

$OL = p$, in fig. 84 $= r_n$, in fig. 47.

= normal component stress on plane *AB*.

Fig. 216.

$$OM = \tfrac{1}{2}p, \quad \text{and} \quad MR = \sqrt{\frac{p^2}{4} + q^2}.$$

These are readily calculated and become known quantities; we also have

$$\tan 2\theta = \frac{RL}{ML} = \frac{\text{arrow } q \text{ (fig. 198)}}{\text{half-arrow } p \text{ (fig. 84)}},$$

giving θ the angle which the plane of greater principal stress makes with the plane of cross-section AB, both planes being normal to the paper. Further

$OM + MR$ = intensity of greater principal stress (of same kind as arrow p, fig. 84);

$OM - MR$ = intensity of smaller principal stress (of opposite kind from arrow p, fig. 84);

and MR = intensity of greatest tangential stress, being on the planes inclined at 45° to the planes of principal stress;

that is, we have the maximum value of the thrust, tension, and shearing stress at the point H.

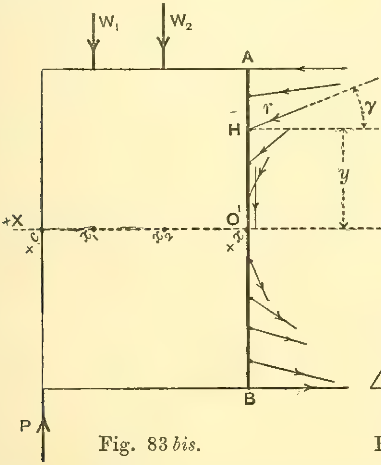


Fig. 83 *bis.*

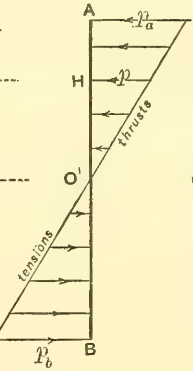


Fig. 84 *bis.*

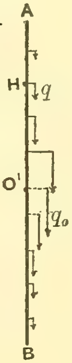


Fig. 85 *bis.*

That the two principal stresses are of opposite kind, follows from the fact that the stress on the plane CD is wholly tangential (fig. 32, p. 46).

If the principal planes be drawn through H for a short distance on each side, that upon which the stress is thrust by a full line, and that upon which it is tension by a dotted line, and if this be done for a number of points H on the elevation; then the full lines and the dotted lines will each form a series of polygons with short sides, and if we take the points H close enough, each polygon becomes a curve. These curves are called lines of principal stress; and the tangents thereto at any point H, where a dotted line and a full line curve intersect, are the planes of principal stress at H. See Rankine's "Applied Mechanics," sect. 310, and his "Treatise on Shipbuilding."

These curves have the following properties:—They cut the neutral axis at 45°; for, considering a point there, $p = 0$, and in fig. 216 R will fall on the line $R'O$ produced upwards, $2\theta = 90°$, and $\theta = \theta' = 45°$. Similarly all lines that meet the end section, that is the section over the point of support, meet it at an inclination of 45°; for, since the bending moment is zero, we again have $p = 0$. All lines meet the upper and under skin at right angles, since $q = 0$ at A and B (fig. 85).

In the elevation of the same beam, these curves will be different for different loads, except when the loads are kept in the same positions on the beam and altered in a fixed ratio; thus, for a rectangular beam—if the load be uniform, then for positions of H ranged on a horizontal line, the arrows p and q, figs. 84 and 85, will both vary, since the bending moment and shearing force alter for each cross-section; if the load be at the middle of span, p will vary, but q will not, since the shearing force is the same at each cross-section.

The planes of principal stress will differ in the elevations of two beams which are loaded alike, and whose cross-sections are different but uniform throughout in each case; for instance, if the cross-sections be a rectangle and a triangle respectively, we have $q_{\text{max.}}$ at the centre of the cross-section in both cases; the neutral axis, however, is at the centre in one case but not in the other; so that, though both be loaded alike, the planes of principal stress will differ in the two elevations.

In designing beams, it will be seen that we have followed the usual practice of considering p_a and p_b at the section of maximum bending moment to be the greatest value of thrust and tensile stress respectively, and q_0 for the section over the greater supporting force to be the greatest intensity of shearing stress. Strictly the points at which these maxima occur are to be defined thus:—Let x, y (fig. 82) be the coordinates of any point H referred to O the centre of the neutral axis as origin; then for the cross-section at x

$$M_x = np_a bh^2, \quad \text{and} \quad p_a = \frac{M_x}{nbh^2};$$

also $$p_y : p_a :: y : m'h; \quad \therefore \; p_y = \frac{yM_x}{m'nbh^3}.$$

We also have $$q' = \frac{F_x}{S}, \quad \text{and} \quad q_0 = \frac{kF_x}{S};$$

k being the ratio of the maximum and the average intensity of the shearing stress for such cross-section; q_y is readily derived by considering the manner in which the stress is distributed. Then (fig. 216)

$$MR^2 = q^2{}_y + (\tfrac{1}{2}p_y)^2,$$

and by finding the values of x and y which make this a maximum, we get the point at which the intensity of the shearing stress is greatest; also

$$(OM \pm MR) = \tfrac{1}{2}p_y \pm \sqrt{q^2{}_y + (\tfrac{1}{2}p_y)^2},$$

and by finding the values of x and y which make this sum and difference a maximum respectively, we get the maximum value of the intensity of the thrust and of the tension.

To find these maxima is difficult even in the easiest cases; for instance, for a beam of uniform rectangular cross-section, loaded at the centre, we have

$$p_a = \frac{\frac{1}{2}W(c-x)}{nbh^2} = \frac{3W(c-x)}{bh^2}; \quad p_y = \frac{y}{\frac{1}{2}h}p_a = \frac{6W(c-x)y}{bh^3};$$

$$q' = \frac{\frac{1}{2}W}{bh} = \frac{W}{2bh}, \quad \text{and} \quad q_0 = \tfrac{3}{2}q' = \frac{3W}{4bh};$$

$$q_y = \frac{4q_0}{h^2}\left(\frac{h^2}{4} - y^2\right) = \frac{3W}{4bh^3}(h^2 - 4y^2);$$

and $MR^2 = \left(\dfrac{3W}{4bh^3}\right)^2 \{(h^2 - 4y^2)^2 + 16y^2(c-x)^2\}$;

for every value of y, $\overline{MR}^2$ is greatest when $x = 0$, and then

$$\overline{MR}^2 = \left(\frac{3W}{4bh^3}\right)^2 \{(h^2 - 4y^2)^2 + 16c^2y^2\}$$

$$= \left(\frac{3W}{4bh^3}\right)^2 \{(2c^2 + 4y^2 - h^2)^2 - 4c^2(c^2 - h^2)\},$$

the greatest value of this is obtained by putting

$$y = \pm\frac{h}{2}, \quad \text{then} \quad \overline{MR} = \frac{3Wc}{2bh^2}$$

at the surface of the beam at centre of span.

The quantity p_y is greatest when

$$x = 0, \quad \text{and} \quad y = \frac{h}{2}; \quad \text{then} \quad p_y = \frac{6W(c-x)y}{bh^3} = \frac{3Wc}{bh^2};$$

and since p_y and $\overline{MR}$ thus have their greatest values at the same point, viz., at the surface at the middle of the beam, the greatest principal stress is there situated, and its amount is

$$\overline{OM} + \overline{MR} = \frac{3Wc}{bh^2}.$$

EXAMPLE.

227. A beam of constant rectangular section 9 ins. broad, 20 ins. deep, and 20 ft. span, bears a load of 96 tons uniformly distributed. At a point in the section half-way between the centre and left end, and half-way between the neutral axis and upper skin, find the greatest intensities of thrust, of tension, and of shearing stress; and find the inclinations of the planes of principal stress at the point.

$$M_5 = 2160 \text{ inch-tons}; \quad \tfrac{1}{6}p_a bh^2 = M_5; \quad \therefore\ p_a = \frac{6 \times 2160}{9 \times 20^2} = 3\cdot6,$$

and $$p_y = \tfrac{1}{2}p_a = 1\cdot8 \text{ tons per square inch.}$$

$$F_5 = 24 \text{ tons}; \quad q' = \frac{F}{S} = \frac{24}{9 \times 20} = \frac{2}{15}; \quad q_0 = \tfrac{3}{2}q' = \cdot2;$$

and $$q_y = \tfrac{3}{4}q_0 = \cdot15 \text{ ton per square inch}; \quad MR^2 = \cdot15^2 + \cdot9^2.$$

Therefore $$MR = \cdot91 \text{ ton per square inch max. intensity of shearing stress.}$$

$$OM + MR = \cdot9 + \cdot91 = 1\cdot81 \text{ ton per square inch max. (thrust)};$$

$$OM - MR = \quad - \cdot01 \text{ ton per square inch max. (tension).}$$

$$\text{Tan } 2\theta = \frac{RL}{ML} = \frac{\cdot15}{\cdot90} = 0\cdot1\dot{6}; \quad \therefore\ 2\theta = 9^\circ\,28';$$

$\theta = 4^\circ\,44'$, the inclination of the plane of greater principal stress to the cross-section.

Determination of internal stress by the polariscope.—We have much pleasure in reproducing the following paper.* I do not mean to give a detailed scientific explanation of the theory of light, but, with a view to make my remarks understood by those who have either not studied the subject or have forgotten it, I must make a few remarks thereon.

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* "How to use the Polariscope in the Practical Determination of Internal Stress and Strain." By the late Professor Peter Alexander. A Paper read before the Mathematical and Physical Section of the Philosophical Society of Glasgow, 3rd February, 1887, and reprinted here by the kind permission of the Society.

The following quotation from Spottiswoode's "Polarization of Light" is very clear and concise:—"It is considered as established that light is due to the vibrations of an elastic medium, which, in the absence of any better name, is called ether. The ether is understood to pervade all space and all matter, although its motions are affected in different ways by the molecules of the various media which it permeates. The vibrations producing the sensation of light take place in planes perpendicular to the direction of the ray. The paths or orbits of the various vibrating ethereal molecules may be of any form consistent with the mechanical constitution of the ether; but on the suppositions usually made, and none simpler have been suggested, the only forms possible are the straight line, the circle and the ellipse. But in ordinary light the orbits at different points of the ray are not all similarly situated, although there is reason to believe that in general the orbits of a considerable number of consecutive molecules may be similarly situated; yet in a finite portion of the ray there are a sufficient number of variations of situation to prevent any preponderance of average direction.

"This being assumed, the process of polarization is understood to be the bringing of all the orbits throughout the entire ray into similar positions. And in the case of the tourmaline plate, the orbits are all reduced to straight lines, which consequently lie in one and the same plane. For this reason the polarization produced by tourmaline, as well as by most other crystals, is called rectilinear, or, more commonly, plane polarization. This property of tourmaline may also be expressed by saying that it permits only rectilinear vibrations, parallel to a particular direction, determined by its own internal structure, to traverse it."

A ray of light is analogous to a series of pulses of transverse vibration, travelling along a stretched cord or wire. If the particles of the cord vibrate in straight lines all in one plane, the series of waves is analogous to a ray of plane polarized light. If the particles of the cord vibrate in stationary elliptic orbits similar to each other and similarly situated, the set of waves will be analogous to a ray of elliptically polarized light. If the ellipses have equal axes they are circles, and the set of waves is analogous to a ray of circularly polarized light. If the particles vibrate in elliptic (including circular and rectilinear) orbits, the ellipses not being stationary, but rotating about the line along which the waves are propagated, the set of waves will be analogous to a ray of ordinary light. This motion of the ellipses in their own planes, about a line perpendicular to their planes, is called *apsidal* motion, and, as a consequence of the propagation of this motion from particle to particle of the cord or medium, the ellipses will not be similarly situated, but the whole series will form a surface which will be the same as that of an elliptic cylinder which has been twisted uniformly about its axis.

If two lines be chosen perpendicular to each other and to the line of propagation of the vibrations, any set of pulses of transverse vibration is equivalent to two simultaneous sets of transverse rectilinear vibrations parallel, respectively, to the two chosen lines, that is to say, to two rays of plane polarized light travelling together having their planes of vibration perpendicular to each other.

When a ray of light enters an æolotropic transparent substance it is usually resolved into two rays of plane polarized light vibrating in planes perpendicular to each other, parallel to the directions of greatest and least elasticity of the substance. The exceptions to this are those in which the ray was plane polarized, and had its plane of vibration parallel to the axis of greatest or least elasticity of the substance before entering the same, in which cases the ray suffers no change except, perhaps, a change of direction.

A polariscope consists of two parts, namely, a polarizer and an analyzer. The polarizer changes a ray of common light into a ray of plane polarized light, by suppressing one of the two component plane polarized rays which the former is equivalent to. The analyzer may be the same as the polarizer, but its function is different. It is used to examine the nature of a ray of light entering it.

If the polarizer and analyzer be similarly situated, the ray of light which has been polarized by the former will pass unchanged through the latter. If, now, the analyzer be rotated about its axis through a right angle, it will suppress the ray which has been polarized by the polarizer, and darkness will be the result. Again, if a plate of æolotropic transparent matter, with parallel faces, be introduced between the polarizer and analyzer with its faces perpendicular to the ray passing from the former to the latter, the ray will in general suffer a change, because its component plane polarized rays, with vibrations parallel to the directions of greatest and least elasticity of the introduced plate, will travel through the plate with different velocities, and therefore the one will suffer retardation relatively to the other. Consequently the resultant ray which leaves the plate will not, in general, be the same as that which enters it; and therefore when it enters the analyzer it will be resolved into two components, one with its vibrations parallel to those of the ray which leaves the polarizer, and the other with its vibrations perpendicular to those of the former, and the analyzer will suppress the former component, but will allow the latter to pass through it. Hence the light will be partially restored. Should, however, the plate be introduced with its axis of greatest elasticity or least elasticity parallel to the vibrations of the ray leaving the polarizer, it will allow the ray to pass through unchanged, and consequently the analyzer will entirely suppress the ray, and there will still be darkness.

If the plate be not very small, and if instead of a single ray of light we have a cone or sheaf of rays, it may happen that at some parts of the plate the direction of greatest or least elasticity is parallel to the direction of the vibrations of the rays, and at other parts not so, which will be the case if the lines of stress in the plate are curvilinear. Should this be so, the whole plate will appear luminous, except along certain lines, which are the *loci* of points at which the axis of greatest or least elasticity is parallel to the vibrations of the rays leaving the polarizer, which lines will appear black. If the plate be now rotated about a line through its centre and perpendicular to its faces, the black lines will assume new positions in the plate.

To illustrate this, I shall show you the black line in a rectangular bar of glass subjected to stress, similar to that of a beam supported at both ends, and uniformly loaded throughout its length. You will observe that when the length is parallel to the direction of vibration of the polarized ray, the black line lies along the upper and lower edges of the bar, and perpendicularly crosses the middle of the bar, as is indicated in fig. 217.

It shows the forms and positions of the black line for nine positions of the bar, from the horizontal position to the vertical position and corresponding to the first eight points of the compass.

It will be observed that the central point of the bar is *always* black; consequently we conclude that there is *no* stress there. At every other point of the black *loci* lines of stress cross them horizontally and vertically, that is to say, parallel and perpendicular to the direction of vibration of the polarized ray.

If we wish to draw lines of stress we can do so as as follows:—Draw all these black *loci* in one rectangle on a piece of cardboard as in fig. 218.

Let the card be mounted so as to be capable of rotation about a horizontal axis passing perpendicularly through the centre of the rectangle. Let the card now be rotated slowly about this axis, and simultaneously let a tracing point (fig. 219), carried round with the card, move horizontally so as always to cross each black *locus* at the instant when the rectangle is in the position corresponding to this *locus*. The line thus traced on the cardboard will be approximately a line of stress. By this method a network of lines of stress can be drawn in the

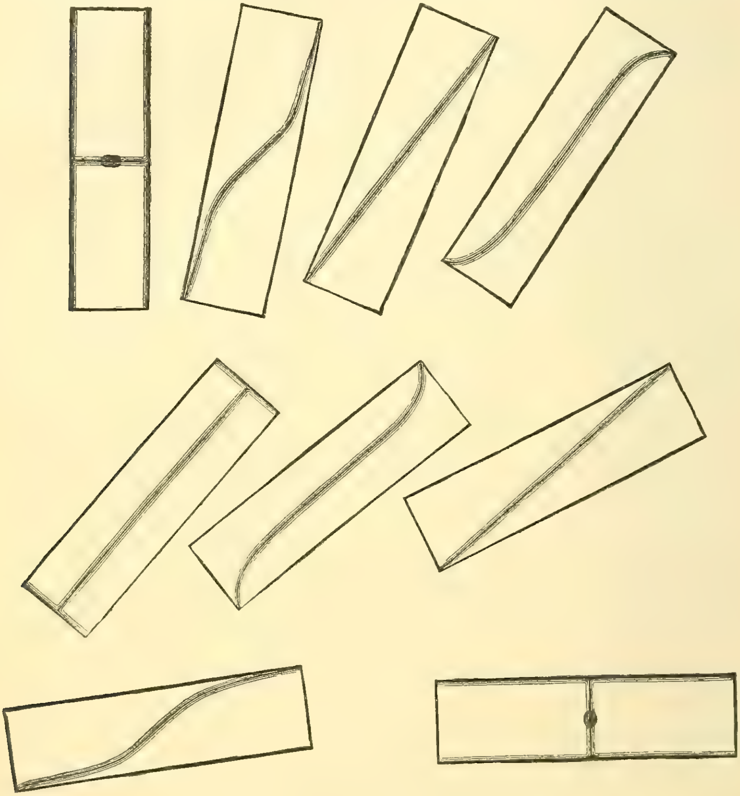

Fig. 217.

rectangle on the cardboard. This network (fig. 218) will show what is the state of internal stress of the glass bar and consequently that of the uniformly loaded beam.

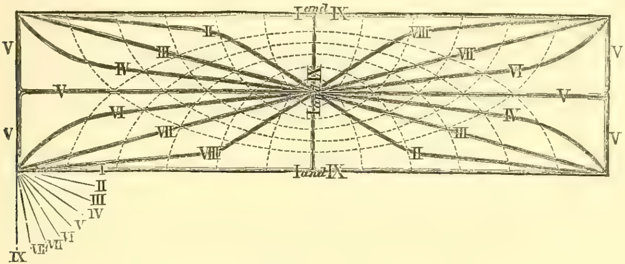

Fig. 218.

The black *loci*, I, II, &c., correspond to the angular positions, I, II, &c., of the card. The dotted lines are the lines of internal stress.

A more accurate way of drawing the lines of stress is to rotate the card in front of the lantern simultaneously with the rotation of the glass bar, so as always to have the image of the glass bar occupying the same position on the card, and let the tracing point, while carried round by the card, at the same time move horizontally, so as always to be on the black *locus* projected on the card. The line drawn in this way will be an accurate delineation of a line of stress. This motion of the tracing point may be accomplished as follows:—Let the tracer be a small toothed wheel, *T*, capable of rotation about a fixed axis, *AA′*, at the end of a tracing pen, *P*, and let a heavy load, *W*, be attached rigidly to the pen at the end of a rod, *D*, so as to compel the axis, *AA′*, of the tracing wheel to be vertical when in use.

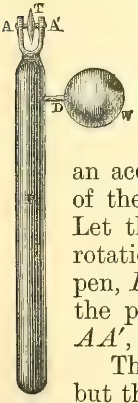

Fig. 219.

The tracing point will thus be incapable of any motion but that communicated to it by the rotation of the card, and the horizontal motion given to it by the rotation of the tracing wheel as the tracing point is made to keep on the black *locus*.

I append a rough drawing of the instrument by which I imitate the stresses to which parts of structures are subjected:—

Fig. 220 shows side and end elevations, and plan of the instrument. *EFKH* denote the glass bar, *RW* a glass bar which receives the pressure from the screw *SS′* and communicates it to the glass bar through bits of indiarubber, such as

MN, interposed between the brass and glass. B is the screw at the top of the instrument by which the pressure is communicated. The end view shows the openings at the end through which the bars of glass and brass are introduced. PP' and Q are pins upon which the ends of the glass bar are supported. If we wish to imitate a beam fixed at the ends we must introduce other pins, T and V, above the ends of the glass bar. Fig. 220 is intended to indicate the arrangement for imitating a beam supported at both ends and subjected to a uniformly distributed load. This is approximately imitated by introducing a long strip of rubber EF between the brass and glass bars. To imitate loads supplied at isolated points of the beam, I introduce little narrow bits of rubber at these points between the glass and brass. For example, fig. 221 shows how I imitate the stress of a beam loaded at its centre and supported at its ends, by introducing a little bit of rubber Z between the brass bar RW and the glass bar $EFKH$, right under the screw SS'. Uniformly bending stress may be imitated by introducing little bits of rubber between the brass and glass at X_1 and X_2.

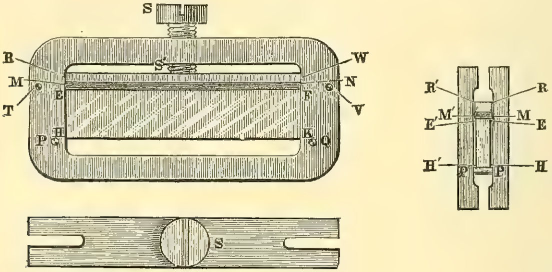

Fig. 220.

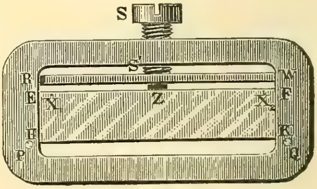

Fig. 221.

Other forms of stress may be similarly imitated.

The black *loci* can easily be studied by turning the apparatus round with the hand in an ordinary table polariscope, but in the lantern polariscope the rotation must be communicated by a contrivance similar to that which rotates the lantern kaleidoscope.

The idea of turning the study of these black *loci* to practical use for the determination of the state of internal stress of parts of a structure was suggested to me some three or four years ago by a civil engineer, Mr. James Love, who was superintending the manufacture of certain works by a Glasgow firm for a London firm of civil engineers. His hopes of the usefulness of the method were so great that he entered on a systematic study of higher mathematics and optics, so as to be able to work it out.

I have lost sight of Mr. Love, but I have thought the matter out, and this is the result divested of mathematical symbolism.[*]

We think the remarkable similarity of the lines of stress, as drawn by Professor Peter Alexander's practical method, to those plotted by Rankine, go a long way in establishing the practical accuracy of the Bernoulli-Eulerian theory[†] of the strains due to simple flexure when the amount of strain is very small. In Thomson and Tait, too, the curvatures as given by that theory are shown to agree perfectly with the exact mathematical results for small strains. Again, the fact that the *resilience* of the beam, or energy stored in it, exactly equals the amount of work done by the load in descending through the *deflection*, where both resilience and deflection are calculated by the approximate theory, is an appeal to the conservation of energy.

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[*] Consult *Nature* of July 16, 1891, for Carus Wilson's note on glass beams.

[†] See Williamson's *Elasticity* for quotations from Pearson's paper "On the Flexure of heavy Beams subjected to continuous Systems of Load," in the *Quarterly Journal of Mathematics*, 1889, pp. 63–110.

CHAPTER XVII.

CURVATURE, SLOPE, AND DEFLECTION.

At any point in a plane curve, the *direction* of the curve is that of the tangent at the point; and the *curvature* is the rate of change of direction at the point.

If two points be taken on a circle, the change of direction as you pass along the arc from one to the othor is the angle which the tangent at one of the points makes with the tangent at the other; this angle is equal to the angle at the centre subtended by the arc between the points; since this change of direction takes place uniformly, the rate of change is found by dividing the total change by the arc; the total change as just stated is the angle at the centre, and this angle when expressed in circular measure is the ratio of the arc to the radius; dividing this ratio by the arc, we then have for every point of a circle,

$$\text{curvature} = \frac{1}{r}\cdot \tag{1}$$

If we take a point in any plane curve, we can find a circle which coincides with the curve for a short arc in the vicinity of the point; and if ρ be the radius of that circle, then the curvature of the circle everywhere being $1 \div \rho$, it is clear that for that particular point of the plane curve,

$$\text{curvature} = \frac{1}{\rho} \tag{2}$$

For any point in the neutral axis of a beam (or cantilever), we have (fig. 81),

$$\text{curvature} = \frac{1}{\rho} = \frac{\left(\frac{\gamma}{ds}\right)}{y} = \frac{\text{strain on any horizontal layer}}{\text{dist. from neut. axis to layer}}$$

$$= \frac{\left(\frac{a}{ds}\right)}{y_a} = \frac{\left(\frac{\beta}{ds}\right)}{y_b} = \frac{\text{strain on either skin}}{\text{dist. of skin from neut. axis}}\cdot$$

Now the strain on a horizontal layer is equal to the longitudinal stress on the layer divided by E the modulus of elasticity of the material (see page 8); hence

$$\frac{1}{\rho} = \frac{\text{stress on any horizontal layer}}{\text{dist. of layer from neut. axis}} \times \frac{1}{E} = \frac{p}{y} \cdot \frac{1}{E} = \frac{p_a}{y_a} \cdot \frac{1}{E};$$

p_a being the normal stress at the skin on the cross-section, and y_a the distance of the skin from the neutral axis as shown in fig. 84; but page 283, we have

$$\frac{p_a}{y_a} = \frac{M}{I}; \tag{3}$$

hence at any point of the neutral axis, the curvature due to any load which induces the bending moment M on the cross-section passing through the point, is

$$\frac{1}{\rho} = \frac{1}{E} \cdot \frac{M}{I}, \tag{4}$$

I being the moment of inertia of the cross-section, and E the modulus of elasticity of the material. Choosing as origin that point where the neutral axis crosses the section of greatest bending moment, we have the curvature at that point for a beam or a cantilever,

$$\frac{1}{\rho_0} = \frac{1}{E} \cdot \frac{M_0}{I_0}; \tag{4a}$$

and if we wish this to correspond to the proof *or* working load, it is only necessary to make M_0 the bending moment due to the one *or* the other. These values of M_0 can be easily obtained from equation 3 by substituting for p_a the value f corresponding to the proof *or* working strength of the material. For this cross-section of greatest bending moment, putting y_0 instead of y_a,

$$\frac{1}{\rho_0} = \frac{1}{E} \cdot \frac{f}{y_0} = \frac{1}{E} \cdot \frac{f}{m'h};$$

so that

$$\frac{1}{E} \cdot \frac{M_0}{I_0} = \frac{1}{E} \cdot \frac{f}{y_0}; \quad \text{or} \quad \frac{f}{y_0} \cdot \frac{I_0}{M_0} = 1.$$

Multiplying the value for $\frac{1}{\rho}$ by this quantity, which being unity will not alter the value, we have

$$\frac{1}{\rho} = \left(\frac{1}{E} \cdot \frac{M}{I}\right)\left(\frac{f}{y_0} \cdot \frac{I_0}{M_0}\right) = \frac{f}{Ey_0} \cdot \frac{MI_0}{M_0 I}. \tag{5}$$

The slope of a beam or cantilever at any point whose abscissa is x_1, the origin being at the point where the neutral axis is horizontal, is found as follows:—Let S be any point between the origin and that point; then at fig. 81 we had for the increment of slope between T and S, supposing these points to be indefinitely close,

$$di = \frac{1}{\rho} \times dx; \tag{6}$$

if we add these increments di from point to point between the origin where the slope is zero and the point x_1, it gives us the slope at that point; and

$$i_{x_1} = \int_0^{x_1} \left(\frac{1}{\rho}\right) dx; \tag{7}$$

the right-hand side expresses, in the language of the Integral Calculus, the summation of the products equal respectively to these increments.

In the case of a beam symmetrically loaded, the origin will be at the centre of the span, and in practice, for any load, the origin will be sensibly in the same position; in the case of a cantilever, the origin will be at the point of support.

At the point of support of a beam, and at the free end of a cantilever, we thus have *the slope*,

$$i_c = \int_0^{c} \left(\frac{1}{\rho}\right) dx. \tag{7a}$$

that is the integral with respect to x of the general expression for the curvature between the limits c and 0.

Deflection of the neutral axis of a beam or cantilever at any point whose abscissa is x_2. Let x_1 be the abscissa of S (figs. 80 and 81; in the case of a beam let u be the height of S above the lowest point of the neutral axis, in the case of a cantilever let u be the depth below the highest point, and let du be the small difference of level between S and T, two points indefinitely close; then

$$\frac{du}{dx_1} = \tan i_{x_1} = i_{x_1} \text{ (the angle being small)},$$

or the difference of height of S and T is

$$du = i_{x_1} \,.\, dx_1; \tag{8}$$

and summing all these increments from point to point, between the point $x = 0$ (the lowest point) and the point x_2, we have the height of x_2 above 0,

$$u_x = \int_0^{x_2} (i_{x_1})\, dx_1. \tag{9}$$

The height of the end above the centre of a beam, or above the free end of a cantilever, is given by the equation

$$u_c = \int_0^c (i_{x_1})\, dx_1; \tag{9a}$$

this in the case of a beam is the deflection of the centre below the points of support, and in the case of a cantilever it is the deflection of the free end below the fixed end; so that *the deflection*

$$v_0 = \int_0^c (i_{x_1})\, dx_1. \tag{10}$$

Again the deflection at any point x_2 is

$$v_{x_2} = v_0 - u_{x_2} = v_0 - \int_0^{x_2} (i_{x_1})\, dx_1. \tag{10a}$$

For i_{x_1} we may substitute the value in equation 7, and

$$v_0 = \int_0^c \int_0^{x_1} \left(\frac{1}{\rho}\right) dx \,.\, dx_1; \tag{11}$$

$$v_{x_2} = v_0 - \int_0^{x_2} \int_0^{x_1} \left(\frac{1}{\rho}\right) dx \,.\, dx_1. \tag{11a}$$

Beam of uniform section, load at centre.—For all cases of *uniform* section I is constant; and for load at centre, we have, fig. 107,

$$\frac{M}{M_0} = \frac{W}{2}(c - x) \div \tfrac{1}{2}Wc = \left(1 - \frac{x}{c}\right); \quad \frac{1}{\rho} = \frac{f}{Ey_0}\left(1 - \frac{x}{c}\right). \tag{12}$$

$$\begin{aligned} i_c &= \int_0^c \left(\frac{1}{\rho}\right) dx = \frac{f}{Ey_0}\int_0^c \left(1 - \frac{1}{c}x\right) . \, dx \\ &= \frac{f}{Ey_0}\left[x - \frac{1}{c} . \frac{x^2}{2}\right]_0^c = \frac{f}{Ey_0}\left(c - \frac{1}{c}\frac{c^2}{2} - 0\right) = \frac{1}{2} . \frac{f}{E} . \frac{c}{y_0}. \end{aligned} \tag{13}$$

In every case the expression for i_c *the slope* can be arranged into these three factors; a numerical factor depending on the form of the cross-section, in this case $\frac{1}{2}$, and for which Rankine puts m''; a factor $\frac{f}{E}$ depending on the material, and a factor $\frac{c}{y_0}$ depending on the dimensions. The general expression is

$$i_c = m''\frac{f}{E} . \frac{c}{y_0}: \tag{13a}$$

and m'' is called the numerical coefficient for the slope.

The deflection

$$\begin{aligned} v_0 &= \int_0^c \int_0^{x_1} \left(\frac{1}{\rho}\right) dx . dx_1 = \frac{f}{Ey_0}\int_0^c \int_0^{x_1}\left(1 - \frac{1}{c}x\right) dx . dx_1 \\ &= \frac{f}{Ey_0}\int_0^c \left[x - \frac{1}{c}\frac{x^2}{2}\right]_0^{x_1} dx_1 = \frac{f}{Ey_0}\int_0^c \left(x_1 - \frac{1}{c}\frac{x_1^2}{2} - 0\right) dx_1 \\ &= \frac{f}{Ey_0}\left[\frac{x_1^2}{2} - \frac{1}{2c}\frac{x_1^3}{3}\right]_0^c = \frac{f}{Ey_0}\left(\frac{c^2}{2} - \frac{1}{2c}\frac{c^3}{3} - 0\right) = \frac{1}{3}\frac{f}{E}\frac{c^2}{y_0}. \end{aligned} \tag{14}$$

In every case the expression for *the* deflection can be arranged into these three factors; a numerical factor depending on the form of the section, in this case $\frac{1}{3}$, and for which Rankine puts n''; a factor $\frac{f}{E}$ depending on the material, and a factor $\frac{c^2}{y_0}$ depending on the dimensions. The general expression is

$$v_0 = n''\frac{f}{E}\frac{c^2}{y_0}; \tag{14a}$$

and n'' is called the numerical coefficient for *the* deflection.

In this case, the numerical coefficients are

$$m'' = \tfrac{1}{2}; \quad n'' = \tfrac{1}{3}.$$

Observe that f is the working or proof strength of the material, according as you may want the working or proof values of i_c and v_0. If f be given, as is generally the case, in lbs. per square inch, then E is to be in lbs. per square inch, and y_0 and c in inches; if we then calculate ρ or ρ_0, v or v_0, they will be in inches also.

From equation (12), if we put $x = 0$, we get

$$\rho_0 = \frac{Ey_0}{f}. \tag{15}$$

This will be the expression for ρ_0 in every case, and we shall not repeat it.

The inclinations i, when calculated from these formulæ, are in circular measure.

Beam of uniform section, uniform load.—From fig. 110,

$$\frac{M}{M_0} = \frac{W}{4c}(c^2 - x^2) \div \tfrac{1}{4}Wc = \left(1 - \frac{x^2}{c^2}\right)$$

$$\frac{1}{\rho} = \frac{f}{Ey_0} \cdot \frac{MI_0}{M_0 I} = \frac{f}{Ey_0}\left(1 - \frac{x^2}{c^2}\right); \tag{16}$$

$$i_c = \int_0^c \left(\frac{1}{\rho}\right) dx = \frac{f}{Ey_0}\int_0^c \left(1 - \frac{1}{c^2}\,x^2\right) dx$$

$$= \frac{f}{Ey_0}\left[x - \frac{1}{c^2}\frac{x^3}{3}\right]_0^c = \frac{f}{Ey_0}\left(c - \frac{1}{c^2}\frac{c^3}{3} - 0\right) = \frac{2}{3}\frac{f}{E}\cdot\frac{c}{y_0}; \tag{17}$$

$$v_0 = \int_0^c \int_0^{x_1} \left(\frac{1}{\rho}\right) dx \,.\, dx_1 = \frac{f}{Ey_0}\int_0^c \int_0^{x_1} \left(1 - \frac{1}{c^2}x^2\right) dx \,.\, dx_1$$

$$= \frac{f}{Ey_0}\int_0^c \left[x - \frac{1}{c^2}\frac{x^3}{3}\right]_0^{x_1} dx_1 = \frac{f}{Ey_0}\int_0^c \left(x_1 - \frac{1}{c^2}\frac{x_1^{\,3}}{3} - 0\right) dx_1$$

$$= \frac{f}{Ey_0}\left[\frac{x^2}{2} - \frac{1}{3c^2}\frac{x^4}{4}\right]_0^c = \frac{f}{Ey_0}\left(\frac{c^2}{2} - \frac{1}{3c^2}\frac{c^4}{4} - 0\right) = \frac{5}{12}\frac{f}{E}\cdot\frac{c^2}{y_0}; \tag{18}$$

The numerical coefficients are

$$m'' = \tfrac{2}{3};\ n'' = \tfrac{5}{12}.$$

Cantilever of uniform section, load at end.—From fig. 108,

$$\frac{M}{M_0} = W(c - x) \div Wc = \left(1 - \frac{x}{c}\right)$$

exactly as in the case of a beam loaded at the centre; and the numerical coefficients are

$$m'' = \tfrac{1}{2};\ n'' = \tfrac{1}{3}.$$

Cantilever of uniform section, uniform load.—I cancels I_0; from fig. 114

$$\frac{M}{M_0} = \frac{W}{2c}(c - x)^2 \div \tfrac{1}{2}Wc = \left(1 - \frac{x}{c}\right)^2$$

$$\frac{1}{\rho} = \frac{f}{Ey_0}\frac{MI_0}{M_0 I} = \frac{f}{Ey_0}\left(1 - \frac{x}{c}\right)^2; \tag{19}$$

$$i_c = \int_0^c \left(\frac{1}{\rho}\right) dx = \frac{f}{Ey_0}\int_0^c \left(1 - \frac{2}{c}x + \frac{1}{c^2}x^2\right) dx$$

$$= \frac{f}{Ey_0}\left[x - \frac{2}{c}\frac{x^2}{2} + \frac{1}{c^2}\frac{x^3}{3}\right]_0^c = \frac{f}{Ey_0}\left(c - \frac{2}{c}\frac{c^2}{2} + \frac{1}{c^2}\frac{c^3}{3} - 0\right)$$

$$= \frac{1}{3}\frac{f}{E}\cdot\frac{c}{y_0}; \tag{20}$$

$$v_0 = \int_0^c\int_0^{x_1}\left(\frac{1}{\rho}\right) dx \,.\, dx_1 = \frac{f}{Ey_0}\int_0^c\int_0^{x_1}\left(1 - \frac{2}{c}x + \frac{1}{c^2}x^2\right) dx \,.\, dx_1$$

$$= \frac{f}{Ey_0}\int_0^c \left[x - \frac{2}{c}\frac{x^2}{2} + \frac{1}{c^2}\frac{x^3}{3}\right]_0^{x_1} . \, dx_1$$

$$= \frac{f}{Ey_0}\int_0^c \left(x_1 - \frac{1}{c}x_1^2 + \frac{1}{3c^2}x_1^3 - 0\right) dx_1$$

$$= \frac{f}{Ey_0}\left[\frac{x_1^2}{2} - \frac{1}{c}\frac{x_1^3}{3} + \frac{1}{3c^2}\frac{x_1^4}{4}\right]_0^c$$

$$= \frac{f}{Ey_0}\left(\frac{c^2}{2} - \frac{1}{c}\frac{c^3}{3} + \frac{1}{3c^2}\frac{c^4}{4} - 0\right) = \frac{1}{4}\frac{f}{E}\cdot\frac{c^2}{y_0}; \tag{21}$$

The numerical coefficients are

$$m'' = \tfrac{1}{3};\ n'' = \tfrac{1}{4}.$$

Beam of uniform section, bending moment constant.—If a beam be symmetrically placed on, and extend beyond its two points of support; and if the two projecting parts be loaded symmetrically, while the intermediate portion is unloaded; the bending moment on the portion of the beam between the points of support is constant, and we have I and I_0, M and M_0, cancelling each other.

$$\frac{1}{\rho} = \frac{f}{Ey_0}; \tag{22}$$

a constant quantity, so that the neutral axis is circular.

$$i_c = \int_0^c \left(\frac{1}{\rho}\right) dx = \frac{f}{Ey_0}\int_0^c dx = \frac{f}{Ey_0}\left[x\right]_0^c = \frac{f}{Ey_0}(c-0) = \frac{f}{E}\cdot\frac{c}{y_0}; \tag{23}$$

$$\begin{aligned} r_0 &= \int_0^c\int_0^{x_1}\left(\frac{1}{\rho}\right) dx\,.\,dx_1 = \frac{f}{Ey_0}\int_0^c\int_0^{x_1} dx\,.\,dx_1 \\ &= \frac{f}{Ey_0}\int_0^c \left[x\right]_0^{x_1} dx_1 = \frac{f}{Ey_0}\int_0^c (x_1 - 0)\,dx_1 \\ &= \frac{f}{Ey_0}\left[\frac{x_1^{\,2}}{2}\right]_0^c = \frac{f}{Ey_0}\left(\frac{c^2}{2} - 0\right) = \frac{1}{2}\cdot\frac{f}{E}\cdot\frac{c^2}{y_0}; \end{aligned} \tag{24}$$

The numerical coefficients are

$$m'' = 1;\ n'' = \tfrac{1}{2}.$$

Beam of uniform section, loaded with two equal weights at equal distances from the centre of span.—Let W be the total load, x_r and $-x_r$ the abscissæ of the loads. $P = \frac{1}{2}W$; $M_0 = \frac{1}{2}W(c - x_r)$, and the bending moment is constant along the central portion of span; for values of x between x_r and c, $M_x = \frac{1}{2}W(c - x)$. For the central portion of span, $\dfrac{M_x}{M_0} = 1$; for the end portion $\dfrac{M_x}{M_0} = \dfrac{c - x}{c - x_r}$;

$$\frac{1}{\rho} = \frac{f}{Ey_0}\frac{M}{M_0} = \frac{f}{Ey_0}. \tag{25}$$

For values of x from 0 to x_r,

$$\frac{1}{\rho} = \frac{f}{Ey_0}\frac{1}{c - x_r}(c - x). \tag{25a}$$

For values of x from x_r to c.

$$i_c = \int_0^c \left(\frac{1}{\rho}\right).\,dx = \frac{f}{Ey_0}\left\{\int_0^{x_r} dx + \frac{1}{c - x_r}\int_{x_r}^{c}(c - x)\,dx\right\}$$

$$= \frac{f}{Ey_0}\left\{[x]_0^{x_r} + \frac{1}{c - x_r}\left[cx - \frac{x^2}{2}\right]_{x_r}^{c}\right\}$$

$$= \frac{f}{Ey_0}\left\{(x_r - 0) + \frac{1}{c - x_r}(c^2 - \tfrac{1}{2}c^2 - cx_r + \tfrac{1}{2}x_r^2)\right\}$$

$$= \frac{1}{2}\left(1 - \frac{x_r}{c}\right)\frac{f}{E}\frac{c}{y_0}; \tag{26}$$

$$v_0 = \int_0^c\int_0^{x_1}\frac{1}{\rho}.\,dx\,.\,dx_1 = \int_0^{x_r}\int_0^{x_1}\frac{1}{\rho}\,dx\,.\,dx_1 + \int_{x_r}^{c}\int_0^{x_1}\frac{1}{\rho}\,dx\,.\,dx_1;$$

$$\frac{Ey_0}{f}v_0 = \int_0^{x_r}[x]_0^{x_1}\,dx_1 + \int_{x_r}^{c}\left\{x_r + \frac{1}{c - x_r}[cx - \tfrac{1}{2}x^2]_{x_r}^{x_1}\right\}dx_1$$

$$= \int_0^{x_r}[x_1 - 0]\,dx_1 + \frac{1}{c - x_r}\int_{x_r}^{c}(cx_1 - \tfrac{1}{2}x_1^2 - x_r^2)\,dx_1$$

$$= [\tfrac{1}{2}x_1^2]_0^{x_r} + \frac{1}{c - x_r}\left[\frac{c}{2}x_1^2 - \tfrac{1}{6}x_1^3 - \tfrac{1}{2}x_r^2 x_1\right]_{x_r}^{c}$$

$$= \tfrac{1}{2}x_r^2 + \frac{1}{c - x_r}\left[\frac{c^3}{2} - \frac{c^3}{6} - \frac{c}{2}x_r^2 - \frac{c}{2}x_r^2 + \tfrac{1}{6}x_r^3 + \tfrac{1}{2}x_r^3\right];$$

$$v_0 = \left\{\frac{2c^3 - 3cx_r^2 + x_r^3}{6(c - x_r)}\right\}\frac{f}{Ey_0} = \left\{\tfrac{1}{3} + \frac{x_r}{3c} - \tfrac{1}{6}\frac{x_r^2}{c^2}\right\}\frac{f}{E}\frac{c^2}{y_0}$$

$$= \left\{\tfrac{1}{2} - \tfrac{1}{6}\left(1 - \frac{x_r}{c}\right)^2\right\}\frac{f}{E}\frac{c^2}{y_0}. \tag{27}$$

Thus, for an uniform beam, loaded at two points which are equidistant from the centre, and so that the stress on the outer fibres of the middle cross section is f, we have

$$m'' = \tfrac{1}{2}\left(1 + \frac{x_r}{c}\right);\quad n'' = \tfrac{1}{2} - \tfrac{1}{6}\left(1 - \frac{x_r}{c}\right)^2.$$

Beam or cantilever of uniform strength and uniform depth.—For the proof or working load, the bending moment at each section equals the proof or working moment of resistance to bending there; since, however, the depth is constant, the moment of resistance to bending at each cross-section is proportional to the breadth, that is the breadth at such sections is proportional to M; and since the depth is constant, I for each section is proportional to the breadth; hence $\frac{M}{I}$ is constant at every section,

$$\frac{MI_0}{M_0 I} = 1, \quad \text{and} \quad \frac{1}{\rho} = \frac{f}{Ey_0}; \tag{28}$$

as in the preceding case of a beam of uniform section and bending moment constant; so that for beams or cantilevers to resist *any load*, and made of uniform strength by varying the breadth only, the numerical coefficients for the slope and deflection are

$$m'' = 1; \; n'' = \tfrac{1}{2}.$$

Beam of uniform strength and uniform breadth, loaded at the centre.—See text at figs. 96, 176, &c., where it is explained that the elevation is obtained by degrading the bending moment diagram.

Since M is proportional to h^2, and I to h^3, we have $\frac{I}{M}$ proportional to h; and

$$\frac{MI_0}{M_0 I} = \frac{h_0}{h} = \sqrt{\frac{M_0}{M}} = \sqrt{\frac{c}{c - x}};$$

since from fig. 107, we have

$$\frac{M_0}{M} = \tfrac{1}{2}Wc \div \frac{W}{2}(c - x);$$

$$\frac{1}{\rho} = \frac{f}{Ey_0}\frac{MI_0}{M_0 I} = \frac{f\sqrt{c}}{Ey_0}(c - x)^{-\frac{1}{2}}. \tag{29}$$

$$i_c = \int_0^c \left(\frac{1}{\rho}\right) dx = -\frac{f\surd c}{Ey_0}\int_0^c (c - x)^{-\frac{1}{2}}(-\, dx)$$

$$= -\frac{f\surd c}{Ey_0}\left[\frac{(c - x)^{\frac{1}{2}}}{(\frac{1}{2})}\right]_0^c = -\frac{2f\surd c}{Ey_0}\left[(c - x)^{\frac{1}{2}}\right]_0^c$$

$$= -\frac{2f\surd c}{Ey_0}\{0^{\frac{1}{2}} - c^{\frac{1}{2}}\} = 2\frac{f}{E}\cdot\frac{c}{y_0}. \tag{30}$$

$$v_0 = \frac{f\sqrt{c}}{Ey_0}\int_0^c\int_0^{x_1}(c-x)^{-\frac{1}{2}}(-dx)(-dx_1)$$

$$= \frac{f\sqrt{c}}{Ey_0}\int_0^c\left[\frac{(c-x)^{\frac{1}{2}}}{(\frac{1}{2})}\right]_0^{x_1}(-dx_1) = \frac{2f\sqrt{c}}{Ey_0}\int_0^c\left[(c-x)^{\frac{1}{2}}\right]_0^{x_1}(-dx_1)$$

$$= \frac{2f\sqrt{c}}{Ey_0}\int_0^c\left\{(c-x_1)^{\frac{1}{2}} - c^{\frac{1}{2}}\right\}(-dx_1)$$

$$= \frac{2f\sqrt{c}}{Ey_0}\left\{\int_0^c(c-x_1)^{\frac{1}{2}}(-dx_1) + c^{\frac{1}{2}}\int_0^c dx_1\right\}$$

$$= \frac{2f\sqrt{c}}{Ey_0}\left[\frac{(c-x_1)^{\frac{3}{2}}}{(\frac{3}{2})} + c^{\frac{1}{2}}x_1\right]_0^c = \frac{4f\sqrt{c}}{3Ey_0}\left[(c-x_1)^{\frac{3}{2}} + \tfrac{3}{2}c^{\frac{1}{2}}x_1\right]_0^c$$

$$= \frac{4f\sqrt{c}}{3Ey_0}\left\{(0 + \tfrac{3}{2}c^{\frac{3}{2}}) - (c^{\frac{3}{2}} + 0)\right\} = \frac{2}{3}\frac{f}{E}\frac{c^2}{y_0}. \qquad (31)$$

The numerical coefficients are $m'' = 2$, $n'' = \frac{2}{3}$.

Cantilever of uniform strength and uniform breadth loaded at the end.

$$\frac{MI_0}{M_0I} = \frac{h_0}{h} = \sqrt{\frac{M_0}{M}} = \sqrt{\frac{c}{c-x}},$$

since from fig. 108, we have

$$M_0 \div M = Wc \div W(c-x).$$

The value of $\frac{1}{\rho}$ is the same as for the preceding case, and the coefficients for the slope and deflection are $m'' = 2$, $n'' = \frac{2}{3}$.

Beam of uniform strength and uniform breadth uniformly loaded.

$$\frac{MI_0}{M_0I} = \frac{h_0}{h} = \sqrt{\frac{M_0}{M}} = \frac{c}{\sqrt{c^2 - x^2}},$$

since from fig. 110, we have

$$M_0 \div M = \tfrac{1}{4}Wc \div \frac{W}{4c}(c^2 - x^2).$$

$$\frac{1}{\rho} = \frac{f}{Ey_0} \cdot \frac{MI_0}{IM_0} = \frac{fc}{Ey_0} \frac{1}{\sqrt{c^2 - x^2}}. \tag{32}$$

$$i_c = \int_0^c \left(\frac{1}{\rho}\right) dx = \frac{fc}{Ey_0} \int_0^c \frac{dx}{\sqrt{c^2 - x^2}} = \frac{fc}{Ey_0} \left[\sin^{-1} \frac{x}{c}\right]_0^c$$

$$= \frac{fc}{Ey_0} \{\sin^{-1}(1) - \sin^{-1}(0)\} = \frac{\pi}{2} \cdot \frac{f}{E} \cdot \frac{c}{y_0}. \tag{33}$$

$$v_0 = \int_0^c \int_0^{x_1} \left(\frac{1}{\rho}\right) dx\, dx_1 = \frac{fc}{Ey_0} \int_0^c \int_0^{x_1} \frac{dx}{\sqrt{c^2 - x^2}} dx_1$$

$$= \frac{fc}{Ey_0} \int_0^c \left[\sin^{-1} \frac{x}{c}\right]_0^{x_1} dx_1 = \frac{fc}{Ey_0} \int_0^c \left\{\sin^{-1} \frac{x_1}{c} - \sin^{-1} 0\right\} dx_1$$

$$= \frac{fc}{Ey_0} \int_0^c \sin^{-1} \left(\frac{x_1}{c}\right) dx_1 ; \text{ integrating by parts}$$

$$= \frac{fc}{Ey_0} \left[x_1 \sin^{-1} \left(\frac{x_1}{c}\right) - \int x_1 \cdot \frac{dx_1}{\sqrt{c^2 - x_1^2}}\right]_0^c$$

$$= \frac{fc}{Ey_0} \left[x_1 \sin^{-1} \left(\frac{x_1}{c}\right) + \tfrac{1}{2} \int (c^2 - x_1^2)^{-\frac{1}{2}} (-2x_1 dx_1)\right]_0^c$$

$$= \frac{fc}{Ey_0} \left[x_1 \sin^{-1} \left(\frac{x_1}{c}\right) + \tfrac{1}{2} \frac{(c^2 - x_1^2)^{\frac{1}{2}}}{(\frac{1}{2})}\right]_0^c$$

$$= \frac{fc}{Ey_0} \left[x_1 \sin^{-1} \left(\frac{x_1}{c}\right) + (c^2 - x_1^2)^{\frac{1}{2}}\right]_0^c$$

$$= \frac{fc}{Ey_0} \{(c \,.\, \sin^{-1} 1 + 0) - (0 + c)\}$$

$$= \frac{fc}{Ey_0} \left(\frac{\pi}{2} c - c\right) = \left(\frac{\pi}{2} - 1\right) \frac{f}{E} \frac{c^2}{y_0}. \tag{34}$$

The numerical coefficients are

$$m'' = \frac{\pi}{2} = 1{\cdot}5708 ; \qquad n'' = \left(\frac{\pi}{2} - 1\right) = 0{\cdot}5708.$$

Proportion of the greatest depth of a beam to the span.—Putting the working strength of the material as the value of f, we have, as a general formula,

$$v_0 = n'' \frac{f}{E} \frac{c^2}{y_0};$$

and since $y_0 = m'h_0$, we have

$$\frac{v_0}{2c} = \frac{n''}{4m'} \frac{f}{E} \frac{2c}{h_0}.$$

Now $\frac{v_0}{2c}$ is the ratio of the deflection to the span, and its reciprocal $\frac{2c}{v_0}$ represents the stiffness of the beam; $\frac{h_0}{2c}$ is the ratio of depth of beam at centre to span, and we have

$$\frac{2c}{v_0} = \frac{4m'}{n''} \frac{E}{f} \frac{h_0}{2c};$$

or, the stiffness of a beam is proportional to the ratio of the depth at centre to span.

For instance, to give a working stiffness 1000 to a wrought iron beam of uniform symmetrical section uniformly loaded; we have the ratio of depth at centre to span (Rankine, *C.E.* §170)

$$\frac{h_0}{2c} = \tfrac{1}{4} \frac{n''}{m'} \cdot \frac{f}{E} \cdot \frac{2c}{v_0}$$

$$= \tfrac{1}{4} \frac{(\frac{5}{12})}{(\frac{1}{2})} \cdot \frac{10000 \text{ lbs. per square inch}}{30000000 \text{ lbs. per square inch}} \cdot 1000 = \frac{1}{14 \cdot 4}.$$

That is to secure the degree of stiffness 1000 required for the wrought-iron beam, the depth must not be less than a fourteenth of the span. In the same way it may be shown that, in general, to give to beams the degree of stiffness that practice shows to be necessary, and that is usually prescribed, the depth at centre must bear to the span a ratio varying from $\frac{1}{8}$th to $\frac{1}{14}$th, according to the material, form, and manner of loading.

Slope and deflection under any load less than the proof load.—For the proof load, the stress on the skin furthest from the neutral axis at the cross-section of maximum bending moment is f, and for a smaller load it is p_a (fig. 84); now, though the load is less than the proof load, yet being distributed in the same way, the ratio $M:M_0$ is in no way changed; I is the same as before, and we have E and y_0 constant; so that

$$i'_c = m'' \frac{p_a}{E} \frac{c}{y_0}; \quad v'_0 = n'' . \frac{p_a}{E} . \frac{c^2}{y_0} .$$

But since

$$p_a = \frac{mWl}{nb_0h^2_0}, \quad \text{and} \quad y_0 = m'h,$$

we have

$$i'_c = \frac{mm''}{m'n} \frac{Wlc}{Eb_0h_0^3}; \quad v'_0 = \frac{mn''}{m'n} \frac{Wlc^2}{Eb_0h_0^3} .$$

For a beam $l = 2c$, and for a cantilever $l = c$; so that for any load for a beam

$$i'_c = \frac{2mm''}{m'n} \frac{Wc^2}{Eb_0h_0^3}; \quad v'_0 = \frac{2mn''}{m'n} \frac{Wc^3}{Eb_0h_0^3},$$

and for a cantilever

$$i'_c = \frac{mm''}{m'n} \frac{Wc^2}{Eb_0h_0^3}; \quad v'_0 = \frac{mn''}{m'n} \frac{Wc^3}{Eb_0h_0^3} .$$

The coefficients

$$\frac{mm''}{m'n} \quad \text{and} \quad \frac{mn''}{m'n}$$

assume values which depend on the cross-section and the system of loading.

For similar beams similarly loaded we thus have v'_0, the deflection under any load less than the proof load, proportional to W the total load, and to c^3 or the cube of the length; and inversely as b_0 the breadth at centre of span, and h_0^3 the cube of depth at centre of span.

That v'_0 is proportional to W does not follow from Hooke's Law, but has been established upon the supposition that the slope is so small that the tangent and circular measure of *the slope* are sensibly equal. In bending small pieces of wood in a machine which registers the load and the deflection as the coordinates of a line, it is found that the line is straight even when the piece of wood is bent to a considerable amount; this proves that the formula above is a close approximation, even when the slope is considerable; it must not be forgotten, however, that the formula is *only* approximate when the slope is great. It has been stated that where such a machine's register ceases to be straight, the elastic limit has been passed; no such conclusion can be drawn; the only just conclusion to draw is that since the slope is visibly great (say 8° or 10°), the formula above has ceased to be a close approximation. Were a second approximation made, it would be found that v'_0 was not exactly proportional to W; and so long as the register did not depart from the curve which is the locus of that new equation, it would be inaccurate to infer that the elastic limit had been passed.

Deflection of beam supported on three props.—Let HK be a beam of uniform cross-section, bearing an uniform load of amount U; and let it be supported on three props, one at each end and one at the centre. Let W be the reaction of the central prop, and $P = Q$, the reaction of each end prop; then $P + Q + W = U$.

Fig. 222.

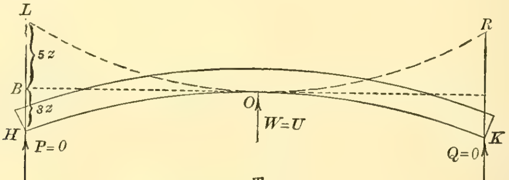
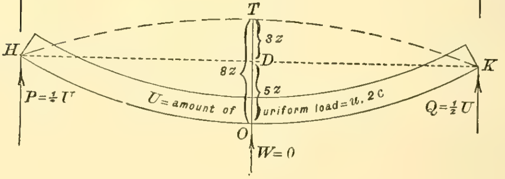


Fig. 223.



In fig. 222, let $W = U$, then $P = Q = 0$, and OH is a cantilever of uniform cross-section fixed at O and uniformly loaded, for which $n'' = \frac{1}{4}$; in fig. 223, let $W = 0$, then $P = Q = \frac{1}{2}U$, and HK is a beam of uniform cross-section uniformly loaded, for which $n'' = \frac{5}{12}$; hence

$$BH \text{ (fig. 222)} : DO \text{ (fig. 223)} :: 3 : 5.$$

Putting $BH = 3z$, then $DO = 5z$.

In fig. 222, if we suppose the end props to be pushed up till they support all the load, then HOK will assume the form LOB, where $HL = 8z$; and in fig. 223, if we suppose the central prop to be pushed up till it just supports all the load, then HOK will assume the form HTK, where $OT = 8z$. Hence we have the following theorems.

Theorem.—If to an uniform cantilever (OH, fig. 222) loaded uniformly, we apply at the end a load ($P = \frac{1}{2}U$) equal to that uniform load, it will produce an additional deflection in the direction of the applied load ($HL = \frac{8}{3}BH$) equal to eight-thirds of that due to the uniform load alone.

In the figure, $P = \frac{1}{2}U$ is applied upwards at H, but the theorem holds for P applied downwards, since deflection is sensibly proportional to load; it being understood that the total deflection in no case exceeds the proof deflection.

Cor.—A load P', applied to the end of an uniform cantilever loaded uniformly, will produce an additional deflection in the direction of P'; and the amount of this additional deflection is proportional to the load P'.

Theorem.—If to an uniform beam (HOK, fig. 223) loaded uniformly we apply at the centre a load ($W = U$) equal to the uniform load, it will produce an additional deflection in the direction of the applied load ($OT = \frac{8}{5}DO$) equal to eight-fifths of that due to the uniform load alone.

In the figure, $W = U$ is applied upwards at O, but the theorem holds for W applied downwards, as for the previous theorem.

Cor.—A load W' applied at the centre of an uniform beam loaded uniformly will produce an additional deflection in the direction of W'; and the amount of this additional deflection is proportional to the load W'. See Thomson and Tait's "Natural Philosophy," first edition, §§ 618, 619.

Uniform beam uniformly loaded and supported on three props at the same level, one at each end and one at the centre (fig. 224).—This figure is obtained by pushing up the central prop in fig. 223 till O coincides with D. Then since the reaction of the central prop in fig. 224 has produced the deflection OD, which is five-eighths of OT, it follows that $W = \frac{5}{8}U$, and therefore $P = Q = \frac{3}{16}U$.

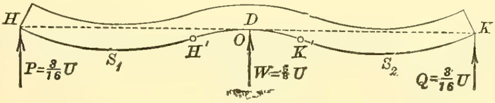


Fig. 224.

From fig. 117,

$$\overline{OS}_1 = \frac{W}{U}\overline{OH} = \tfrac{5}{8}\overline{OH}; \quad \text{hence} \quad \overline{OH}_1 = \tfrac{1}{4}\overline{OH}.$$

At centre O,

$$M_0 = -\tfrac{1}{32}Ul, \text{ max.}; \quad F = -\tfrac{5}{16}U, \text{ max.}$$

At S_1 and S_2 the maximum deflection occurs, and as proved in the next article,

$$n'' = \tfrac{1}{4} \times \frac{175}{1024} = \cdot 043.$$

The central prop increases the strength four times and the stiffness nearly ten times.

Uniform beam uniformly loaded fixed at one end and supported at the other (fig. 225).—The left half of fig. 224 represents such a beam. Let $W = \frac{1}{2}U$ the uniform load, and $l = OH$ the span; then $P = \frac{3}{8}W$; the shearing force at O is the remainder of load, viz. $\frac{5}{8}W$; the shearing force changes sign at S where the bending moment has a maximum value; and the locus of the shearing force diagram is a straight line, since the load is uniform.

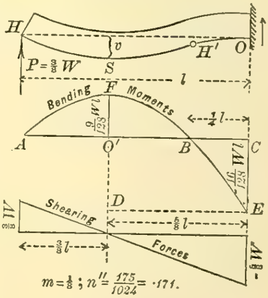


Fig. 225.

From fig. 117 the bending moment diagram is $AFBE$ a parabola with axis vertical and apex on vertical through S.

$$O'F = M_s = \frac{3W}{8}\cdot\frac{3l}{8} - \frac{3W}{8}\cdot\frac{3l}{16} = \frac{9}{128}Wl$$

a positive maximum; and since $AFBE$ is a parabola

$$FD : FO' :: DE^2 : O'B^2 :: 25 : 9;$$

therefore $$FD = \tfrac{25}{9}FO',$$

and $$CE = O'D = FD - FO' = -\tfrac{16}{9}O'F = -\tfrac{1}{8}Wl,$$

that is $M_0 = -\frac{1}{8}Wl$, the greatest value; this quantity may also be calculated directly by taking moments about O.

This solution is exact, and all the results are readily got by remembering that OH' is a quarter of the span. The approximate solution indicated in Rankine's "Applied Mechanics," sec. 308, assumes H' to be sensibly on the same level as H.

To find the deflection at S; considering the beam HH' alone, we have for the deflection at S, since $\overline{SH} = \frac{3}{8}l = \frac{3}{4}c$;

$$v_1 = \frac{5}{12}\frac{f_s}{E}\cdot\frac{\overline{SH}^2}{y_0} = \frac{5}{12}\frac{f_s}{Ey_0}\cdot\frac{9c^2}{16} = \frac{15}{64}\frac{f_s}{E}\cdot\frac{c^2}{y_0},$$

where f_s is the intensity of stress at the skin at S.

Next consider the cantilever OH', taking into account its uniform load alone. If the cantilever were of length SH, its deflection would be $\frac{3}{5}v_1$; and since the deflection of a cantilever is proportional to the cube of its length and to the load, we have for the deflection of OH' for uniform load alone,

$$v_2 = \left(\frac{OH'}{SH}\right)^4 \cdot \tfrac{3}{5}v_1 = \left(\tfrac{1}{4} \div \tfrac{3}{8}\right)^4 \cdot \tfrac{3}{5}v_1 = \tfrac{16}{135}v_1;$$

further, if a load equal to that on OH' be put at the end, it will produce an additional deflection $\frac{8}{3}v_2$; the load at end of OH', viz., that on SH', will produce a proportionate deflection, and we have for the deflection due to load at end of OH',

$$v_3 = \left(\frac{SH'}{OH'}\right)\tfrac{8}{3}v_2 = \left(\tfrac{3}{8} \div \tfrac{1}{4}\right)\tfrac{8}{3}v_2 = 4v_2;$$

and the total deflection of $OH' = 5v_2 = \frac{16}{27}v_1$.

The total deflection of the point S is therefore

$$v = v_1 + \frac{1}{2}\frac{16}{27}v_1 = \frac{35}{27}v_1 = \frac{35}{27}\cdot\frac{15}{64}\frac{f_s}{E}\cdot\frac{c^2}{y_0} = \frac{175}{576}\frac{f_s}{E}\frac{c^2}{y_0}.$$

Let f_0 be the proof strength of the material; when the beam is loaded with the proof load, then f_0 = intensity of stress on skin at O the fixed end, and

$$f_s : f_0 :: M_s : M_0 :: 9 : 16;$$

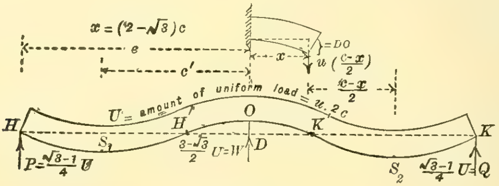
hence $f_s = \frac{9}{16}f_0$; substituting this, we have

$$v = \frac{175}{576}\cdot\frac{9}{16}\frac{f_0}{E}\cdot\frac{c^2}{y_0} = \frac{175}{1024}\cdot\frac{f_0}{E}\cdot\frac{c^2}{y_0};$$

so that $m = \dfrac{1}{8}$, and $n'' = \dfrac{175}{1024} = \cdot 171$.

Uniform beam uniformly loaded and fixed at both ends (fig. 227).—Suppose the central of the three props that support HK, fig. 226, to push up till O is above the level of H and K such a distance that H' and K', the points of contrary flexure, shall be on the same level as H and K. Let $x = OK'$ the distance of the point of contrary flexure from O, and let u be the intensity of the uniform load.

Consider OK' alone. It is a cantilever fixed at O, of length x, loaded uniformly with intensity u, and loaded at K' the free end with $\frac{u}{2}(c - x)$, half the load spread on $K'K$. For the uniform load alone, compare OK' with OK (fig. 222), whose deflection is $3z$; their deflections are proportional to the cubes of their lengths and to the loads; hence for uniform load alone the deflection of OK' (fig. 226) is



$x = (2 - \sqrt{3})c$
e
c'
x
$= DO$
$u\left(\frac{c-x}{2}\right)$
$\frac{c-x}{2}$
U = amount of uniform load $= u.2c$
O
H
H
K
K
S_1
$\frac{3-\sqrt{3}}{2}$
$U=W$
D
$P = \frac{\sqrt{3}-1}{4}U$
S_2
$\frac{\sqrt{3}-1}{4}U = Q$

Fig. 226.

$$\left(\frac{OK'}{OK}\right)^4 \times 3z = \left(\frac{x}{c}\right)^4 . 3z;$$

OK' being shown in fig. 226, and OK in fig. 222. By theorem at fig. 222, a load at the end of this cantilever, and of amount ux, would produce an additional deflection $8\left(\frac{x}{c}\right)^4 z$; the load $\frac{1}{2}u(c - x)$ will produce a proportional deflection; and we have for the deflection due to the load at end of OK',

$$\frac{4(c - x)}{3x}\left(\frac{x}{c}\right)^4 . 3z;$$

and for the total deflection of the point K',

$$OD = \frac{x^3}{c^4}(4c - x)z.$$

To pass from fig. 223 to fig. 226, the central prop has been pushed up through OD (fig. 223) together with DO (fig. 226); that is through

$$5z + \frac{x^3}{c^4}(4c - x)z = \left\{5 + \frac{(4c - x)x^3}{c^4}\right\} z;$$

and by the converse of corollary to theorem at fig. 223, the reaction on the central prop will be

$$W = \frac{1}{8z}\left\{5 + \frac{(4c - x)x^3}{c^4}\right\} z \times U.$$

Now, $OS_1 = \frac{1}{2}(c + x)$; and since (fig. 117) $OS_1 = \frac{W}{U}c$, we have

$$\tfrac{1}{2}(c + x) = \tfrac{1}{8}\left(5 + \frac{(4c - x)x^3}{c^4}\right)c;$$

solving this equation we find

$$OK' = x = c(2 - \sqrt{3}),$$

which determines the position of the points of contrary flexure. Substituting this for x in the expression for W as given above, we have the reaction of central prop

$$W = \frac{3 - \sqrt{3}}{2} U.$$

If we take $OK'K$, half of this beam, and suppose the end at O fixed, and the end K to be supported by the cantilever $O'K$ similar to OK' in all respects; we have (fig. 227) a beam fixed at both ends and uniformly loaded; its semi-span is

$$c' = OS_1 = \frac{3 - \sqrt{3}}{2} c;$$

but
$$OK' = (2 - \sqrt{3})c = \left(1 - \frac{1}{\sqrt{3}}\right)c';$$

hence the distance from the centre to a point of contrary flexure is

$$SK' = \frac{1}{\sqrt{3}} c' = \cdot 289 l',$$

where l' is the span.

Let W' be the total load on the beam (fig. 227); then the shearing force diagram will vary from $\frac{1}{2}W'$ to $-\frac{1}{2}W'$ from left to right end. The bending moment diagram is the same parabola as for the beam supported at the ends, except that it passes through the points of contrary flexure;

$$FO'' : FD' :: O''B^2 : D'L^2,$$

or $O''F$ is one-third, while $O''D$ is two-thirds of the maximum bending moment for the beam supported only; hence at fixed end

$$M_{\text{max.}} = -\tfrac{2}{3}\,\tfrac{1}{8}\,W'l' = -\tfrac{1}{12}\,W'l'.$$

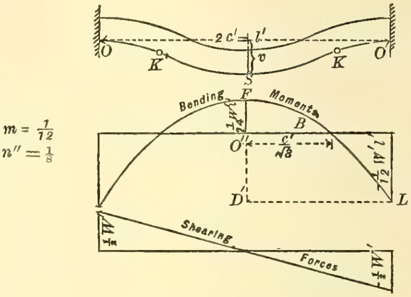


Fig. 227.

To find the deflection; consider first the part KK', for it

$$v_1 = \frac{5}{12}\frac{f_s}{E}\cdot\frac{SK^2}{y_0} = \frac{5}{12}\cdot\frac{f_s}{Ey_0}\frac{c'^2}{3} = \frac{5}{36}\frac{f_s}{E}\cdot\frac{c'^2}{y_0},$$

where f_s is the stress on the skin at S. Consider next the cantilever OK', taking into account the uniform load alone; a cantilever of length SK' would deflect $\frac{3}{5}v_1$, and taking account of its length, we have for the deflection of OK' for uniform load alone

$$v_2 = \left(\frac{OK'}{SK}\right)^4 \cdot \frac{3}{5}v_1.$$

A load at end of OK', equal to the uniform load on it, would produce an additional deflection of $\frac{8}{3}v_2$; but the load at end of OK' is that on $K'S$, and it will produce a proportionate deflection; so that we have for the deflection of OK' for load at end

$$v_3 = \left(\frac{SK'}{OK'}\right)\frac{8}{3}v_2 = \left(\frac{OK'}{SK'}\right)^3\frac{8}{5}v_1.$$

Hence the total deflection of S is

$$v = v_1 + v_2 + v_3 = \left\{1 + \frac{3}{5}\left(\frac{OK'}{SK'}\right)^4 + \frac{8}{5}\left(\frac{OK'}{SK'}\right)^3\right\}v_1.$$

Now $$\frac{OK'}{SK'} = \left(1 - \frac{1}{\sqrt{3}}\right) \div \frac{1}{\sqrt{3}} = (\sqrt{3} - 1),$$

$$\left(\frac{OK'}{SK'}\right)^4 = (28 - 16\sqrt{3}), \quad \text{and} \quad \left(\frac{OK'}{SK'}\right)^3 = (6\sqrt{3} - 10);$$

$$v = \left(1 + \frac{84}{5} - 16\right) v_1 = \frac{9}{5} v_1 = \frac{1}{4} \frac{f_s}{E} \frac{c'^2}{y_0} = \frac{1}{8} \frac{f_0}{E} \frac{c'^2}{y_0};$$

since M at the end and at the centre are in the ratio of 2 and 1.

We thus have $m = -\frac{1}{12}$; and $n'' = \frac{1}{8}$.

From the above, we see that fixing the ends of an uniform beam which is loaded uniformly increases the

strength in the ratio $\frac{1}{8} : \frac{1}{12}$, or $3 : 2$;

stiffness in the ratio $\frac{5}{12} : \frac{1}{8}$, or $10 : 3$.

As already shown at fig. 117, the *strength* is economised to the greatest extent if by means of hinges, or in some other way, we shift the point of contrary flexure to a distance $c \div \sqrt{2}$ from the centre; we then increase the

strength in the ratio $2 : 1$;

stiffness in the ratio $\dfrac{5}{12} : \dfrac{\sqrt{2}}{6}$, or $10 : 5{\cdot}66$ (see below).

Maximum Stiffness.—For some position of the hinges, the stiffness will be a maximum; in order to find this point, let β be the distance of the hinges from the centre, and f_s the stress on the skin at centre; then as above we have

$$v = \left\{1 + \tfrac{3}{5}\left(\frac{c - \beta}{\beta}\right)^4 + \tfrac{8}{5}\left(\frac{c - \beta}{\beta}\right)^3\right\} \tfrac{5}{12} \frac{f_s}{E} \frac{\beta^2}{y_0}$$

$$= \frac{3c^3 - 4c^2\beta - 6c\beta^2 + 12\beta^3}{12c\beta^2} \frac{f_s}{E} \cdot \frac{c^2}{y_0};$$

where f_s is the proof strength so long as $M_{\text{max.}}$ is at the centre, that is so long as $\beta \geqq \frac{c}{\sqrt{2}}$; putting

$\beta = c$, we have $n'' = \frac{5}{12}$. (Hinges at ends); $\beta = \frac{e}{\sqrt{2}}$, we have

$n'' = \frac{\sqrt{2}}{6}$. (Hinges at ·707c from centre.)

When $\beta < \frac{c}{\sqrt{2}}$, then $M_{\text{max.}}$ is at the end, and it is necessary to substitute for f_s in terms of f_0 the proof stress on skin at end; thus, if

$$\beta = \frac{e}{\sqrt{3}}, \quad \text{put} \quad f_s = \tfrac{1}{2}f_0,$$

and the above expression gives

$$v = \tfrac{1}{4}\frac{f_s}{E}\frac{c^2}{y_0} = \tfrac{1}{8}\frac{f_0c^2}{Ey_0}; \text{ or } n'' = \tfrac{1}{8}.$$

The stiffness is a maximum when v is a minimum; this occurs when $\beta = \frac{1}{2}c$, sensibly giving

$$v = \tfrac{1}{3}\frac{f_sc^2}{Ey_0} = \tfrac{1}{9}\frac{f_0c^2}{Ey_0}; \quad \text{or} \quad n'' = \tfrac{1}{9} \text{ sensibly a minimum.}$$

Fixing the ends and placing the two hinges at the quarter points of span increases the *stiffness* in the ratio $\frac{5}{12} : \frac{1}{9}$, or 4 : 1 nearly.

Some economy of a material whose strength to resist tension is, say less than that to resist thrust, can be secured by making the constant cross-section of such a form that the upper skin at end section, and under skin at central section, shall simultaneously come to their working stress. Thus for a material half as strong to resist tension as thrust, and with the hinges fixed so that $\beta = \frac{c}{\sqrt{3}}$, a section whose neutral axis is half as far from the upper as it is from the under skin, would give considerable economy; as the upper and under skin at end sections, and under skin at central section would all come to their working strength at one time.

Beam of uniform strength, uniform depth, fixed at the ends.—Suppose that $OK'SKO'$, fig. 227, is the beam. From equation 28, p. 345, the curvature is constant so that OK' is part of a circle whose centre is on the vertical through O, and SK' is part of a circle of the same radius whose centre is on the vertical through S; by symmetry $SK' = OK'$, and K and K' the points of contrary flexure are midway between the centre and the ends of the span. Since the beam is of uniform strength and depth, the breadth varies as the bending moment; hence the plan should correspond with the bending moment diagram. For the case of an uniform load, the plan will correspond with the bending moment diagram in fig. 227, the curve being drawn on *both* sides of a centre line, the two parabolas passing through the quarter points of the span; the breadths are then to be reduced till that at any point (say the end) is just sufficient to give the necessary resistance to bending, and we have the plan of the beam; see Rankine's "Applied Mechanics," fig. 144. Since $O''B$, fig. 227, is half of $D'L$, and since the the load is uniform, $O''F$ is one-quarter, and $O''D'$ the maximum bending moment is three-quarters, of FD' the amount of the maximum bending moment for the same beam *not* fixed at the ends. The plan must allow sufficient breadth at the points of contrary flexure in order to be able to resist the shearing force at these points; on K', for instance, the shearing force will be the load on SK', and the breadth at K' must be sufficient to resist this amount.

EXAMPLES.

228. A beam 24 feet span, of uniform symmetrical section as shown in fig. 207, is made of wrought iron whose working strength $f = 4$ tons per square inch, and whose modulus of elasticity $E = 11600$ tons per square inch. Find the radii of curvature at intervals of 4 feet, when loaded uniformly with the working load.

Taking equation 16, page 341, we have

$$\frac{1}{\rho} = \frac{f}{Ey_0}\left(1 - \frac{x^2}{c^2}\right) = \frac{4}{11600 \times 10}\left(1 - \frac{x^2}{144}\right),$$

where x and c are to be in one name,

$$\rho = 29000 \frac{144}{144 - x^2}, \ x \text{ being in feet.}$$

The radii of curvature are

$$\rho_0 = 29000; \quad \rho_4 = 32625; \quad \rho_8 = 52200 \text{ inches}; \quad \rho_{12} = \text{infinity}.$$
$$= 2420; \quad = 2720; \quad = 4350 \text{ feet}.$$

The reason that ρ is in inches is because y_0 is in inches, and the proportions derived at fig. 81 show clearly that ρ and y are in one name. We had to put y_0 in inches, because for the material f and E are given in tons on the square inch.

229. Calculate *the* slope and deflection in the previous example

$$i_c = \tfrac{2}{3}\frac{f}{E}\frac{c}{y_0} = \tfrac{2}{3}\cdot\frac{4}{11600}\cdot\frac{144}{10} = \cdot 0033\,; \quad \therefore\ i_c = 0^\circ\,11' \text{ nearly};$$

$$v_0 = \frac{5}{12}\frac{f}{E}\cdot\frac{c^2}{y_0} = \frac{5}{12}\cdot\frac{4}{11600}\cdot\frac{(144)^2}{10} = 0\cdot 30 \text{ inch} = 0\cdot 025 \text{ foot}.$$

This is nearly one-thousandth of the span, which is about the extreme ratio of working deflection to span allowable in practice (Rankine's "Applied Mechanics," sect. 302); and this degree of stiffness is secured by making the depth of this wrought-iron girder a $\frac{1}{14.4}$th part of the span.

230. Find the slope and deflection in example 228 by means of a graphical solution.

As in fig. 228, which is drawn to a scale of 20 feet to an inch, draw verticals; one to represent the vertical through the centre of span; the others to be drawn on each side of the centre, and at 2, 6, 10, and 12 feet therefrom; the last two will be through the extremities of span.

From c_1 any point in the central vertical, with radius 24·2 feet, that is a hundredth part of ρ_0, describe the arc $B'AB$ between the verticals through B and B'; produce Bc_1 to c_2, so that $c_2B = 27\cdot 2$, a hundredth part of ρ_4, and about c_2 describe the arc BC; similarly $c_3C = 43\cdot 5$; and from D, DE is drawn at right angles to Dc_3. On the other side of the point A construct $AB'E'$; draw the horizontal chord EOE', and the tangent $E'K$. Then EAE' represents the curve assumed by the neutral axis of the beam when under the proof load, but with its vertical dimensions exaggerated 100 fold.

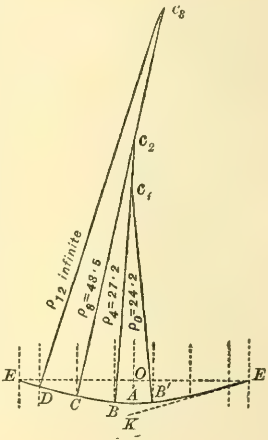


Fig. 228.

$$\text{The deflection}\qquad v_0 = \frac{1}{100}\times OA = \frac{1}{100}\times 2\cdot 5 \text{ (by scale)} = \cdot 025 \text{ foot},$$

$$i_c = \tan i_c = \frac{OK}{100} \div OE' = \frac{4\cdot 0}{100} \div 124 = \cdot 003 = 0^\circ\,11'.$$

231. If the beam in example 228 be loaded with one ton per foot of span, find the deflection.

Let W' be the working load, then $mW'l$ equals the working value of M_0; that is $\frac{1}{8}W'(24\times 12) = 1948$ (see fig. 207); therefore $W' = 54\cdot 1$ tons.

Now the load in this example is 24 tons, $\frac{4}{9}$ths of W' the working load; and since deflection is proportional to load, we have

$$\text{Deflection} = v'_0 = \tfrac{4}{9}\times \cdot 30 = 0\cdot 13 \text{ inch} = 0\cdot 011 \text{ foot}.$$

Otherwise, if the working deflection had not been already calculated, we have

$$\text{Deflection} = v'_0 = \left(\frac{2mn''}{m'n}\right)\frac{Wc^3}{Eb_0h_0^3} = \frac{2mn''}{n'b_0h_0^3}\,\frac{Wc^3}{E} = \frac{2mn''}{I_0}\cdot\frac{Wc^3}{E}\,; \qquad I_0 = 4870$$

$$= \frac{2\left(\frac{1}{8}\right)\left(\frac{5}{12}\right)}{4870}\cdot\frac{24\times(144 \text{ in.})^3}{11600} = \cdot 13 \text{ in.} = \cdot 011 \text{ foot}.$$

232. Find the working deflection for a wrought-iron beam of uniform strength and uniform breadth, and loaded uniformly; the span is 24 feet, and the upper half of the cross-section at centre of beam is shown in fig. 207; $f = 4$ tons, and $E = 11600$ tons per square inch.

$$v_0 = n''.\frac{f}{E}.\frac{c^2}{y_0} = \left(\frac{\pi}{2} - 1\right)\frac{f}{E}.\frac{c^2}{y_0} = \cdot 57 \times \frac{4}{11600}.\frac{144^2}{10} = \cdot 41 \text{ in.} = \cdot 034 \text{ feet.}$$

233. A wrought-iron rectangular beam 20 feet span, 16 inches deep, and 4 inches broad, is loaded at the centre. Calculate the proof deflection if the proof strength of the iron be 7 tons, and its modulus of elasticity 12000 tons per square inch.

$$v_0 = n''.\frac{f}{E}.\frac{c^2}{y_0} = \tfrac{1}{3}.\frac{7}{12000}.\frac{120^2}{8} = \cdot 35 \text{ inch} = \cdot 029 \text{ feet.}$$

234. Find the resilience of the above beam (see Chap. I.).

Let W be the proof load,

$$M_0 = \mathsf{M}_0, \quad \text{or} \quad mWl = nfbh^2; \quad \tfrac{1}{4}W.240 = \tfrac{1}{6} \times 7 \times 4 \times 16^2;$$

therefore $\quad W = 20$ tons nearly.

Resilience $= \frac{1}{2}$ proof load × proof deflection $= \frac{1}{2} \times 20$ tons $\times \cdot 029$ feet

$= \cdot 29$ ft.-tons $= 650$ ft.-lbs.

235. Find a general expression for the resilience of a rectangular beam loaded at the centre.

Let W be the proof load, and v_0 the proof of deflection, then

$$v_0 = \left(\frac{2mn''}{m'n}\right)\frac{Wc^3}{Ebh^3}.$$

$$\text{Resilience} = \tfrac{1}{2}Wv_0 = \left(\frac{mn''}{m'n}\right)\frac{c^3}{Ebh^3} \times W^2; \quad \text{but } m.W.2c = nfbh^2;$$

$$\therefore \quad W = \frac{nfbh^2}{2mc}; \quad \therefore \text{ resilience} = \frac{mn''}{m'n}.\frac{c^3}{Ebh^3}.\left(\frac{nfbh^2}{2mc}\right)^2 = \frac{nn''}{8mm'}.\frac{f^2}{E}.2cbh.$$

The first factor depends on the form of cross-section and the distribution of load, and the second upon the material; the third is the volume of the beam.

Hence for a rectangular beam of a given material loaded at the centre with the proof load, the resilience is directly proportional to its volume; a result corresponding with that obtained for direct stress, Chap. I. Suppose for any given material we take a one-inch cube; then, when loaded at the centre as beam, we have

$$\text{resilience per cubic inch} = \left(\frac{nn''}{8mm'}\right)\frac{f^2}{E};$$

for a rectangular section $n = \frac{1}{6}$, $m' = \frac{1}{2}$; for load at centre

$$m = \tfrac{1}{4}, \quad \text{and} \quad n'' = \tfrac{1}{3}; \quad \frac{nn''}{8mm'} = \frac{1}{18},$$

and resilience per cubic inch $= \dfrac{1}{18}.f.\dfrac{f}{E} = \dfrac{1}{18} \times$ proof stress × proof strain.

In the case of wrought-iron, for which the proof stress $f = 7$, and $E = 12000$ tons per square inch, and remembering that the deflection is in inches,

$$\text{resilience per cubic inch} = \frac{1}{18} \times \frac{7^2}{12000} = \cdot 000227 \text{ inch-tons} = \cdot 0423 \text{ ft.-lbs.}$$

For the previous example, the volume of the beam is 15360 cubic inches ; multiplying this quantity by ·0423 we find the result given there.

236. For a rectangular timber beam of uniform section and uniformly loaded, find the ratio of depth to span, so that the working deflection may be a six-hundredth part of the span. The working strength of the wood is one ton, and its modulus of elasticity is 800 tons per square inch.

$$\frac{h_0}{2c} = \tfrac{1}{4}\frac{n''}{m'} \cdot \frac{f}{E} \cdot \frac{2c}{v_0} = \tfrac{1}{4}\,\frac{\frac{5}{12}}{\frac{1}{2}}\,\frac{1}{800} \cdot 600 = \cdot 156.$$

That is, the depth is to be between a sixth and a seventh of the span.

237. What stiffness will be secured for an uniform rectangular beam of the same timber, loaded at centre, by making the depth an eighth of the span.

$$\frac{v_0}{2c} = \tfrac{1}{4}\frac{n''}{{}'m} \cdot \frac{f}{E} \cdot \frac{2c}{h_0} = \tfrac{1}{4} \cdot \frac{\frac{1}{3}}{\frac{1}{2}} \cdot \frac{1}{800} \cdot 8 = \frac{1}{600}.$$

That is, the working deflection will be a 600th part of the span.

238. Find m'' and n'', the numerical coefficients for slope and deflection, for a beam of uniform section loaded with two equal weights at points which trisect the span.

In the equations at foot of page 344, substitute $\frac{1}{3}c$ for x_r ; and

$$\textit{Ans.}\ m'' = \tfrac{1}{2}\left(1 + \frac{\frac{1}{3}c}{c}\right) = \tfrac{2}{3}. \quad n'' = \tfrac{1}{2} - \tfrac{1}{6}\left(1 - \frac{\frac{1}{3}c}{c}\right)^2 = \tfrac{23}{54}.$$

239. A beam of uniform cross-section is loaded with two equal weights symmetrically placed on the span ; the amount of each weight is for each position such that the outer fibres of the middle cross-section bear the working stress f. Compare the amount of deflection when the abscissæ of the left weight are as follows:—

$$x_r = 0, \quad x_r = \tfrac{1}{3}c, \quad \text{and} \quad x_r = \tfrac{2}{5}c.$$

Ans. For $x_r = 0$, $n'' = \frac{1}{3}$; for $x_r = \frac{1}{3}c$, $n'' = \frac{23}{54}$; for $x_r = \frac{2}{5}c$, $n'' = \frac{22}{50}$.

CHAPTER XVIII.

FIXED AND MOVING LOADS ON A UNIFORM GIRDER WITH ENDS FIXED HORIZONTALLY.

Unsymmetrical Fixed Loads.

We now consider the deflection of a uniform beam and the slopes or *tips-up* at the freely hinged ends due to loads placed unsymmetrically upon it.

On fig. 229 let a load W be concentrated at D a distance z from the centre of the span. D divides the span into two segments m and n. The bending moment diagram is ACB. The height of C is h, and of any other point is y. The origin is at Q the lowest point of the beam which divides the span into segments a and b. The deflection is $u_a = u_b$, and the tips-up are α and β. The span is $l = 2c$.

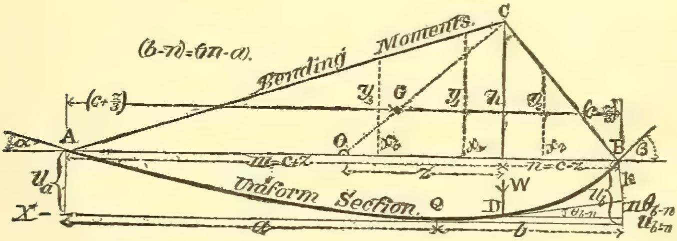


Fig. 229.

In Chapter, XVII. equation (5), I cancels I_0, and all the factors are constant except M the bending moment, y may be supposed to include this constant in every case.

$$\frac{y_1}{h} = \frac{a + x_1}{m}; \quad \frac{y_2}{h} = \frac{b - x_2}{n}; \quad \frac{y_3}{h} = \frac{a - x_3}{m}.$$

The slope at D and its height above Q are

$$\theta_{b-n} = \frac{h}{m}\int_0^{b-n} (a + x_1)\, dx = \frac{h}{m}\left[ax + \frac{x^2}{2}\right]_0^{m-a} = \frac{h}{2}\frac{m^2 - a^2}{m}.$$

$$u_{b-n} = \frac{h}{m}\int_0^{b-n}\int_0^x (a + x_1)\, dx dx = \frac{h}{m}\left[\frac{ax^2}{2} + \frac{x^3}{6}\right]_0^{m-a}$$

$$= \frac{h}{6m}(m - a)^2 (m + 2a).$$

And $k = \frac{h}{n}\int_{b-n}^{b}\int_{b-n}^{x} (b - x_2)\, dx dx = \frac{h}{n}\int_{b-n}^{b}\left[bx - \frac{x^2}{2}\right]_{b-n}^{x} . dx = \frac{h}{3}n^2.$

Also $u_a = \frac{h}{m}\int_0^a\int_0^x (a - x_3)\, dx dx = \frac{h}{3}\frac{a^3}{m}.$

Equating u_a and u_b we have (fig. 229),

$$u_a = u_b = u_{b-n} + n\theta_{b-n} + k,$$

$$2a^3 = (m - a)^2 (m + 2a) + 3n(m^2 - a^2) + 2n^2 m,$$

$$(m + n)(m^2 + 2mn - 3a^2) = 0,$$

$$3a^2 = m(l + n) = (c + z)(3c - z). \qquad (1)$$

Substituting we have the deflection of the beam

$$v_d = u_a = u_b = \frac{ha^3}{3m} = \frac{h\sqrt{3}}{27}(3c - z)\sqrt{(3c - z)(c + z)}. \qquad (2)$$

$$\beta = \theta_b = \theta_{b-n} + \frac{h}{n}\int_{b-n}^{b} (b - x_2)\, dx = \frac{h}{2m}(m^2 - a^2) + \frac{h}{n}\left[bx - \frac{x^2}{2}\right]_{b-n}^{b}$$

$$= \frac{h}{2}\left(m - \frac{a^2}{m} + n\right) = \frac{h}{2}\left(l - \frac{a^2}{m}\right).$$

But by equation (1) $a^2 \div m = (l + n) \div 3$, so that

$$\beta = \frac{h}{6}(2l - n) = \frac{h}{6}(3c + z). \tag{3}$$

$$\alpha = \theta_a = \frac{h}{m}\int_0^a (a - x_3)\,dx = \frac{h}{2}\frac{a^2}{m} = \frac{h}{6}(l + n) = \frac{h}{6}(3c - z). \tag{4}$$

And
$$\frac{\alpha}{\beta} = \frac{3c - z}{3c + z} = \frac{c - \bar{x}}{c + \bar{x}}, \tag{5}$$

where $\bar{x}$ is the distance horizontally of G the centre of gravity of the bending moment diagram from the centre of the span. Also

$$(\alpha + \beta) = hc = \text{area } ACB. \tag{6}$$

The following are perfectly general:—

Theorems.—If a girder of uniform section and hinged at the ends be loaded in any manner between them—(a) The sum of the *tips-up* at the ends is proportional to the area of the bending moments diagram, and (b) their ratio is the same as that of the segments into which the perpendicular from the centre of gravity of the bending moment diagram divides the span, but in the inverse order.

Equations (6) and (5) prove these theorems for one concentrated load.

Let there be two such concentrated loads so that $\alpha = \alpha_1 + \alpha_2$ and $\beta = \beta_1 + \beta_2$; then, from equation (5),

$$\frac{\alpha_1}{\alpha_1 + \beta_1} = \frac{c - \bar{x}}{2c} \quad \text{or} \quad 2c\alpha_1 = A_1 c - A_1\bar{x} = A_1 c - G_1,$$

where A_1 is the area of the bending moment diagram for W_1 alone, while G_1 is its geometrical moment about the centre of the span. For W_{2e} alone on the girder, we have $2c\alpha_2 = A_2 c - G_2$, and adding $2c\alpha = Ac - G$ for the joint bending moment diagram. Similarly, for any number of loads.

Putting $z = 0$, in equations (4) and (2), gives us

$$i_c = \tfrac{1}{2}hc \quad \text{and} \quad u_c = \tfrac{1}{3}hc^2, \tag{7}$$

which are just the same as equations (13) and (14) of Chapter XVII. for the uniform beam with load at centre if we put

$$h = f \div Ey_0.$$

Again, put $z = c$, and from equations (3) and (4), we have

$$\beta = \tfrac{2}{3}hc \quad \text{and} \quad \alpha = \tfrac{1}{3}hc. \tag{8}$$

The bending moment diagram is now right-angled at B; and as W is at B, and over the abutment, there is no strain on the beam, and α and β are both zero, but still just before W went off they were in that ratio.

There is another way of loading the beam so that the bending moment diagram shall be right-angled at B, that is, by applying a couple at the end B.

Suppose then a right-handed couple M_Q applied at B, the hinge A will hold down with a force P, and the hinge B will push up with $Q = -P$. In this manner the two hinges will constitute a reacting couple, with the span as an arm, and of opposite sense, but equal in moment to M_Q; so that

$$Q = -P = M_Q \div 2c.$$

The ends of the beam will *droop* by α and β at the ends A and B, the *droop* β being double the *droop* α by equation (8). Their sum will be the area of bending moment diagram, viz. the right-angled triangle of height M_Q and base $2c$. In the same way let a left-handed couple M_P be applied at the end A. The joint bending moment diagram due to both couples will be of height M_P and M_Q at A and B, with a straight locus HK, fig. 230, between them. If these two end couples be the only loads on the beam, M_Q being bigger than M_P, then the hinge A must hold down and the hinge B hold up with equal forces $P = -Q$, forming a reacting couple with the span as arm to balance $M_Q - M_P$, so that $P = -Q = (M_Q - M_P) \div 2c$. The beam will be everywhere convex upwards, its two ends drooping so that the *droops* α and β shall have their sum equal to the area of the bending moment diagram $AHKB$, and bear to each other the same ratio into which the perpendicular from g divides the span only in the inverse order.

Suppose now that the uniform girder hinged at the ends is loaded between them in any way, as for instance, merely with the weight W (fig. 230), trisecting the span. The two ends will tip up. By applying suitable end couples M_P and M_Q which alone would produce *droops* exactly equal to those *tips-up*, we may destroy the *tips-up* and make the ends level.

The conditions are that M_P and M_Q are given by AH and BK when HK is drawn across the bending moment diagram ABC, in such a way that the areas $AHKB$ and ACB shall be equal, and have their centres of gravity g and G in one vertical line.

We have the following graphical solution for a uniform girder with its ends held horizontal and loaded between them in any way. First draw the bending moment diagram for the extern loads as if the girder were hinged at the ends; next draw a straight line cutting across it so that the four-sided figure shall include the same area over the span as the bending moment diagram, and have its centre of gravity on the same vertical. This line, HK on fig. 230, is the new base cutting the locus in two points S and T between which the bending moments make the girder convex downwards, beyond which they make it convex upwards. From S to T ordinates are to be measured *up* to SCT for positive bending moments, and from S to H and T to K downwards to SA and TB for negative bending moments.

BEAM OF UNIFORM SECTION FIXED HORIZONTALLY AT THE ENDS AND SUBJECT TO A ROLLING LOAD.

Let W be at any distance z (fig. 230) from the centre O, then ACB, with C directly under W, is the bending moment diagram for the ends hinged. Let HK map out the area $AHKB = ACB$, and let g and G, their centres of gravity, be in a vertical line.

For shortness, put $h = AH$, and $k = BK$; also put m for the height of C from AB, and $OA = OB = c$, the half span. For areas equal we have $h + k = m$. For g under G take moments about A, first dividing $AHKB$ into two triangles by the diameter HB; their areas are hc and kc; their levers about A are $\frac{1}{3}$ of $2c$ and $\frac{2}{3}$ of $2c$, while the area of ACB is mc, and its lever $c + \frac{1}{3}z$, so that

$$mc\left(c + \tfrac{1}{3}z\right) = hc \times \tfrac{2}{3}c + kc \times \tfrac{4}{3}c; \quad m(3c + z) = 2c(h + 2k);$$

therefore $$k = \frac{c + z}{2c}m, \quad \text{and} \quad h = \frac{c - z}{2c}m.$$

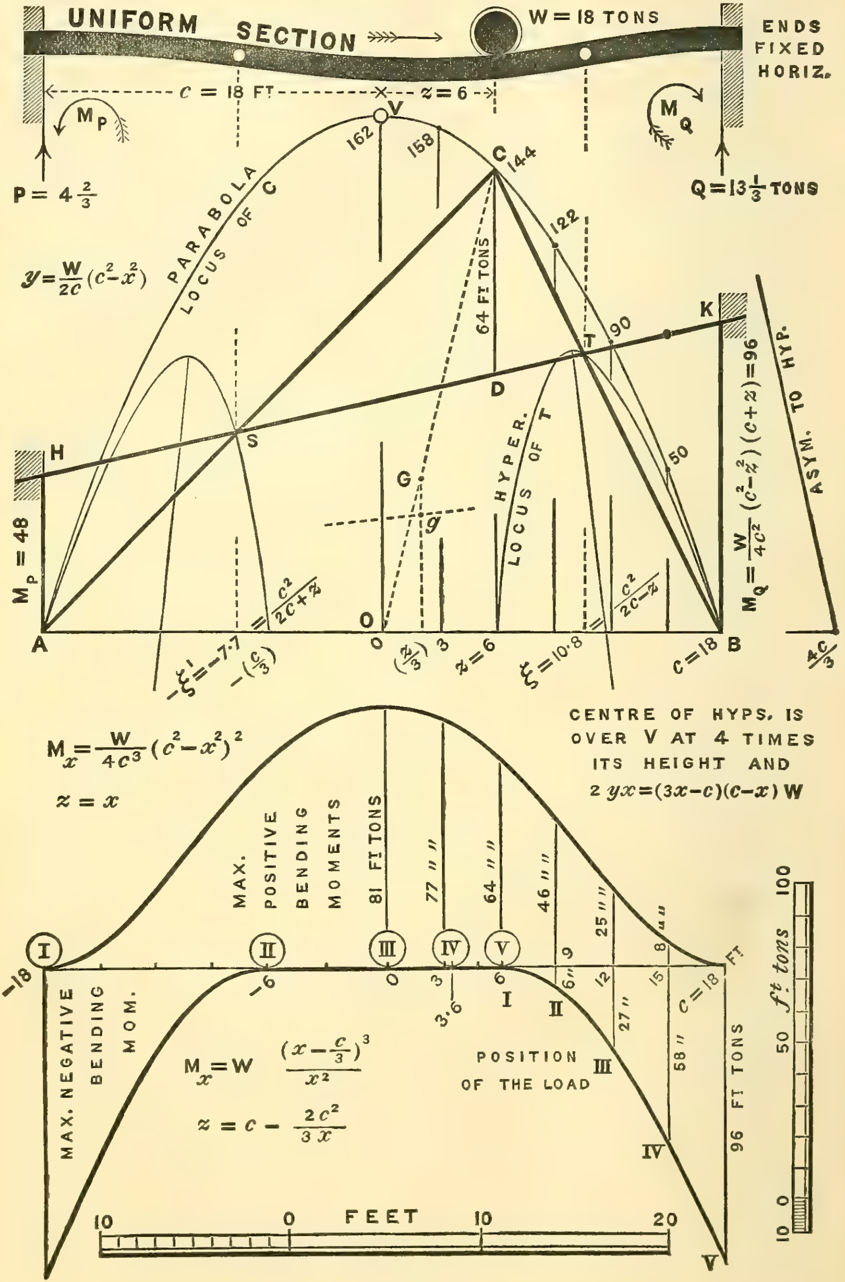

UNIFORM SECTION
W = 18 TONS
ENDS FIXED HORIZ.
c = 18 FT
z = 6
M_P
M_Q
$P = 4\frac{2}{3}$
$Q = 13\frac{1}{3}$ TONS
V
162
158
C 144
122
90
50
PARABOLA LOCUS OF C
$y = \frac{W}{2c}(c^2 - x^2)$
64 FT TONS
K
T
D
H
S
G
g
O
HYPER. LOCUS OF T
$M_Q = \frac{W}{4c^2}(c^2 - z^2)(c + z) = 96$
ASYM. TO HYP.
$M_P = 48$
$\frac{c^2}{2c+z}$
$\frac{c^2}{2c-z}$
A
B
$-\xi' = -7{\cdot}7$
$-(\frac{c}{3})$
0
$(\frac{z}{3})$
3
z = 6
$\xi = 10{\cdot}8$
c = 18
$\frac{4c}{3}$
CENTRE OF HYPS. IS OVER V AT 4 TIMES ITS HEIGHT AND
$2yx = (3x - c)(c - x)W$
$M_x = \frac{W}{4c^3}(c^2 - x^2)^2$
$z = x$
MAX. POSITIVE BENDING MOMENTS
81 FT TONS
77 " "
64 " "
46 " "
25 " "
8 " "
I
II
III
IV
V
-18
-6
0
3
6
6"
9
12
15
c = 18 FT
3·6
I
II
III
IV
V
27 "
58 "
96 FT TONS
MAX. NEGATIVE BENDING MOM.
$M_x = W\frac{(x - \frac{c}{3})^3}{x^2}$
$z = c - \frac{2c^2}{3x}$
POSITION OF THE LOAD
10
0
FEET
10
20
100
50
0
10
ft. tons

Fig. 230.

The equation to HK is

$$\frac{y_1 - h}{x + c} = \frac{k - h}{2c}$$

or
$$y_1 = \frac{k - h}{2c} x + \frac{k + h}{2} = \frac{zm}{2c^2} x + \frac{m}{2},$$

$$y_1 = \frac{m}{2c^2}(zx + c^2).$$

The equation to BC is

$$\frac{y_2 - 0}{x - c} = \frac{m - 0}{z - c}, \quad y_2 = m\frac{c - x}{c - z}.$$

Let us take the negative bending moment at any point x to the right of T. It will be the intercept from TK down to TB, and

$$M_{\text{neg.}} = y_1 - y_2 = \frac{m}{2c^2(c - z)}\left\{(c - z)(c^2 + xz) - 2c^2(c - x)\right\}.$$

Now, m, the bending moment under W, is the height of C above AB, or

$$m = \frac{W}{2c}(c^2 - z^2),$$

and substituting and simplifying

$$M_{\text{neg.}} = \frac{W}{4c^3}(c + z)^2((2c - z)x - c^2). \tag{9}$$

On the girder above T and S are shown virtual hinges or points of contrary flexure. To find ξ, the abscissa of T, substitute $M_{\text{neg.}}$ = zero, and $x = \xi$ into (9), and change sign of z for ξ', then

$$\xi = \frac{c^2}{2c - z}; \quad \text{also } -\xi' = \frac{c^2}{2c + z}. \tag{10}$$

Now ξ is a maximum when $z = -c$, that is, when W is just at left end, and substituting this value of z, we get the maximum value of ξ to be $\frac{1}{3}c$.

Hence, for the *middle third* of the span, the bending moment cannot be negative at any point for any position of the load.

For negative values of M, then, we require $x > \frac{c}{3}$. In equation (9) let x be any constant value between $\frac{1}{3}c$ and c, while z varies,

$$\frac{4c^3}{W}\frac{dM}{dz} = (c + z)(3x(c - z) - 2c^2),$$

$$\frac{4c^3}{W}\frac{d^2M}{dz^2} = - 6zx - 2c^2.$$

First, $z = -c$ makes $\frac{dM}{dz} = 0$, and $\frac{4c^3}{W}\frac{d^2M}{dz^2} = 6c\left(x - \frac{c}{3}\right)$ a *positive* quantity, as x is between $\frac{c}{3}$ and c. Hence $z = -c$ makes $M_{\text{neg.}}$ a minimum.

Second, with $z = c - \frac{2c^2}{3x}$, $\frac{dM}{dz} = 0$, and $\frac{4c^3}{W}\frac{d^2M}{dx^2} = -6c\left(x - \frac{c}{3}\right)$ a negative quantity, since $x > \frac{c}{3}$. Substituting this value of z in equation (9), we have

$$M_x = W\frac{\left(x - \frac{c}{3}\right)^3}{x^2}\text{, a negative maximum.} \tag{11}$$

When the load is at a point

$$z = c - \frac{2c^2}{3x}. \tag{12}$$

On fig. 230 there are laid *down* to scale the values of M_x at I, II, III, IV, and V, for the values of x, namely, 2, 3, 4, 5, and 6 twelfths of the span. The corresponding positions of the rolling load are shown with circles round the Roman numerals agreeable to equation (12). The maximum of negative maxima is at the end when $x = c$, and $z = \frac{1}{3}c$. At the top of fig. 230, the load W is shown in this position, namely, *trisecting the span*, and the maximum of negative maxima bending moments at the end is

$$M_c = \frac{8}{27}Wc = \frac{4}{27}Wl \text{ maximum.} \tag{13}$$

By substituting for m in the preceding expression for h and k we have

$$M_Q = BK = \frac{W}{4c^2}(c^2 - z^2)(c + z). \tag{14}$$

$$M_P = AH = \frac{W}{4c^2}(c^2 - z^2)(c - z). \tag{14a}$$

We come now to consider the positive bending moments. Changing the sign of (9) we have

$$M'_x = \frac{W}{4c^3}(c + z)^2 (c^2 - (2c - z)x) \text{ from } C \text{ to } T,$$

or as long as x lies between z and $\xi = \frac{c^2}{2c - z}$. Now M'_x is only admissible while $z \leq x$; and as it continually increases with z, M'_x is clearly greatest for $z = x$, that is, when the load is at the section. As the diagram is of the like form right and left of W or C, it follows that the positive maximum bending moment occurs at any point when the load W is over it. Substituting $z = x$ we have

$$M_x = \frac{W}{4c^3}(c^2 - x^2)^2, \text{ a positive maximum.} \tag{15}$$

Cor.—If the girder were only hinged at the ends, the maximum bending moment is $M_0 = \frac{1}{4} Wl$, when fixed horizontally at the ends by equation (13) it is $\frac{4}{27} Wl$; hence fixing a uniform girder at the ends increases its strength in the ratio 27 to 16 to resist a rolling load. This increase is 69 per cent. With the load confined to the central point the increase would be 100 per cent., for HK would then be horizontal, and ACB isosceles, and for the two bending moment diagrams of equal area we would then have $AH = CD = BK$, and all three maxima *half* of that for the ends hinged.

It can readily be proved that the loci of S and T are the pair of hyperbolas shown on fig. 230. The equation to the locus of T is $2yx = (3x - c)(c - x)W$. The principal axes of the pair meet on the vertical OV produced four times that amount.

It is interesting to notice from fig. 230, that at a point $\frac{1}{6}$ of the span from the right end, the positive and negative bending moments 25 and 27 are nearly equal. That is, for the end sixth of the span the negative maximum bending moment is greater than the positive. This point might be found from the equation

$$\frac{(c^2 - x^2)^2}{4c^3} = \frac{\left(x - \frac{c}{3}\right)^3}{x^2}.$$

It will be found that by trial and error $x = \frac{2}{3}c$ gives

$$\frac{25}{4 \times 27} = \frac{1}{4} \text{ nearly.}$$

Levy gives this point as $\cdot 17l$ from the end, a trifle more than $\frac{1}{6}$ of span, as the diagram (fig. 230) shows it to be.*

Shearing Force.—For the position of the load shown on figs. 230, 231, the left supporting force Q is composed of two parts, one a major part of W, namely $W(c + z) \div 2c$, just as if the girder were simply hinged at the ends. The other part is $(M_Q - M_P) \div 2c$ due to the couple at one end being greater than the other, so that

$$Q = W\frac{c + z}{2c} + \frac{W}{4c^3} z(c^2 - z^2).$$

Now Q is the instantaneous shearing force at every point in the shorter segment of the girder from W to the right abutment B. It is clear, then, for a section $x = z$, that is, for a section indefinitely close to W on its right side, that Q is the maximum shearing force. Hence

$$F_x = W\frac{c + x}{2c} + \frac{W}{4c^3} x(c^2 - x^2) \text{ a maximum,}$$

the load being just to the same side of the section as the more remote abutment

$$F_x = \frac{W}{4c^3}(c + x)^2(2c - x) \text{ maximum.} \qquad (16)$$

---

* In *La Statique Graphique*, second edition, vol. II., p. 124, Levy gives equations (20) and (19) derived by constructing graphically "lines of influence." He finds that (19) supersedes (20) in magnitude for a 17th part of the span from each end. In Levy's equations (20) and (19) the origin is at the end of the girder; substituting $(c - x)$ for his x, and $2c$ for his l, these equations (20) and (19) are identical with our equations (11) and (15) given above in their symmetrical form, the origin at the centre, and the loci of which are plotted on the lower part of our fig. 230.

The solution in the same form as Levy's is given in Du Bois on *The Stresses in Framed Structures*, second edition.

Now $F_c = W$ at the ends just as if the girder had been hinged at the ends. So also at the centre $F_0 = \frac{1}{2} W$, as indeed is evident, for with W at the end there are no end couples; while with W at the centre the end couples are equal and do not modify the supporting forces. Hence fig. 138 is an approximation to the shearing force diagram.

On fig. 231 is shown the diagram of maximum shearing forces plotted from equation (16).

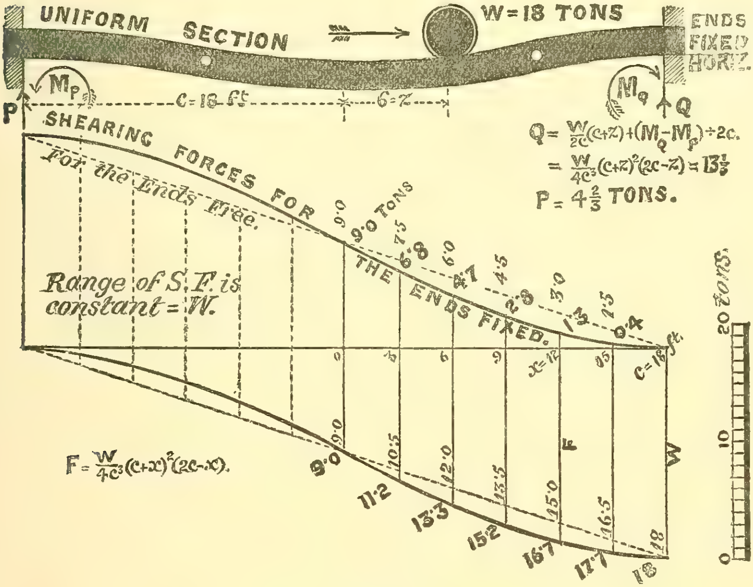


Fig. 231.

Levy shows this diagram in his fig. 25, p. 125 (see footnote). From his equation (21 *bis*), identical with our equation (16), he infers that the locus starts horizontally at each end. It is readily shown by differentiating (16) when the differential coefficient of the variable part is $3(c^2 - x^2)$, then $x = \pm c$ makes this zero, so that the slope at end is zero, and at centre is 45°.

It is important to observe that at any section the positive maximum shearing force is instantly succeeded by the negative as the load passes it, so that their arithmetical sum or the *range* of the shearing force, as we have called it, is W, and is in no way altered by the fixing of the ends of the girders.

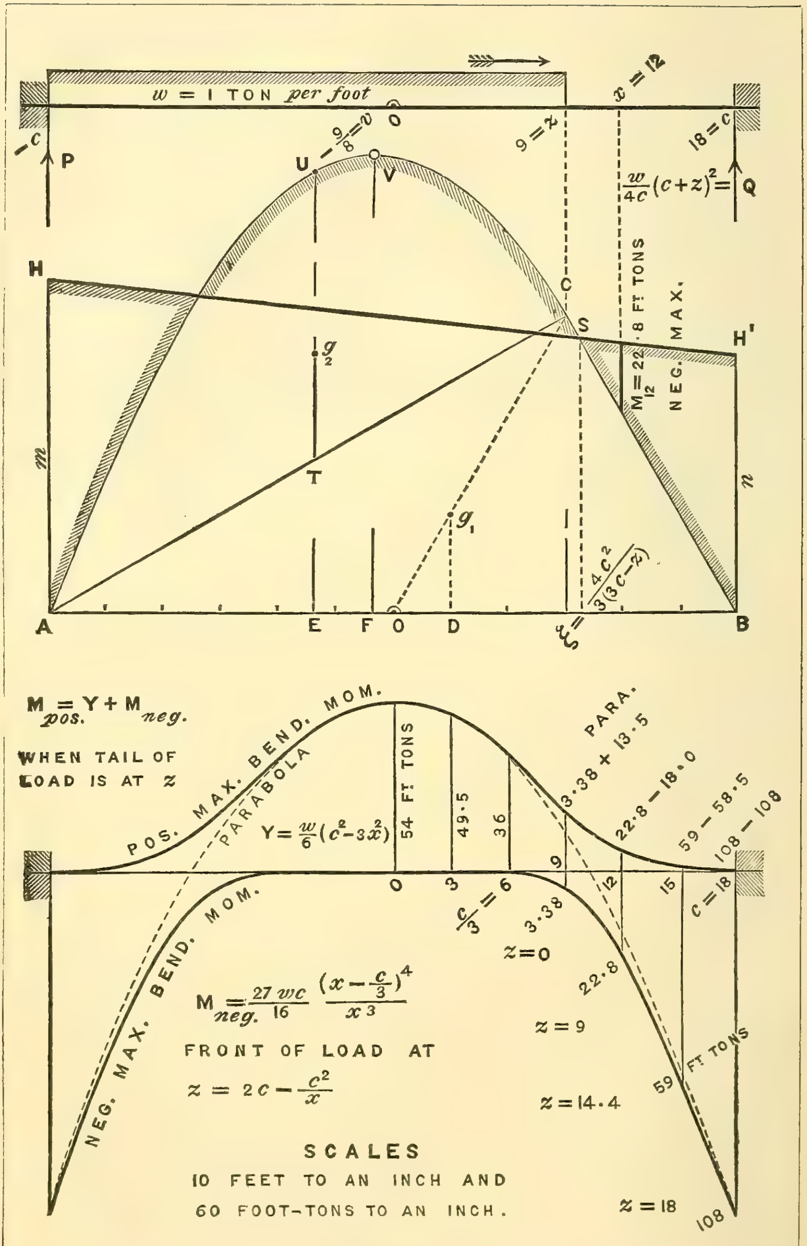

w = 1 TON per foot
x = 12
O
−c
P
U
$v = -\frac{9}{8}$
V
$9 = z$
$18 = c$
$\frac{w}{4c}(c+z)^2 = Q$
H
C
S
H'
g_2
$M_{12} = 22.8$ FT TONS
NEG. MAX.
m
T
n
g_1
$\frac{4c^2}{3(3c-z)}$
A
E
F
O
D
$\xi =$
B
$M_{pos.} = Y + M_{neg.}$
WHEN TAIL OF LOAD IS AT z
POS. MAX. BEND. MOM.
PARABOLA
$Y = \frac{w}{6}(c^2 - 3x^2)$
54 FT TONS
49.5
36
PARA.
3.38 + 13.5
22.8 − 18.0
59 − 58.5
108 − 108
0
3
6
9
12
15
$c = 18$
$\frac{c}{3} = 6$
3.38
z = 0
NEG. MAX. BEND. MOM.
22.8
$M_{neg.} = \frac{27wc}{16} \frac{(x - \frac{c}{3})^4}{x^3}$
z = 9
FRONT OF LOAD AT
$z = 2c - \frac{c^2}{x}$
z = 14.4
59
FT TONS
SCALES
10 FEET TO AN INCH AND
60 FOOT-TONS TO AN INCH.
z = 18
108

Fig. 232.

BEAM OF UNIFORM SECTION FIXED HORIZONTALLY AT THE ENDS AND SUBJECT TO A UNIFORM ADVANCING LOAD.

For convenience the uniform load may be taken as 1 ton per foot-run.

Consider first the negative bending moments. For the middle third of the span they are zero at each point; for the load, however it be disposed, may be looked upon as a number of concentrated loads, no one of which can produce a negative bending moment on the middle third by the last problem (fig. 230). See top of page 370.

Let x be the abscissa of any section on the third next the right abutment, measured from the centre as origin, so that x lies between the values $\frac{1}{3}c$ and c, where c is the semi-span. Let the load advance from the left abutment till the *front* is at the section z from the centre, where z is measured in the same direction as x but is *less than* x.

On fig. 232, $AVCB$ is the instantaneous bending moment diagram for the ends hinged: compare fig. 126. AVC is parabolic, but CB is a straight locus. By theorems (a), (b), p. 365, HH' is drawn across so that the areas $AHH'B$ and $AVCB$ are equal and have their centres of gravity on one vertical line.

As z is less than x, the new base HH', for ends fixed, cuts at S on the straight part CB. For ends hinged

$$Q = \frac{w}{4c}(c+z)^2,$$

the height of C above AB is

$$\frac{w}{4c}(c+z)^2(c-z),$$

while the height at x to SB is

$$y = \frac{w}{4c}(c+z)^2(c-x).$$

The bending moment diagram $AUCB$ for the extern load divides into two parts a triangle ACB of area $\frac{1}{4}w(c+z)^2(c-z)$, and lever about A of $AD = (c + \frac{1}{3}z)$, and a parabolic part AUC of area $\frac{2}{3}\overline{TU}(c+z)$, and lever about A of $AE = \frac{1}{2}(c+z)$, where $TU = \frac{1}{8}w(c+z)^2$, just as for a span $(c+z)$ uniformly loaded.

Again, the bending moment diagram *AHH′B*, for the end couples m and n that destroy the *tips up*, divides into two triangles; *AHH′* of area mc and lever about A of $\frac{2}{3}c$, and *AH′B* of area nc and lever $\frac{4}{3}c$.

Equating the areas of these two bending moment diagrams and also their moments about A, satisfies the two conditions laid down by preceding *Theorems* (*a*) and (*b*), we have the two equations

$$(m + n)c = \frac{w}{4}(c + z)^2(c - z) + \frac{w}{12}(c + z)^3,$$

$$(m + n)\frac{2c^2}{3} = \frac{w}{4}(c + z)^2(c - z)\left(c + \frac{z}{3}\right) + \frac{w}{24}(c + z)^4.$$

So that

$$m = \frac{w}{48c^2}(c + z)^2(11c^2 - 10cz + 3z^2), \tag{17}$$

$$n = \frac{w}{48c^2}(c + z)^3(5c - 3z). \tag{18}$$

The height to *HH′* at the section x is

$$y' = n + \frac{c - x}{2c}(m - n),$$

$$y' = \frac{w}{48c^3}(4c^2(2c - z) - 3(c - z)^2x)(c + z)^2.$$

So that at x

$$M_{\text{neg.}} = y' - y = \frac{w}{48c^3}(c + z)^3(3(3c - z)x - 4c^2). \tag{19}$$

If $x = \xi$, when M is zero, we have for S the virtual hinge

$$\xi = \frac{4c^2}{3(3c - z)}.$$

Putting u for the part of equation (19) that varies with z, then

$$\frac{du}{dz} = 12(c + z)^2(2cx - c^2 - xz),$$

$$\frac{d^2u}{dz^2} = 12(c + z)(3cx - 2c^2 - 3xz).$$

Now
$$z = 2c - \frac{c^2}{x}, \tag{20}$$

which, observe, is less than x as premised, makes

$$\frac{du}{dz} = 0, \quad \text{and} \quad \frac{d^2u}{dz^2} = 36c(c+z)\left(\frac{c}{3} - x\right),$$

a negative quantity as we are only dealing with values of x which are greater than $\frac{1}{3}c$, in that we have disposed of the middle third of the span already as far as *negative* bending moments are concerned. Hence this value of z makes $M_{\text{neg.}}$ a maximum for all points in the third of the span towards the right abutment. Substituting and reducing, we have

$$M_{\text{neg.}} = \frac{27wc}{16} \frac{\left(x - \frac{c}{3}\right)^4}{x^3} \text{ max.}, \tag{21}$$

when the front of the load is at z, equation 20.

We come now to consider the positive bending moments. Let the load cover the span, and simultaneous at all sections the bending moment is given by the dotted parabola on fig. 232 as already explained, measuring up and down from the horizontal base across it. The equation to this parabola with the centre of that base as origin is

$$y = \frac{w}{6}(c^2 - 3x^2). \tag{22}$$

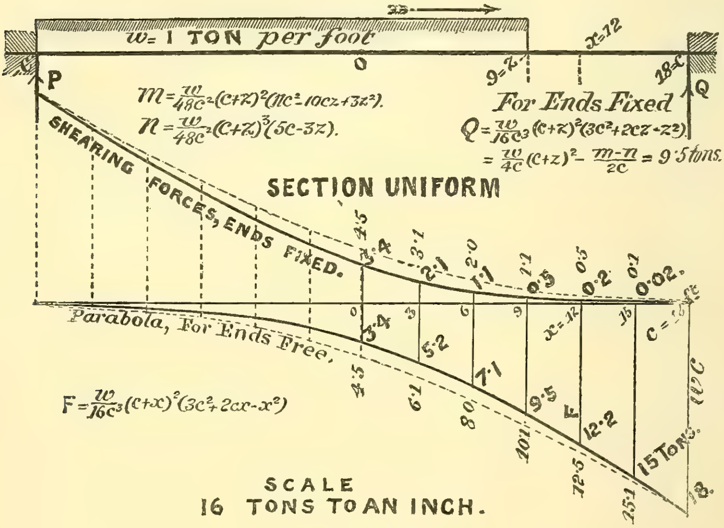
See fig. 91 for such equations, and fig. 227 for the conditions.

Now y is positive in the middle third and even further out from the middle, but we shall only consider the middle third. Now removing any part of the load is the same as if we had left the load all over the span and applied a load of equal intensity *upward* at a part, and this upward load being itself negative cannot cause any positive bending moment in the middle third of the span. Hence the dotted parabola itself gives the maximum bending moment at each section in the middle third, the load being all over the span

$$M_{\text{pos.}} = \frac{w}{6}(c^2 - 3x^2) \text{ max.} \tag{23}$$

for values of x from 0 to $\frac{1}{3}c$ only.

For the third of the span next the right abutment, let x be the abscissa of any section as before. Now we have shown that equation (20) $M_{\text{neg.}}$ is greatest for the load extending from the left abutment to $z = 2c - \frac{c^2}{x}$. If this load were acting upwards the bending moment at x would still be the maximum, but of course positive. But before applying this upward load, let the whole span be loaded, and the upward load removes the load from left abutment to z and leaves it from z to the right abutment. Hence the maximum positive bending moment is the



w= 1 TON per foot
x
x=12
c
P
0
9=z
18=c
Q
$m = \frac{w}{48c^2}(c+z)^2(11c^2 - 10cz + 3z^2)$.
$n = \frac{w}{48c^2}(c+z)^3(5c-3z)$.
For Ends Fixed
$Q = \frac{w}{16c^3}(c+z)^2(3c^2+2cz-z^2)$
$= \frac{w}{4c}(c+z)^2 - \frac{m-n}{2c} = 9{\cdot}5$ tons.
SHEARING FORCES, ENDS FIXED.
SECTION UNIFORM
4·5
3·4
3·1
2·1
2·0
1·1
1·1
0·5
0·5
0·2
0·1
0·02.
ft
0
3
6
9
x=12
15
c=18
Parabola, For Ends Free.
3·4
5·2
7·1
9·5
F
12·2
15 TONS.
4·5
6·1
8·0
10·1
12·5
15·1
18.
1/2 C
$F = \frac{w}{16c^3}(c+x)^2(3c^2+2cx-x^2)$
SCALE
16 TONS TO AN INCH.

Fig. 233.

same as $M_{\text{neg.}}$ only measured upwards from the dotted parabola (fig. 232), and occurs when the short segment from z is loaded. Hence for values of x from $\frac{1}{3}c$ to c

$$M_{\text{pos.}} = \frac{w}{6}(c^2 - 3x^2) + \frac{27wc}{16}\frac{\left(x - \frac{c}{3}\right)^4}{x^3}; \qquad (24)$$

a maximum when the tail of the load is at z, equation 20.

The same argument, on page 377, shows the negative or downward intercepts between HH' and the arc AU, all to be maxima for the whole span loaded. These maxima only reach down to the dotted parabola, and are superseded by the locus of equation 21.

Shearing Force.—For the ends fixed (fig. 233), we have

$$Q = \frac{w(c+z)^2}{4c} - \frac{m-n}{2c} = \frac{w}{16c^3}(c+z)^2(3c^2+2cz-z^2);$$

and this is the shearing force at x for the position of the load shown. Now the differential coefficient of Q with respect to z is positive, or Q continually increases with z till $z = x$. Again $z > x$ means $z = x$, and an additional load between x and the right abutment a share of which is thrown on the left abutment, so that the increase of Q is more than cancelled when from Q we subtract this whole extra load as we must for the shearing force; hence the shearing force at any section x has maximum positive and negative values when the segment from x to one or other abutment is loaded, exactly as was the case for the ends hinged (see fig. 133), only now

$$F_{\text{pos.}} = \frac{w}{16c^3}(c+x)^2(3c^2+2cx-x^2). \qquad (25)$$

EXAMPLES.

240. Graphical construction of the bending moment and shearing force diagrams for a uniform girder of 36 feet span, fixed horizontally at the ends, and loaded with 12 tons at 9 feet from the left end, and 6 tons at 21 feet from the left end (fig. 234).

The load line is laid down to a scale of 10 tons to an inch. From a trial pole a link polygon constructed among the four forces P, W_1, W_2, and Q, laid down at 10 feet to an inch, and a vector from the pole parallel to the closing side determines the joint between the reactions, making P scale $11\frac{1}{2}$ and Q scale $6\frac{1}{2}$ tons, being the values for hinged ends. They may also be found by taking moments about the left hinge. Through this joint the base of the shearing force diagram is drawn horizontally, and the diagram completed for hinged ends.

A new pole on the horizontal through the joint is chosen at a distance 6 feet, a horizontal base laid off for the bending moment diagram (with hinged ends), which is then constructed as a link polygon to the new pole.

Thus far it will be seen that the load and girder correspond to the moving models (figs. 149, 150), with the locomotive standing so that its heavy wheel is 9 feet from the left abutment.

To return to fig. 234 it will be found that the scale for bending moments is 60 foot-tons to an inch, that is, six times as fine as the ton scale.

For the ends fixed we must draw a base, cutting across the bending moment diagram, so as to include the same area over the base, and have the centres of gravity in one vertical (see *Theorems* (*a*) and (*b*), page 365).

To do this graphically the bending moment diagram is divided into *three triangles* and one rectangle. The first triangle has a base 9 feet and a height 103·5 foot-tons, being the bending moment under W_1, either as scaled off or as calculated. The last triangle has a base 15 feet and a height 97·5 foot-tons. The dimensions of the remaining triangle and the rectangle are readily seen.

It is convenient to reckon 18 feet the *half span* as a horizontal unit for these areas, which will then be found to be $A_1 = 26$, $A_2 = 2$, $A_3 = 65$, and $A_4 = 41$. The lines of action are drawn *down* through the respective centres of gravity. Next the lines of action A_5 and A_6 are drawn upwards through the trisecting points of the span to represent the areas of the two triangles, into which the bending moment diagram for the pair of end couples can always be divided. A load line of the areas is laid down to the foot-ton scale, a link polygon drawn among the lines of action when a vector from the pole drawn parallel to its closing side divides the areas A_5 and A_6 from each other. Since the half-span is unity, it will be seen that A_5 and A_6 are M_5 and M_6 on the foot-ton scale. Their values scale 51 and 83 foot-tons, and are pricked up at the ends and the base drawn across.

To modify the shearing force diagram $M_6 - M_5 = 31$ foot-tons, dividing this by 36 feet, the lever between the supports, we get $\frac{8}{9}$ths of a ton as the common force by which the right end must hold down and left hold up to resist the left-handed couple $M_6 - M_5$. Drawing a new base for the shearing force diagram $\frac{8}{9}$ths of a ton lower down decreases the one and increases the other support by this amount.

241. To find the righting end couple M_5 (fig. 234) by calculation. For equal areas and equal moments about 0,

$$M_6 + M_5 = 26 + 2 + 65 + 41 = 134,$$

$$12M_6 + 24M_5 = 26 \times 6 + 2 \times 13 + 65 \times 15 + 41 \times 26,$$

and $M_5 = 51$, $M_6 = 83$ ft.-tons.

242. A uniform girder is loaded on the left half only with a uniform load: find the righting end couples to hold the ends horizontal.

Substituting for m and n in equations (17) and (18), we get

$$m = \tfrac{11}{48}\, wc^2 \quad \text{and} \quad n = \tfrac{5}{48}\, wc^2.$$

Now the bending moment at centre for hinged ends is Q a quarter of total load, multiplied by c half span; that is, the height of C (fig. 232) is now $\frac{1}{4}wc^2$ or $\frac{12}{48}wc^2$. Hence, if on fig. 232, we suppose the load to be on left half only then m and n are $\frac{11}{12}$ and $\frac{5}{12}$ respectively, of the height of C the middle point of the bending moment diagram.

243. The case of a uniform girder, uniformly loaded on the left half only, is so important that we shall calculate the righting end couples directly.

First since Q is a quarter and P three-quarters of the load, it follows that the shearing force diagram crosses the base at a point $\frac{3}{8}$th of the span from A (fig. 232). Also AE is one-half of span so that U and C are the same height and $ET = TU$ are each half the height of C. With C the half span as unity for horizontal areas, the area AUC is $\frac{2}{3}TU$ or $\frac{1}{3}$rd height of C. Putting k for the height of C, area AUC is $\frac{1}{3}k$ and its lever about A is $\frac{1}{2}c$ while area ACB is k, and its lever about A is c. Again for $AHH'C$ divided into two triangles by AH', the area of AHH' is m, its lever $\frac{1}{3} \times 2c$; the other $AH'B$ has an area n and lever $\frac{2}{3} \times 2c$. Equating areas and moments of areas about A

$$m + n = \frac{k}{3} + k; \qquad m\,\frac{2c}{3} + n\,.\,\frac{4c}{3} = \frac{k}{3}\,.\,\frac{c}{2} + kc; \qquad m = \frac{11}{12}\,k \quad \text{and} \quad n = \frac{5}{12}\,k.$$

The graphical construction is shown at top of fig. 249.

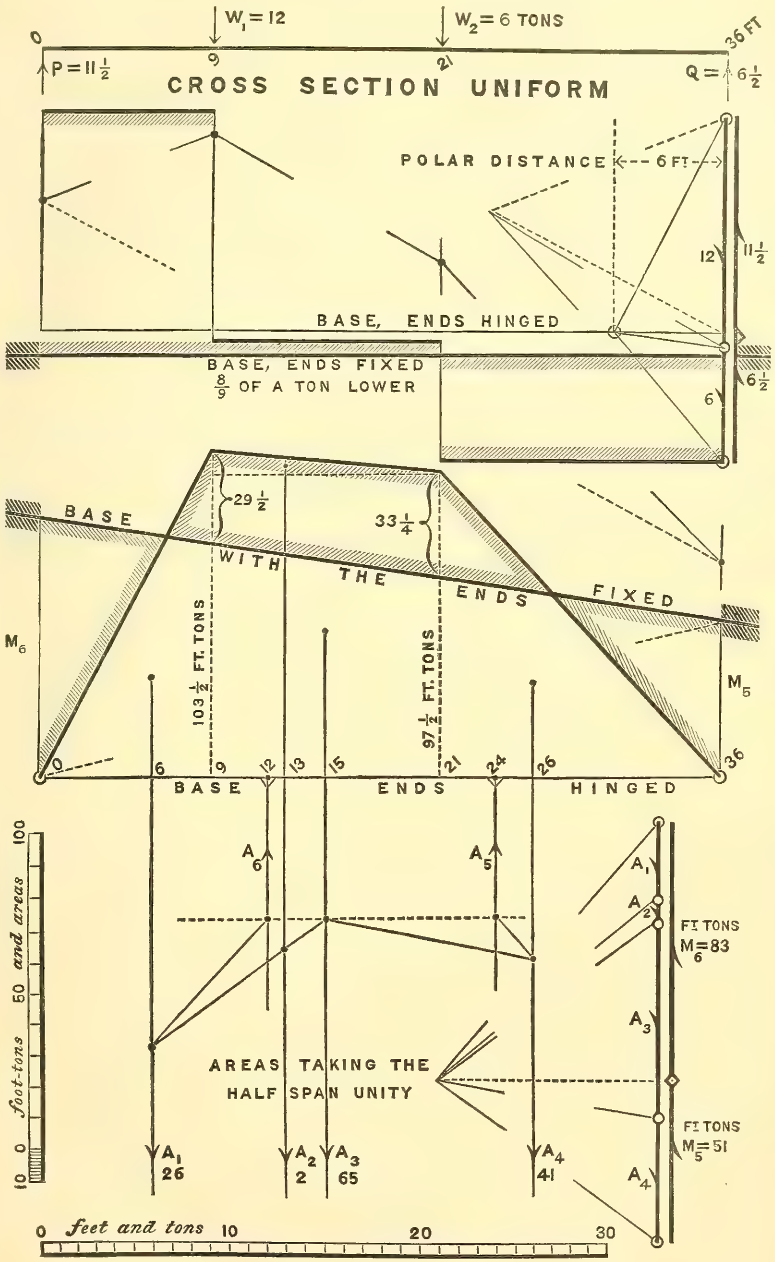

W₁ = 12
W₂ = 6 TONS
0
9
21
36 FT
P = 11 1/2
Q = 6 1/2
CROSS SECTION UNIFORM
POLAR DISTANCE
6 FT
12
11 1/2
BASE, ENDS HINGED
BASE, ENDS FIXED
8/9 OF A TON LOWER
6
6 1/2
29 1/2
33 1/4
BASE WITH THE ENDS FIXED
M₆
M₅
103 1/2 FT. TONS
97 1/2 FT. TONS
0
6
9
12
13
15
21
24
26
36
BASE ENDS HINGED
A₆
A₅
A₁
A₂
FT TONS
M₆ = 83
A₃
AREAS TAKING THE
HALF SPAN UNITY
FT TONS
M₅ = 51
A₄
A₁ 26
A₂ 2
A₃ 65
A₄ 41
100
50
0
10
foot-tons and areas
0 feet and tons 10
20
30

Fig. 234.

CHAPTER XIX.

SKELETON SECTIONS FOR BEAMS AND STRUTS; COMBINED THRUST WITH BENDING AND TWISTING WITH BENDING; LONG STRUTS.

THIN HOLLOW CROSS-SECTIONS.

LET t be the uniform thickness of a thin hollow cross-section of any form, and let n' and n be the numerical coefficients respectively of the moment of inertia and the moment of resistance to bending of a solid cross-section of the same form; let B, H be the breadth and depth of the rectangle circumscribing the section, and b, h the breadth and depth of the rectangle circumscribing the hollow; then

$$\begin{aligned} I_0 &= n'(BH^3 - bh^3) = n'B(H^3 - h^3), \text{ since } B = b \text{ nearly} \\ &= n'B(H-h)(H^2 + Hh + h^2) \\ &= n'B \,.\, 2t \,.\, 3H^2 \text{ since } H = h \text{ nearly} = n' \,.\, 6BH^2t. \end{aligned}$$

$$\begin{aligned} \mathsf{M} &= \frac{f}{m'H} I_0 = \left(\frac{n'}{m'}\right) 6f \,.\, B \,.\, H \,.\, t \\ &= n \,.\, 6fBHt \quad \text{(1st approx.)} \end{aligned} \qquad (a)$$

Thin hollow rectangle, $\mathsf{M} = \frac{1}{6} \,.\, 6fBHt = fBHt.$

Thin hollow circle, $\mathsf{M} = \frac{\pi}{32} 6fBHt = {\cdot}6fd^2t$, ($d$ = diam.)

A closer approximation for any form is obtained thus:—

$$\begin{aligned} I_0 &= n'\{BH^3 - bh^3\} = n'\{(B-b)H^3 + b(H^3 - h^3)\} \\ &= n'\{(B-b)H^3 + b(H-h)(H^2 + Hh + h^2)\} \\ &= n'\{2t \,.\, H^3 + b \,.\, 2t' \,.\, 3Hh\} \text{ (approx.)} \\ &= 2n'H\{tH^2 + 3bt'h\} \text{ (approx.)}; \end{aligned}$$

$$\therefore \quad \mathsf{M} = 2nf(tH^2 + 3t'bh) \text{ (approx.)}, \qquad (b)$$

where t is the thickness of each side, and t' the thickness of the top or bottom.

To design a thin hollow cross-section—choose the depth H the proper fraction of the span to give the required degree of stiffness; assume B a suitable fraction of H to give sufficient lateral stiffness, and the above is a simple equation from which to find t; or t may be assumed a multiple of the thickness that the metal plates are usually manufactured, and B found from the formulæ. Now find the moment of resistance to bending, and the resistance to shearing, by the accurate formulæ; and if this differs from M and F, alter t or B for further approximation.

CROSS-SECTIONS OF EQUAL STRENGTH.

When a beam is made of a material whose strengths to resist tension and thrust are different, the area of the upper flange is made different from that of the lower (fig. 235), in order that both flanges may be brought to the proof *or* working stress at the same time. In the case of cast iron, the strengths to resist tension and thrust are as 1 and 6; and on this account the area of the upper flange (compressed) is about one-sixth that of the lower flange (extended). This form of cross-section was first proposed by Mr. Hodgkinson. On account of the liability of cast iron to crack if unequally cooled, sudden changes of thickness of metal are to be avoided; on this account the top of the web may be made of the same thickness as the top flange, and the bottom of the web of the same thickness as the bottom flange.

Double T *cross-section* (fig. 235).—The position of the neutral axis, and the moment of inertia about that axis, in terms of the areas and depths of the three rectangles, are found as follows, the notation being:

	Areas.	Depths.
Upper flange, . . .	A_1,	h_1.
Web, ...	A_2,	h_2.
Lower flange, . . .	A_3,	h_3.
	——	—
Totals, .	A	h

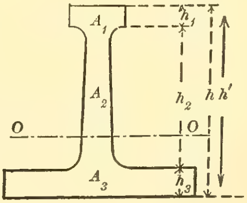


Fig. 235.

Exact Solution.—The height of the neutral axis above the lower side of the cross-section is obtained thus—

$$A y_b = A_1\left(\frac{h_1}{2} + h_2 + h_3\right) + A_2\left(\frac{h_2}{2} + h_3\right) + A_3\frac{h_3}{2};$$

$$y_b = \frac{h}{2} - \frac{(h_1 + h_2)A_3 - (h_2 + h_3)A_1 - (h_3 - h_1)A_2}{2A}. \qquad (1)$$

The moment of inertia for each rectangle is shown on page 283; and letting Σ as before denote the "sum," the moment of inertia for the cross-section is

$$I_0 = \Sigma I + \Sigma A \,.\, x_0^2; \quad \Sigma I = \Sigma\frac{bh^3}{12} = \frac{A_1h_1^2 + A_2h_2^2 + A_3h_3^2}{12};$$

$$\Sigma A x_0^2 = \Sigma bhx_0^2 = A_1\left[y_b - \left(h_2 + h_3 + \frac{h_1}{2}\right)\right]^2 + A_2\left[y_b - \left(h_3 + \frac{h_2}{2}\right)\right]^2$$

$$+ A_3\left[y_b - \frac{h_3}{2}\right]^2$$

$$= A_1\left\{\frac{-A_2(h_1 + h_2) - A_3(h_1 + 2h_2 + h_3)}{2A}\right\}^2$$

$$+ A_2\left\{\frac{A_1(h_1 + h_2) - A_3(h_2 + h_3)}{2A}\right\}^2$$

$$+ A_3\left\{\frac{A_1(h_1 + 2h_2 + h_3) + A_2(h_2 + h_3)}{2A}\right\}^2;$$

$$\therefore \quad I_0 = \frac{A_1h_1^2 + A_2h_2^2 + A_3h_3^2}{12} + \frac{1}{4A}\{A_1A_2(h_1 + h_2)^2$$

$$+ A_1A_3(h_1 + 2h_2 + h_3)^2 + A_2A_3(h_2 + h_3)^2\}. \qquad (2)$$

The moment of resistance to bending is as before

$$M = \frac{f_b I_0}{y_b}. \qquad (3)$$

Approximate Solution.—(Rankine's "Civil Engineering," § 163, 164). When h_1 and h_3 are *small* compared with h_2, we may leave them out of the exact formulæ; and we obtain

$$y_b = \frac{h}{2} - \frac{(A_3 - A_1) h_2}{2A}, \tag{4}$$

$$I_0 = \frac{A_2 h_2^2}{12} + \frac{h_2^2}{4A} (A_1 A_2 + 4A_1 A_3 + A_2 A_3). \tag{5}$$

Put h' for $h_2 + \dfrac{h_1 + h_3}{2}$, that is for the distance between the centres of gravity of the flanges; and let A_2 be the area of the cross-section of the vertical web measured from centre to centre of the top and bottom flanges; then, *nearly*

$$y_b = \frac{h'}{2} - \frac{(A_3 - A_1) h'}{2A} = \frac{h'}{2}\left(\frac{2A_1 + A_2}{A}\right). \tag{6a}$$

$$y_a = \frac{h'}{2}\left(\frac{2A_3 + A_2}{A}\right). \tag{6b}$$

$$I_0 = h'^2 \left\{\frac{A_2}{12} + \frac{1}{4A} (A_1 A_2 + 4A_1 A_3 + A_2 A_3)\right\}. \tag{7}$$

Since $\qquad y_a : y_b : h :: f_a : f_b : f_a + f_b,$

we have $\qquad y_a : y_b : h' :: f_a : f_b : f_a + f_b$ approx.

From equations (6) we have

$$y_a : y_b : h' :: 2A_3 + A_2 : 2A_1 + A_2 : 2A;$$

$$\therefore \quad f_a : f_b : f_a + f_b :: 2A_3 + A_2 : 2A_1 + A_2 : 2A \text{ approx.,}$$

and

$$\frac{2A_3 + A_2}{2(A_1 + A_2 + A_3)} = \frac{f_a}{f_a + f_b}. \tag{8}$$

From this equation we can eliminate A_1 and A_3, and

$$A_1 = \frac{f_b}{f_a} A_3 + \frac{f_b - f_a}{2f_a} A_2. \tag{9}$$

$$A_3 = \frac{f_a}{f_b} A_1 + \frac{f_a - f_b}{2f_b} A_2. \tag{10}$$

Substituting this value of A_3 in equation (7), we obtain

$$I_0 = h'^2 \left\{ \frac{A_1A_2 + A_2^2 + A_2A_3 + 3A_1A_2 + 12A_1A_3 + 3A_2A_3}{12\,(A_1 + A_2 + A_3)} \right\}$$

$$= h'^2 \left\{ \frac{4A_1A_2 + 4A_2\left(\frac{f_a}{f_b}A_1 + \frac{f_a - f_b}{2f_b}A_2\right) + 12A_1\left(\frac{f_a}{f_b}A_1 + \frac{f_a - f_b}{2f_b}A_2\right) + A_2^2}{12\left(A_1 + A_2 + \frac{f_a}{f_b}A_1 + \frac{f_a - f_b}{2f_b}A_2\right)} \right\}$$

$$= \frac{h'^2}{f_a + f_b} \left\{ f_aA_1 + (2f_a - f_b)\frac{A_2}{6} \right\}. \qquad (11)$$

$$= \frac{f_b}{y_b} I_0 = \frac{f_a + f_b}{h} I_0 = \frac{f_a + f_b}{h'} I_0 = h' \left\{ f_aA_1 + (2f_a - f_b)\frac{A_2}{6} \right\} \qquad (12a)$$

$$= h' \left\{ f_bA_3 + (2f_b - f_a)\frac{A_2}{6} \right\}. \qquad (12b)$$

In designing a beam to resist a given bending moment, the depth h' is taken at a fraction, say $\frac{1}{8}$th to $\frac{1}{14}$th of the span so as to ensure stiffness; the thickness of the web is then fixed by considerations of practical convenience, and so as to give sufficient resistance to shearing; and the area of the upper and lower flange can then be found by equations 12; having thus fixed the value of A_1 and A_3 we can then choose breadths and depths suitable for the flanges.

For the section as thus fixed calculate **M** and **F** by the tabular method shown for figs. 185, 207, &c.; if h_1 and h_3, the depths chosen for A_1 and A_3, are very small, it will be found that **M** and **F** are sensibly what is required, and that the neutral axis sensibly divides the depth of section as f_a and f_b, so that no further calculation is necessary. If, however, one or both of the depths h_1, h_3, be *not* very small, the solution by the tabular method will differ considerably from the data, and further approximation will be necessary. When one of the flanges, as in the case of cast iron, is comparatively deep, the inaccuracy of the results will be considerable, and one or more further approximations may be required. For different examples, the error will be different in amount, and we have no simple means of judging how great this error will be in any particular case.

In the equations 11 and 12 given above, the results for **M** and I are close when h_1 *and* h_3 are small; the result for y_a will be close although h_3 is not small, and that for y_b will be close although h_1 is not small; thus, suppose that h_3 is small, we have

$$y_b = \frac{h}{2} - \frac{(h_1 + h_2)A_3 - (h_2 + h_3)A_1 - (h_3 - h_1)A_2}{2A}$$

$$= \frac{h}{2} - \frac{(h_1 + h_2)A_3 - h_2 A_1 + h_1 A_2}{2A}$$

putting $h' - \frac{1}{2}h_1 = h_2$, and $h' = h$,

$$= \frac{h'}{2} - \frac{h_1(A_2 + A_3) + (h' - \frac{1}{2}h_1)(A_3 - A_1)}{2A}$$

$$= \frac{h'}{2}\left(\frac{A - A_3 + A_1}{A}\right) - h_1 \frac{A_1 + 2A_2 + A_3}{4A}$$

$$= h' \frac{2A_1 + A_2}{2A} - \frac{1}{4}h_1 \frac{A_1 + 2A_2 + A_3}{A_1 + A_2 + A_3}$$

$$= h' \frac{2A_1 + A_2}{2A}, \text{ nearly}; \tag{13}$$

a result similar to that found previously in equation (6*b*), but y_a cannot now be found by interchanging A_1 and A_3.

Common forms for cast-iron beams are shown in fig. 236, the corresponding equations for these T-shaped sections are derived from the above by putting $A_1 = 0$, or $A_3 = 0$, according as f_a or f_b is the greater. Thus for a section of this form, when f_a is greater than f_b, the flange will be required on the extended side; when f_b is greater than f_a, the flange will be required on the compressed side; and we have

$$A_3 = \frac{f_a - f_b}{2f_b} A_2; \tag{14}$$

$$\text{or } A_1 = \frac{f_b - f_a}{2f_b} A_2; \tag{15}$$

as the case may be.

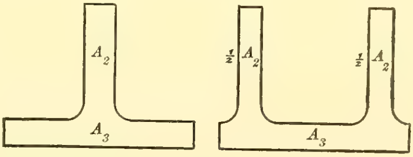


Fig. 236.

Similarly for resistance to bending, we have

$$\mathsf{M} = h'(2f_a - f_b)\frac{A_2}{6}; \tag{16a}$$

$$\text{or} \quad = h'(2f_b - f_a)\frac{A_2}{6}; \tag{16b}$$

as the case may require.

In the case of trough-shaped beams the same formulæ are applicable, if we consider the web A_2 to consist of the two vertical ribs.

Approximate solution—Second method.

PROBLEM.—Two heavy particles A_1 and A_3 (fig. 237) are placed at a known distance h' apart; and another particle A_2, of known mass, is placed midway between them; to find the mass of A_1 and of A_3, so that the centre of inertia of the three particles may divide the distance h' in a given ratio ρ, and that the sum of the moments of inertia about the common centre of inertia may be of a given amount I_0.

Let $\frac{x}{y} = \rho$, and let k be the distance of the centre of inertia of the three particles from A_2, then

$$x = \frac{\rho h'}{\rho + 1};\ y = \frac{h'}{\rho + 1}; \quad \text{and} \quad I_0 = A_1 x^2 + A_2 k^2 + A_3 y^2.$$

Now $$k = x - \frac{h'}{2} = \frac{\rho h'}{\rho + 1} - \frac{h'}{2} = \frac{h'}{2}\frac{\rho - 1}{\rho + 1};\ \therefore \frac{k}{h'} = \tfrac{1}{2}\frac{\rho - 1}{\rho + 1}; \quad (17)$$

if $\frac{1}{2}(A_3 - A_1)$ be taken from A_3 and added to A_1, the common centre of inertia G would coincide with A_2, since the upper and lower particles would then he of equal mass; and we obtain

$$k(A_1 + A_2 + A_3) = h' . \tfrac{1}{2}(A_3 - A_1),$$

or

$$\frac{k}{h'} = \tfrac{1}{2}\frac{A_3 - A_1}{A_1 + A_2 + A_3}. \quad (18)$$

From equations 17 and 18,

$$\frac{A_3 - A_1}{A_1 + A_2 + A_3} = \frac{\rho - 1}{\rho + 1}; \quad \therefore\ A_2 = 2\,\frac{A_3 - \rho A_1}{\rho - 1}; \quad (19)$$

and

$$\rho = \frac{x}{y} = \frac{2A_3 + A_2}{2A_1 + A_2}. \quad (20)$$

$$I_0 = A_1\left(\frac{\rho h'}{\rho + 1}\right)^2 + A_3\left(\frac{h'}{\rho + 1}\right)^2 + 2\,\frac{A_3 - \rho A_1}{\rho - 1}\left(\frac{h'}{2}\,\frac{\rho - 1}{\rho + 1}\right)^2$$

$$= \left(\frac{h'}{\rho + 1}\right)^2\{A_1\rho^2 + A_3 + \tfrac{1}{2}(A_3 - \rho A_1)(\rho - 1)\}$$

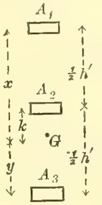
$$= \tfrac{1}{2}\frac{h'^2}{\rho + 1}(\rho A_1 + A_3); \quad (21)$$

and from equations 20 and 21 we can find A_1 and A_3.

Equation 21 may be written more symmetrically thus:

$$\begin{aligned} I_0 &= \frac{h'}{2}\left\{A_1\frac{\rho h'}{\rho+1} + A_3\frac{h'}{\rho+1}\right\} \\ &= \frac{h'}{2}\{A_1 x + A_3 y\} \\ &= h'\{A_1 x + \tfrac{1}{4}(x-y)A_2\} \end{aligned}$$

(22)



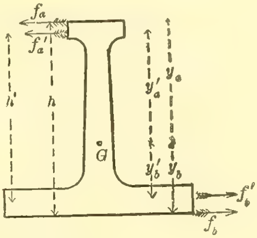


Fig. 237.

by substituting from equation 20,

$$A_3 = \tfrac{1}{2}\left\{\frac{x}{y}(2A_1 + A_2) - A_2\right\}; \text{ hence}$$

$$A_1 = \frac{I_0}{xh'} - \tfrac{1}{4}\left(1 - \frac{y}{x}\right)A_2. \tag{23}$$

Similarly

$$A_3 = \frac{I_0}{yh'} - \tfrac{1}{4}\left(1 - \frac{x}{y}\right)A_2. \qquad Q.E.F. \tag{24}$$

Suppose now that the particles A_1 and A_3 are replaced by flanges of area A_1 and A_3, whose centres are at these points, and the particle A_2 by the web of area A_2, with its centre sensibly midway between the centres of the two flanges; equation 20 will still hold absolutely; equation 21 will be a close approximation to the moment of inertia of the cross-section, the error being that it leaves out of account the moment of inertia of each part about its own neutral axis, a quantity which, for such cross-sections as we are treating, is not large.

If we substitute y'_a for x, and y'_b for y in equations 20 and 22,

$$y'_a : y'_b : h' \;::\; 2A_3 + A_2 : 2A_1 + A_2 : 2A;$$

$$I_0 = h'\{A_1 y'_a + \tfrac{1}{4}(y'_a - y'_b)A_2\} \text{ approx.},$$

and

$$\mathsf{M} = \frac{f'_a I_0}{y'_a} = h'\{f'_a A_1 + \tfrac{1}{4}(f'_a - f'_b)A_2\}; \tag{25}$$

$$= \frac{f'_b I_0}{y'_b} = h'\{f'_b A_3 - \tfrac{1}{4}(f'_a - f'_b)A_2\}. \tag{26}$$

Now

$$f'_a = f_a\frac{y'_a}{y_a} = f_a\frac{y_a - \frac{1}{2}t_a}{y_a} = f_a\left(1 - \frac{t_a}{2y_a}\right), \tag{27}$$

and

$$f'_b = f_b\left(1 - \frac{t_b}{2y_b}\right), \tag{28}$$

where t_a and t_b denote the thickness of the flanges A_1 and A_3

respectively; so that having fixed the thickness of the flanges, we obtain from equations (25) and (26) the area required for A_1 and A_3. Having fixed upon h' from considerations of stiffness, and on A_2 so as to give sufficient resistance to shearing or for other considerations, then equations (25) and (26) will give the areas of the flanges A_1 and A_3, so that the neutral axis shall divide h' in the ratio $f'_a : f'_b$, and, approximately, give M the required moment of resistance to bending; provided that the flanges have their centres at the distance h' apart, and that the web is so disposed that its centre is (sensibly) midway between the centres of the flanges.

When the two strengths of the material, f_a and f_b, are nearly equal, so also are the flanges, and the centre of the web will be sensibly midway. When f_a is much greater than f_b, as is the case with cast iron, then A_3 is much greater than A_1; if the web be made of uniform breadth, its centre will be above the point which is midway between the centres of the flanges; but for practical considerations, it is usual to make the web increase in breadth as we pass from the upper to the lower flange; in which case the centre of the web will be lowered, and thus brought so that its centre is sensibly midway between the centres of the two flanges.

Examples.

244. Find the thickness of metal required for an aqueduct bridge 40 feet span, the waterway being 2 feet square, and the material wrought iron for which $f = 4$ tons per square inch.

The weight of water supported is $4\frac{1}{2}$ tons, and the weight of metal (assumed) $1\frac{1}{2}$ tons; making a further allowance of 2 tons, we have $W = 8$ tons distributed, which gives $M_0 = 480$ inch-tons. Let $B = 24''$, $H = 24''$, then

$$t = \frac{M}{fBH} = \frac{480}{4 \times 24^2} = \frac{1}{5} \text{ in. nearly.}$$

Allowing for rivets, the plates might be taken $\frac{1}{4}''$ thick; but as the plates are liable to rust, the thickness would require to be increased still further, and t might be taken as $\frac{3}{8}''$ in an actual case.

245. Find the moment of resistance to bending of a cast-iron pipe 18″ external diameter, metal 1″ thick, and $f = 2$ tons per square inch, by means of the exact formula; and compare the result with that obtained by the approximate formulæ (*a*) and (*b*).

M = 430 inch tons by exact formula = 390 and 429 by approximate formulæ.

246. Find the thickness of metal required for a cast-iron pipe 24″ external diameter, so that its moment of resistance to bending may be 50 foot-tons, and its resistance to shearing 10 tons; taking $f = 2$ tons per square inch

$$600 = \cdot 6 \times 2 \times 24^2 \times t; \quad \therefore\ t = 0''\cdot 87 \text{ nearly.}$$

On checking the calculation by the accurate formula, and taking $t = 0''\cdot 87$, M = 705 inch-tons; so that $t = 0''\cdot 87$ is more than sufficient; and for shearing this thickness will be several times too great. On account of the difficulty of casting so as to have the thickness quite uniform, an allowance has to be made, and t would probably be taken at about $1''{\cdot}25$.

247. Find, by the approximate formula, the moment of resistance to bending of the cross-section shown in fig. 183, and compare the result with that given in the text at that figure,

$$\mathsf{M} = f\tfrac{1}{3}(2\,.\,900 + 3\,.\,3\,.\,6\,.\,24) = 1032f\,(2 \times 7794f \div 15).$$

248. Design the central cross-section for a wrought iron beam, for which $f_a = 4$, and $f_b = 5$ tons per square inch, suitable for example 171, figs. 99, 100, 153.

Since the span is 42 feet, we may take $h' = 40$ inches; $M = 405{\cdot}75$ ft.-tons $= 4863$ inch-tons; the web may be taken as $\frac{3}{8}''$ thick.

From equation 12 we have

$$4863 = 40\,\{4A_1 + (8-5)\tfrac{15}{6}\}\,; \quad \therefore \quad A_1 = 28{\cdot}5 \text{ sq. in.}$$

$$4863 = 40\,\{5A_3 + (10-4)\tfrac{15}{6}\}\,; \quad \therefore \quad A_3 = 21{\cdot}3 \quad ,,$$

Adopting as a first approximation—breadth of flanges, 21 inches; thickness of top flange, 1·36 in., of bottom do., 1 in.; thickness of web, $\frac{3}{8}$ in., and depth of girder (outside to outside), 41·2 in.; we get by applying the exact method shown at p. 289,

$$\mathsf{M} = \frac{4 \times 21340}{18{\cdot}35} = 4652 \text{ inch-tons,}$$

a result differing from what is required by only 4 per cent.

The upper flange may therefore be taken $1\frac{3}{8}$ in. thick, the lower flange 1 in. thick, and the breadth of each 22 in. This does not take into account the angle irons, and the loss by rivet-holes.

Solve this example by the second approximate formulæ.

In order to fix on t_a and t_b, we will assume f'_a and f'_b equal to f_a and f_b respectively; and then find new breadths to satisfy equations 25 and 26

$$4863 = 40\,\{4A_1 - \tfrac{1}{4}\,(5-4)\,15\}\,;$$

$A_1 = 31$, and similarly $A_3 = 24$ square inches for a first approximation; we may fix on $t_a = 1{\cdot}25$ and $t_b = 1$ inch, and from these obtain suitable breadths,

$$y'_a = \tfrac{4}{9}40 = 17{\cdot}78 \text{ in., and } y'_b = 22{\cdot}22 \text{ in.,} \quad y_a = 18{\cdot}4 \text{ in., and } y_b = 22{\cdot}72 \text{ in.;}$$

$$\therefore \; f'_a = 4\left(1 - \frac{1{\cdot}25}{36{\cdot}8}\right) = 3{\cdot}86, \quad \text{and} \quad f'_b = 4{\cdot}89 \text{ tons.}$$

We have therefore

$$4863 = 40\,\{3{\cdot}86A_1 - \tfrac{1}{4}\,(4{\cdot}89 - 3{\cdot}86)\,15\}\,;$$

$$A_1 = 30{\cdot}5 \text{ square inches,} \quad \text{and} \quad A_3 = 25{\cdot}6 \text{ square inches.}$$

Adopting the following dimensions, upper flange 24·4 in. × 1·25 in., lower flange 25·6 in. × 1 in., and $h = 41{\cdot}12$ in., and solving by the exact method, we obtain

$$\mathsf{M} = \frac{4 \times 24250}{19{\cdot}2} = 5040 \text{ inch-tons,}$$

a quantity differing from the required result by less than 4 per cent.

249. Find suitable dimensions for the top and bottom flanges of a cast iron beam, having given $M = 1200$ inch-tons, $h = 20$ inches, $f_a = 10$ and $f_b = 2$ tons per square inch, and web 1 in. thick.

Let $h' = 18$ inches; then equations 12 and 10,

$$1200 = 18(10A_1 + 54)\,; \quad \therefore \quad A_1 = 1\cdot3 \text{ square inches nearly}\,;$$

$$A_3 = 5A_1 + 36\,; \qquad \therefore \quad A_3 = 42\cdot5 \quad \text{,,} \qquad \text{,,}$$

Taking the upper flange, 1·3 in. × 1 in., and the lower flange 14·2 in. × 3 in., and solving by the exact method, we get

$$y_b = 4\cdot45, \quad I_0 = 1720, \quad M = \frac{2 \times 1720}{4\cdot45}\ 770 \text{ inch-tons}\,;$$

a result very different from that which is required; we will therefore solve this problem by the second approximate method.

Assume $t_a = 1$ in., and $t_b = 3$ in. thick.

In this example $y'_a = 15$, and $y'_b = 3$ in.; so that

$$f'_a = 9\cdot68, \qquad f'_b = 1\cdot\dot{3} \text{ tons}\,;$$

$$1200 = 18\{9\cdot68A_1 + \tfrac{1}{4}(9\cdot68 - 1\cdot33)\,16\}\,;$$

$$A_1 = 3\cdot4, \quad A_3 = 75 \text{ square inches.}$$

Adopting the following dimensions, $A_1 = 3\cdot4$ in. × 1 in., $A_2 = 16$ in. × 1 in., $A_3 = 25$ in. × 3 in.; and solving by the exact formulæ, we obtain

$$M = \frac{2 \times 2460}{3\cdot75} = 1312 \text{ inch-tons,}$$

a result 9 per cent. greater than required.

250. Find suitable dimensions for the top and bottom flanges of a cast-iron beam, having given $M = 800$ inch-tons, $h = 20$ inches, $f_a = 8$, and $f_b = 2$ tons per square inch, and web 1 inch thick.

Let the upper flange be $\frac{1}{2}''$, and the lower flange $2''$ thick, then $h' = 18''\cdot75$, and we have

$$800 = 18\cdot75\{8A_1 + (16 - 2)3\cdot12\}\,; \quad \therefore \quad A_1 = -\tfrac{1}{8},$$

but as negative values are inadmissible, we will take $A_1 = 0$,

$$A_3 = \tfrac{6}{4}\,18\cdot75 = 28\cdot12 \text{ square inches.}$$

We may therefore take the lower flange 14·1 in. broad and 2 in. deep, and the web 18 in. × 1 in.; solving by the exact equation

$$M = \frac{2 \times 1595}{5} = 638 \text{ inch-tons,}$$

a result, as in example 249, very different from that required. Such examples show us that unless the flanges are *thin*, the results are not quite satisfactory.

251. Solve example 250 by the second approximate method, and assuming $t_a = \frac{1}{2}$ in., and $t_b = 2$ in. thick.

In this example $y'_a = 15$ in., and $y'_b = 3\cdot75$ in.; so that $f'_a = 7\cdot87$, and $f'_b = 1\cdot58$ tons,

$$800 = 18\cdot75\{7\cdot87A_1 + \tfrac{1}{4}(7\cdot87 - 1\cdot58)17\cdot5\}\,;$$

$$A_1 = 1\cdot9, \quad \text{and} \quad A_3 = 44\cdot4 \text{ square inches.}$$

Adopting the following dimensions, upper flange 3·8 in. × $\frac{1}{2}$ in., lower flange 22·2 in. × 2 in., and web 17·5 in. × 1 in.; and solving by exact method, we obtain

$$M = \frac{2 \times 2120}{4\cdot75} = 890 \text{ inch-tons,}$$

a result again greater than the one required by about 11 per cent., but much nearer than that obtained by the method in example 250.

ALLOWANCE FOR WEIGHT OF BEAM.

After having designed a beam which is sufficient to bear a given external load, it is necessary to make an allowance for the weight of the beam itself; especially is this the case for beams of long span, as then the weight of the beam bears a considerable proportion to the amount of the external load.

This allowance is readily made by increasing the breadth of the *provisional* beam sufficient for the external load alone; since the breadth is a dimension which appears in the first power in the expression for the resistance to bending.

Consider the weight of the beam, and the external load reduced to its equivalent dead load, as uniformly distributed, a supposition sufficiently exact for our present purpose; let b' denote the breadth, and B' the weight of the provisional beam, computed for W' the external load alone; let b, B, and W denote the same quantities for the *actual* beam sufficient to bear the external load and its own weight; then

$$\frac{b}{b'} = \frac{B}{B'} = \frac{W}{W'}; \qquad 1 - \frac{B'}{W'} = 1 - \frac{B}{W};$$

$$\frac{W}{W'} = \frac{W - B}{W' - B'} = \frac{W'}{W' - B'}.$$

Therefore the breadth of beam required,

$$b = \frac{b' W'}{W' - B'}. \qquad (29)$$

The weight of beam required,

$$B = \frac{B' W'}{W' - B'} \qquad (30)$$

The gross load

$$W = \frac{W'^2}{W' - B'}. \qquad (31)$$

RESISTANCE TO TWISTING AND WRENCHING.

One end of a cylindrical bar is rigidly fixed, and to the other end a couple is applied in a plane at right angles to the axis of the bar; or what is the same thing, as shown in fig. 238, a pair of equal and opposite couples are applied to the ends of the bar; the tendency of these couples is to make the bar rotate about its axis; and if we suppose the bar to consist of fibres originally straight and parallel to the axis, each of these fibres will now have assumed a spiral form. The moment of each couple is called the twisting moment or moment of torsion applied to the bar, and it is constant for each cross-section; on account of the bar being uniform, the stress will be similarly distributed on each cross-section; and since the bar is circular in section the stress at all points equidistant from the axis will be the same.

Suppose two cross-sections to be taken at the distance dx apart; the twisting moment causes the one section to move relatively to the other through an angle di; and if we consider two points originally opposite to each other, that is in the same fibre, one in each section and at a distance r from the axis; then these points, relatively to each other, move laterally through a distance $r \,.\, di$; and since the two sections are dx apart, the rate of twist is

$$r\frac{di}{dx}, \qquad (32)$$

a quantity directly proportional to the distance of the points under consideration from the axis.

We have thus at any point in a cross section, a shearing stress at right angles to the radius drawn to the point, and proportional to that radius in intensity; this may be expressed thus

$$q = Cr\frac{di}{dx}; \qquad (33)$$

where C is the coefficient of transverse elasticity for the material of the cylinder under consideration; Rankine gives

For cast iron, C = 3,000,000 lbs. per square inch (approx.)

For wrought iron, C = 9,000,000 ,, ,, ,, The greatest value of q occurs at the surface of the cylinder; and if f represent the resistance of the material to shearing, and r_1 the radius of the cylinder, then we have

$$q = f\frac{r}{r_1}. \tag{34}$$

If we consider s a small portion of a ring of the cross-section, with its middle point at a distance r from the axis, then $r \,.\, di$ will be its mean length, and we may denote its breadth by dr; its area then is $r \,.\, di \,.\, dr$; the intensity of the shearing stress at s is q; the amount of shearing stress on the small area is therefore

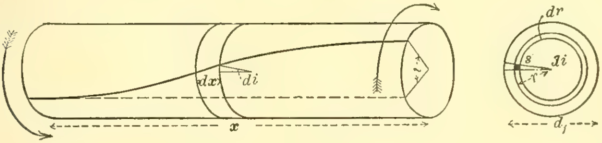


Fig. 238.

$$q \,.\, r \,.\, di \,.\, dr = \frac{f}{r_1} r^2 \,.\, di \,.\, dr;$$

and its moment round the axis, found by multiplying this quantity by r, is therefore

$$\frac{f}{r_1} r^2 (r \,.\, di \,.\, dr).$$

The quantity within brackets is the small area, and r is its distance from the centre; $r^2 (r \,.\, di \,.\, dr)$ is therefore the moment of inertia of the small area about the centre; summing for every such small area, we have the moment of resistance to torsion for the cylinder

$$\mathsf{M} = \frac{f}{r_1} K = \frac{f}{r_1} 2I_0; \tag{35}$$

where K is the moment of inertia of the surface about the centre, and I_0 is the moment of inertia of the same surface about a diameter (see p. 298). We therefore have

$$\mathsf{M} = \frac{f}{r_1}\frac{\pi}{2}r_1^4 = \frac{\pi}{2}fr_1^3 = \frac{\pi}{16}fd_1^3 = \cdot 196 fd_1^3; \qquad (36)$$

where d_1 is the diameter of the cylinder.

For a hollow cylinder, let r_0 and r_1, d_0 and d_1, be the internal and external radii or diameters as the case may be; let I'_0 and I_0 be the moment of inertia about a diameter of a cylinder equal in radius to r_0 and r_1 respectively, then $I_0 - I'_0$ is the moment of inertia about a diameter of the ring under consideration; we have therefore

$$\mathsf{M} = \frac{2f}{r_1}(I_0 - I'_0) = \frac{2f}{r_1}(\tfrac{1}{4}\pi r_1^4 - \tfrac{1}{4}\pi r_0^4)$$

$$= \frac{\pi f}{2}\frac{r_1^4 - r_0^4}{r_1} = \cdot 196 f\frac{d_1^4 - d_0^4}{d_1}. \qquad (37)$$

Comparing these equations with those on page 299, we find that for equal values of the limiting stress f, the resistance of a cylinder, solid or hollow, to wrenching is double its resistance to breaking across.

The working values of the limiting stress f, suitable for shafts, as given by Rankine, are

cast iron, $f = 5000$ lbs. per square inch;

wrought iron, $f = 9000$ lbs. per square inch.

For a cross-section which is not circular, the above formulæ are inapplicable, since the ratio $\frac{q}{r}$ is no longer constant. For a square shaft M. de St. Venant gives as the moment of resistance to torsion

$$\mathsf{M} = 0{\cdot}281 fh^3. \qquad (38)$$

Angle of torsion of an uniform cylindrical shaft.—Let x be the length of the shaft, and i the angle in circular measure through which the one end has turned relatively to the other;

then since the angle of torsion per unit length is constant, we have, from equation (33),

$$\frac{di}{dx} = \frac{i}{x} = \frac{q}{Cr} = \frac{f}{Cr_1}; \quad i = \frac{fx}{Cr_1} = \frac{2fx}{Cd_1}. \tag{39}$$

If f be the working resistance of the material to shearing we have the same angle, whether the shaft be solid or hollow; the values of f and of C for cast and wrought iron have already been stated, and for

$$\text{cast iron,} \qquad i = \frac{1}{300}\,\frac{x}{d_1}; \tag{40a}$$

$$\text{wrought iron,} \quad i = \frac{1}{500}\,\frac{x}{d_1}; \tag{40b}$$

where i is the angle in circular measure through which the one end of a shaft of length x and diameter d_1 has turned relatively to the other end, when the working strain has been produced; the coefficient for cast iron is somewhat uncertain.

When subjected to M any twisting moment not greater than the proof moment, we have for a solid shaft (equation (36))

$$M = \frac{f}{r_1}\,\frac{\pi}{2}\,r_1^4 = \frac{q}{r}\,\frac{\pi}{2}\,r_1^4; \qquad \frac{q}{r} = \frac{2M}{\pi r_1^4};$$

and from equation (39)

$$i = \frac{qx}{Cr} = \frac{2Mx}{\pi r_1^4 C} = 10{\cdot}2\,\frac{Mx}{Cd_1^4}. \tag{41}$$

For a hollow shaft, similarly

$$i = 10{\cdot}2\,\frac{Mx}{C(d_1^4 - d_0^4)}. \tag{42}$$

If we make $x = 1$, or, what is the same thing, if the distance between the two cross-sections which we consider is unity, the stiffness of the shaft will be measured by the reciprocal of i, i being the angle in circular measure through which the two cross-sections have turned relatively to each other, when the skin has has been brought to the proof strain.

For two shafts of the same length and material, but of different diameters, we see from equation (10) that the twisting moment to be applied to each, in order that both may be turned through the same angle of torsion, is proportional to the fourth power of the diameter, the proof stress not being in any case exceeded; thus

$$i = 10{\cdot}2\,\frac{Mx}{Cd^4} = 10{\cdot}2\,\frac{M'x}{Cd'^4}, \quad \text{or} \quad \frac{M}{M'} = \frac{d^4}{d'^4}. \tag{43}$$

Bending and Torsion Combined.

Let the shaft shown in fig. 239 be acted upon by a bending load and a pair of equal twisting couples; and at the point H let M_1 be the moment of the first, and M_2 the moment of the second; then in order to find the amount and direction of the greatest principal stress, we require to combine the greatest direct stress due to bending with the greatest shearing stress due to twisting; this is done by the method of the ellipse of stress.

At the point H, let p be the intensity of thrust (or tension) due to the bending moment M_1, and q the intensity of shearing stress due to the twisting couple M_2; then we have

$$p = \frac{4M_1}{\pi r_1{}^3}; \quad q = \frac{2M_2}{\pi r_1{}^3}. \tag{44}$$

Let p_1 be the greatest intensity of stress (thrust) at the point; then fig. 216 shows the construction required to find its amount; in that figure $\overline{OL} = p$, $\overline{OR'} = q$, and we have

$$p = \overline{OM} + \overline{MR} = \frac{p}{2} + \sqrt{\frac{p^2}{4} + q^2}; \tag{45}$$

the greatest intensity of shearing stress is represented by

$$\overline{MR} = \sqrt{\frac{p^2}{4} + q^2}; \tag{46}$$

and the angle θ made by the greatest stress p_1 with the axis of the shaft is given by the equation

$$\tan 2\theta = \frac{\overline{RL}}{\overline{ML}} = \frac{2q}{p}. \tag{47}$$

By substituting for p and q the values given in equation 44, we have

$$p_1 = \frac{2}{\pi r_1^3} \sqrt{M_1^2 + M_2^2}, \tag{48}$$

and for the greatest intensity of shearing stress

$$\overline{MR} = \frac{2}{\pi r_1^3} \sqrt{M_1^2 + M_2^2}. \tag{49}$$

A very important example of this principle is that of a shaft with a crank attached; in this case we have a force applied to the centre of the crank pin, and resisted by the equal and opposite force at the bearing S. If P represent the force, then the moment of the couple is (fig. 239)

$$M = P \,.\, \overline{SP}; \tag{50}$$

this couple may be resolved into two couples, one a bending couple

$$M_1 = P \,.\, \overline{NS} = M \cos j; \tag{51}$$

the other a twisting couple,

$$M_2 = P \,.\, \overline{NP} = M \sin j. \tag{52}$$

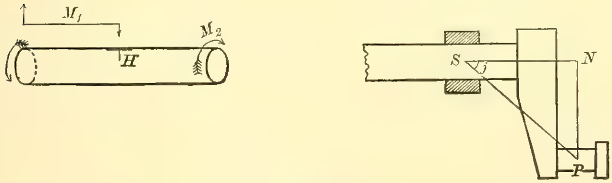


Fig. 239.

The greatest intensity of stress is found by equation 48,

$$\begin{aligned} p_1 &= \frac{2}{\pi r_1^3} \left(M \cos j + \sqrt{M^2 \cos^2 j + M^2 \sin^2 j}\right) \\ &= \frac{2}{\pi r_1^3} (M \cos j + M) = \frac{2}{\pi r_1^3} M (1 + \cos j) \\ &= \frac{5{\cdot}1}{d^3} M (1 + \cos j). \end{aligned} \tag{53}$$

If instead of p_1 we put f the resistance to tension or thrust (the smaller), we get

$$d^3 = \frac{5 \cdot 1}{f} M(1 + \cos j); \tag{54}$$

which enables us to calculate the diameter required for the shaft. If we put f_s for the greatest intensity of the shearing stress, we have

$$d^3 = \frac{5 \cdot 1}{f_s} M; \tag{55}$$

which also enables us to calculate the diameter required; and the greater of the two, one got from equation 54, the other from equation 55, is to be adopted.

The angle made by the principal stress with the axis of the shaft is given by equation 47.

$$\tan 2\theta = \frac{2q}{p} = \frac{M_2}{M_1} = \frac{\sin j}{\cos j} = \tan j; \quad \therefore \theta = \frac{j}{2}. \tag{56}$$

THRUST OR TENSION COMBINED WITH TORSION.

Let the shaft shown in fig. 240 be acted upon by a thrust (or tension) P and a pair of twisting couples of moment M; the stress due to P is uniformly distributed, and that due to M is greatest at the skin; the greatest intensity of stress will therefore be at the skin. If under thrust, the length of the shaft is to be so short compared with its diameter, that the bending action need not be taken into account. At the point H we have a thrust (or tension) $p = \frac{P}{\pi r_1^2}$, and a shearing stress $q = \frac{2M}{\pi r_1^3}$; proceeding as at fig. 216, we have

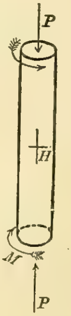


Fig. 240.

$$\overline{OM} = \frac{p}{2} = \frac{P}{2\pi r_1^2}. \tag{57}$$

$$\overline{MR} = \sqrt{\frac{p^2}{4} + q} = \sqrt{\left(\frac{P}{2\pi r_1^2}\right) + \left(\frac{2M}{\pi r_1^3}\right)^2}. \tag{58}$$

The greatest intensity of thrust (or tension) is

$$p_1 = \overline{OM} + \overline{MR}, \tag{59}$$

the angle θ made by p_1 with the axis of the shaft is given by the equation

$$\tan 2\theta = \frac{\overline{RL}}{\overline{ML}} = \frac{2q}{p}. \qquad (60)$$

The greatest intensity of shearing stress is

$$\overline{MR}. \qquad (61)$$

EXAMPLES.

252. A lattice girder 80 feet span bears a load of 100 tons uniformly distributed; depth from centre to centre of flanges 6 feet, and $f = 4$ tons per square inch; the breadth of the flange is 1 ft. 9 in., and is constant, the thickness however varies.

Taking the provisional breadth as 1 ft. 9 in., find how much this has to be increased so as to allow for the weight of the beam itself,

$M_0 = 12{,}000$ inch-tons; $M_{10} = 11{,}250$; $M_{20} = 9000$; $M_{30} = 5250$; $M_{40} = 0$;

since the flange is thin, we have $M = fhb't = 4 \times 72 \times 21 \times t$, and we obtain $t_0 = 2$ in., $t_{10} = 2$ in., $t_{20} = 1{\cdot}5$ in., $t_{30} = 1$ in., and $t_{40} = 0{\cdot}5$ in. say; the average being 1·2 in. nearly. Taking this as the thickness of the flanges, and allowing for bracing, we obtain 10 tons nearly as the weight of the provisional girder; so that $b' = 21$ inches, $B' = 10$ tons, and $W' = 100$ tons; from which we readily find $b = 24$ inches, $B = 11$ tons, and $W = 111$ tons for the actual girder.

253. An aqueduct bridge is 60 feet span and 20 feet broad; the water is 6 feet deep and is carried on iron plates supported on cross girders 6 feet apart; the cross girders are supported on the lower flanges of two plate girders; the weight of the water, cross girders plates and stiffeners is 240 tons, and may be considered uniformly distributed. The flanges of the main girders are to be 80 inches apart centre to centre, the web may be taken as $\frac{3}{8}$th of an inch thick, and $f = 6$ tons per square inch.

Find a provisional breadth for the flanges, and also how much this has to be increased so as to allow for the weight of the actual girder.

Here we find $M_0 = 900$, $M_{10} = 800$, $M_{20} = 500$, and $M_{30} = 0$ foot-tons; from this we find that an area of 22·5 square inches is required for each flange at the centre of girder, and if b' be taken as 18 inches throughout, we have $t_0 = 1\frac{1}{4}$, $t_{10} = 1\frac{1}{4}$, $t_{20} = \frac{3}{4}$, and $t_{30} = \frac{1}{2}$ inch. The average thickness of flange will be 1 inch nearly, and allowing for stiffeners, &c., we find the weight of the provisional beam to be 8 tons nearly; hence we have $b' = 18$ inches, $B' = 8$ tons, and $W' = 120$ tons; from which we obtain (page 393)

$$b = 20 \text{ inches}, B = 9 \text{ tons, and } W = 129 \text{ tons, for each girder.}$$

254. A water wheel of 20 horse power makes 5 revolutions per minute; find the diameter suitable for the malleable iron shaft which transmits this force.

For each revolution 132,000 ft.-lbs. of work are performed; this is equivalent to 21,008 lbs. acting on a wheel of radius one foot, and we have $M = 21{,}008$ ft.-lbs. $= 252{,}096$ inch-lbs.

$$252{,}096 = {\cdot}196 \times 9000\, d_1^3 = 1764\, d_1^3; \quad \therefore \quad d = 5{\cdot}23 \text{ inches.}$$

If this shaft be 12 feet long, what is its angle of torsion when the working moment as above is applied? Take $f = 9000$, and $C = 9{,}000{,}000$; then

$$i = \frac{1}{500}\frac{144}{5{\cdot}23} = {\cdot}05507 = 3^\circ\, 9'.$$

255. The diameter of one shaft is double that of another of the same material; the smaller gave way when subjected to a twisting moment of 2 ft.-tons. What twisting moment will be required to wrench the other? *Ans.* $M = 16$ ft.-tons.

256. A shaft 12 feet long and 6 inches diameter is subjected to a twisting moment of 16 ft.-tons, and the two ends are thus twisted through a certain angle; a second shaft of the same material, 16 feet long and 9 inches diameter, is twisted so that its angle of torsion is exactly the same as that of the first; find the twisting moment required to do this,

$$i = 10{\cdot}2\,\frac{192 \times 144}{C \times 6^4} = 10{\cdot}2\,\frac{M \times 192}{C \times 9^4};$$

therefore

$$\frac{192 \times 144}{1296} = \frac{M \times 192}{6561}; \quad \text{therefore} \quad M = 729 \text{ in.-tons} = 60{\cdot}75 \text{ ft.-tons.}$$

257. What thickness of metal is required for a cast-iron hollow shaft, 10 inches outer diameter, so as to resist a twisting moment of 10 ft.-tons?

Ans. $M = 120 \times 2240 = {\cdot}196 \times 5000\,\frac{10^4 - d_0^4}{10}.$

$d_0^4 = 7258$; therefore $d_0 = 9{\cdot}23$, and the thickness required is $0{\cdot}4$ inch.

258. A malleable iron shaft 20 feet long and 6 inches diameter is subjected to a moment which twists the ends through an angle of 2°; taking C the coefficient of transverse elasticity as 9,000,000, find f, the stress at the skin,

$$i = {\cdot}0349 = \frac{2f.\,240}{9 \times 10^6 \times 6}; \quad \text{therefore} \quad f = 3926 \text{ lbs. per square inch.}$$

259. The inner and outer diameters of a hollow steel shaft are 10 and 12 inches, and $f = 6$ tons per square inch is the working value of the resistance to shearing. What is the twisting moment this shaft is capable of transmitting?

$$\mathsf{M} = {\cdot}196 \times 6\,\frac{12^4 - 10^4}{12} = 1052 \text{ inch-tons.}$$

260. The working tensile strength of the steel for the previous example is 12 tons per square inch, and the crushing strength is greater. What is the moment of resistance to bending of this same shaft?

Ans. $\mathsf{M} = 1052$ inch-tons; since the resistance to wrenching is double the resistance to breaking across when the two values of f are equal.

261. A shaft 9 inches diameter and 12 feet long is supported at its two ends, and loaded at the two points which divide its length into three equal parts with 4 tons at each point; a twisting moment of 20 foot-tons is applied to one end of the shaft while the other is held fixed. Find the greatest intensity of the thrust, tension, and shearing stress; and the angle that the line of greatest principal stress makes with the axis of the shaft.

At any point between the two loads, the bending moment $M_1 = 16$ foot-tons $= 192$ inch-tons; the twisting moment $M_2 = 240$ inch-tons;

$$p = \frac{4 \times 192}{\pi \times 91{\cdot}1} = 2{\cdot}69 \text{ tons}; \quad q = \frac{2 \times 240}{\pi \times 91{\cdot}1} = 1{\cdot}67 \text{ tons};$$

therefore

$$p_1 = 1{\cdot}35 + \sqrt{\frac{7{\cdot}24}{4} + 2{\cdot}80} = 3{\cdot}5 \text{ tons per square inch,}$$

is the greatest value of the intensity of the thrust at the upper point, and of the tension at the lowest point of the skin near the middle of the length of the shaft; the greatest intensity of the shearing stress is 2·15 tons per square inch, and it is situated at the points just mentioned,

$$\tan 2\theta = \frac{3{\cdot}34}{2{\cdot}69} = 1{\cdot}24; \quad \text{therefore} \quad \theta = 25^\circ.$$

262. The crank shaft of an engine is 5 inches diameter; the distance from the centre of the bearing to the point opposite the centre of the crank pin, NS in fig. 239, is 12 inches; the half stroke, NP in figure, is 16 inches; and the pressure applied to the crank pin is 5000 lbs. Find the greatest intensity of thrust, tension, and shearing stress; and θ the angle made by the line of principal stress with the axis of the shaft,

$$PS = 20 \text{ inches}; \quad \therefore \quad M = 100000 \text{ inch-lbs.}; \; M_1 = 60000, \text{ and } M_2 = 80000.$$

$$p_1 = \frac{5{\cdot}1}{125}\,100000\,(1 + \tfrac{3}{5}) = 6530 \text{ lbs. per square inch,}$$

the greatest intensity of thrust and of tension, at the bearing, the one being at the one side and the other being at the other side of the shaft. The greatest intensity of shearing stress is $\frac{5{\cdot}1}{125}$ 100,000 = 4080 lbs. per square inch. The angle

$$\theta = \frac{j}{2} = 27^\circ.$$

263. A shaft 8 inches diameter is subjected to a thrust of 100 tons uniformly distributed over its two ends, and a twisting moment of 30 foot-tons. Find the greatest intensity of thrust and shearing stress, and the angle made by the line of principal stress with the axis of the shaft,

$$p = \frac{100}{50{\cdot}26} = 1{\cdot}99 \text{ tons}; \quad q = \frac{720}{201{\cdot}04} = 3{\cdot}58 \text{ tons per square inch};$$

$$p_1 = 1{\cdot}0 + \sqrt{{\cdot}99 + 12{\cdot}81} = 4{\cdot}71 \text{ tons per square inch};$$

the greatest intensity of thrust; the greatest intensity of shearing stress is 3·71 tons per square inch; and

$$\tan 2\theta = \frac{7{\cdot}16}{1{\cdot}99} = 3{\cdot}6; \quad \theta = 37^\circ.$$

264. Find the diameter of a malleable iron shaft capable of bearing a tension of 50 tons, and a twisting couple whose moment is 25 foot-tons; the resistance of the material to tension and shearing being 5 and 4 tons per square inch respectively;

$$p = \frac{50}{\pi r_1^2} = \frac{15{\cdot}92}{r_1^2}; \quad q = \frac{600}{\pi r_1^3} = \frac{191}{r_1^3};$$

therefore

$$p_1 = 5 = \frac{7{\cdot}96}{r_1^2} + \sqrt{\frac{63{\cdot}36}{r_1^4} + \frac{36500}{r_1^6}};$$

from which we find $r_1 = 3{\cdot}53$ inches. The greatest intensity of shearing stress $= 4 = \sqrt{\frac{63{\cdot}36}{r_1^4} + \frac{36500}{r_1^6}}$; from which we find $r_1 = 3{\cdot}64$ inches; and since this is greater than the former result, it is to be adopted; that is, the diameter required for the shaft is 7½ inches nearly.

THRUST AND INDUCED BENDING.

When thrust is applied to a pillar or strut whose length is great compared with its diameter, it will collapse not by direct crushing but by bending and breaking across.

Let a long thin vertical bar, originally straight be deflected to an extent not greater than the proof deflection by the application of a horizontal external force applied, say at its middle, while the ends are guided so that they cannot move laterally, and let it be held in that position; it will then have a form such as is shown in fig. 241; let the load P be now applied, then when the restraint is withdrawn, the bar will tend to assume its original vertical form, it will remain neutral, or it will collapse according to the amount of P; that is to say, if the moment of P relatively to the centre of the bar, viz. $P \cdot v$, is less than the moment of resistance of the bar to bending, the bar will tend to right itself.

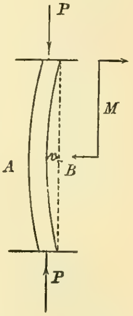


Fig. 241.

The stress on the cross-section AB consists of one part p' due to the load P, and another part p'' due to the bending which takes place in the direction in which the pillar is most flexible; since P is uniformly distributed, we have

$$p' = \frac{P}{S}; \tag{62}$$

where S represents the sectional area of the bar; by equation given on page 269, we have $M = np''bh^2$, and since $M = Pv$

$$p'' \propto \frac{Pv}{bh^2}; \tag{63}$$

where h is the smaller, and b is the larger diameter, when these are unequal; the proof deflection v, page 340, equation (14a) is directly proportional to the square of the length and inversely proportional to the depth; that is to say

$$v \propto \frac{l^2}{h}, \quad \text{and} \quad p'' \propto \frac{Pl^2}{bh^3} \propto \frac{Pl^2}{Sh^2};$$

that is $$p'' \propto p' \frac{l^2}{h^2}; \quad \text{therefore} \quad \frac{p''}{p'} \propto \left(\frac{l}{h}\right)^2; \tag{64}$$

that is, the additional stress due to bending is to the stress due to the direct thrust, as the square of the proportion in which the length of the pillar exceeds the least diameter.

The total intensity of stress $p' + p''$ must not exceed the strength of the material; equating that intensity to f the strength of the material, we have

$$f = p' + p'' = p' + ap' \frac{l^2}{h^2}$$

$$= p' \left\{ 1 + a \left(\frac{l}{h} \right)^2 \right\}$$

$$= \frac{P}{S} \left\{ 1 + a \left(\frac{l}{h} \right)^2 \right\}.$$

$$P = \frac{fS}{1 + a\left(\frac{l}{h}\right)^2}; \qquad (65)$$

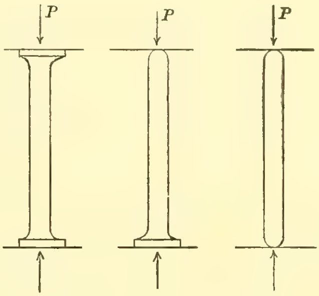


Fig. 242.

where a is a constant coefficient to be determined by experiment. The above investigation is due to Tredgold, and values of the coefficient a, as determined by Professor Gordon, are given below.

For a strut or pillar securely fixed at the ends:—

Wrought-iron solid rectangular section, . . . $a = \frac{1}{3000}$.

*Angle, channel, cruciform, and T-iron (see fig. 243), $a = \frac{1}{900}$.

Cast-iron, hollow, cylinder, ... $a = \frac{1}{400}$.

†For timber struts, oak and pine, ... $a = \frac{1}{250}$.

A pillar rounded at both ends is as flexible as one of double the length fixed at the ends; so that

$$P = \frac{fS}{1 + a\left(\frac{2l}{h}\right)^2} = \frac{fS}{1 + 4a\left(\frac{l}{h}\right)^2}. \qquad (66)$$

The strength of a pillar fixed at one end and rounded at the other is a mean between that of a beam fixed at both ends and one rounded at both ends.

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* As deduced by Unwin.

† As deduced by Weisbach.

The values of f in lbs. per square inch are given by Rankine as follows :—

	Breaking load.	Proof load.	Working load.
Wrought-iron solid rectangular section, . .	36,000	18,000	6,000 to 9,000
Wrought-iron cell, . .	27,000	13,500	4,500 ,, 7,000
Cast-iron cylinders, . .	80,000	26,700	13,300 ,, 20,000
British oak, dry, . .	10,000	—	1,000
American oak, dry, . .	6,000	—	600
Red pine and larch, dry,	5,400	—	550

For green timber the values of f should be halved.

A pillar or strut securely fixed at both ends corresponds with a beam fixed at the ends (fig. 227) ; the points of fracture are the points where in that figure the bending moment is greatest, viz., at the centre and at the ends. A pillar fixed at one end corresponds with the beam shown in fig 225 ; the points of fracture being at the fixed end, and at about one-third of the length from the rounded end. A pillar rounded at both ends corresponds with the beam shown in fig. 110, the point of fracture being at the middle of the length.

The following table gives the results of the above formulæ for pillars of wrought and cast iron, whose diameter and lengths are in different proportions, and whose ends are securely fixed :—

$\frac{l}{h}$	Breaking load, lbs. per sq. in. $= \frac{P}{S}$		$\frac{l}{h}$	Breaking load, lbs. per sq. in. $= \frac{P}{S}$	
	Wrought-iron solid rectangular section.	Cast-iron hollow cylinder.		Wrought-iron solid rectangular section.	Cast-iron hollow cylinder.
10	34,840	64,000	30	27,700	24,620
15	33,490	51,200	35	25,560	19,700
20	31,765	40,000	40	23,480	16,000
26·4	29,230	29,230	50	19,640	11,030

From this table it is seen that, so far as the ultimate strength of such pillars is concerned, cast iron is stronger than, as strong as, or less strong than, wrought iron when the proportion $\frac{l}{h}$ is less than, equal to, or greater than 26·4; this result was first pointed out by Professor Gordon.

For struts in wrought-iron framework, such cross-sections as are shown in fig. 243 may be chosen on account of their stiffness; these are called T, angle, channel, and cruciform iron, respectively; in fixing the proportion $\frac{l}{h}$ for such sections, the value of h is to be taken as the *least* diameter; this is marked in the diagram.

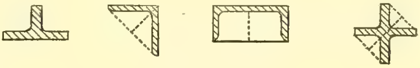


Fig. 243.

Since, however, such cross-sections are not made of very large dimensions, struts above a certain size require to be *built*; in this case they usually consist of four thin plates forming a square and connected at the corners by angle irons; of two thin plates held parallel and at a fixed distance apart, by means of a web and angle irons, or by a lattice work of small diagonal bars; or of two T-irons or channel-irons, with the ribs turned towards each other, and held by a lattice work of small diagonal bars as in the case just stated.

EXAMPLES.

265. A cast-iron column, securely fixed at the ends, external diameter 8 inches, length 20 feet, is to bear a steady load of 30 tons. Find the thickness of metal required.

Here $\frac{l}{h} = 30$, and for that proportion the breaking load is 24,620 lbs. per square inch; taking $\frac{1}{6}$ for a factor of safety, we get the working stress $f = 4100$ lbs. $= 1·8$ tons per square inch; the area of metal required is therefore 17 inches; this gives 6·7 inches for the internal diameter, or $\frac{3}{4}$ inch nearly for the thickness of metal. Since an allowance has to be made for slight irregularities in casting, the thickness of metal should be 1 inch.

266. Find the working load for a cast-iron pillar 12 inches diameter, 40 feet long, metal 1 inch thick, taking 6 as the factor of safety.

Here $\frac{l}{h} = 40$; $S = 34·6$ square inches; $f = 1·2$ tons per square inch; and the steady working load is 42 tons nearly, both ends being securely fixed; but if the load is such as to cause considerable vibration, from one-half to two-thirds of this amount may be taken.

267. What is the crushing load for a malleable iron bar 6 in. × 3 in. × 10 feet long, securely fixed at one end?

$S = 18$ square inches; $\frac{l}{h} = 40$, and the ultimate stress corresponding is 10·5 tons per square inch, when fixed at both ends; when rounded at both ends, this reduces to 5·1 tons; when fixed at one end and rounded at the other, the result is the mean of these quantities, viz. 7·8 tons per square inch.

The crushing load is therefore 140 tons steadily applied.

268. Find the working strength of a strut formed of channel irons ½ inch thick, 6 inches broad, width of each flange (outside) 2½ inches, and length 6 feet, fixed securely at both ends.

Here $S = 5$ square inches; $\frac{l}{h} = 29$ nearly, and the working strength is 1·4 tons per square inch, or 7 tons nearly.

269. What is the working load for a strut of seasoned American oak firmly fixed at the ends, 20 feet long, and 1 foot square?

Ans. $P =$ 33,000 lbs. nearly, or 14½ tons.

270. A strut of red pine whose ends are to be well fixed, is to be 4 inches thick and 6 feet long; the thrust applied to its ends is calculated to be 4 tons. Find the breadth required.

$$S = P\,\frac{1 + a\left(\frac{l}{h}\right)^2}{f} = 4\,\frac{1 + \frac{324}{250}}{\frac{1}{4}} = 37 \text{ square inches nearly,}$$

and the breadth required is therefore 9¼ inches nearly.

Long Steel Struts.

For long struts in wrought iron and steel, the double T-section is of great importance, as most struts, where merely booms of girder or themselves girders, can be *blocked-out* into such a form. The extreme shapes where the flanges are equal, and where one flange is zero, that is, double T-sections equal above and below, and the T-iron sections are especially important. The angle-iron too is often constrained to bend like a T-iron by being braced in other directions.

We will establish a general expression from which to select an approximate one suitable for designing struts. The sizes are shown clearly on fig. 244. As in equation 6,

$$A = t(a + b + qc); \quad m = \frac{2b + qc}{a + b + qc} \cdot \frac{h'}{2}; \quad n = \frac{2a + qc}{a + b + qc} \cdot \frac{h'}{2},$$

and taking a mean of the term in δ^2 repeated twice, once with each of the evident values of δ, for symmetry, we have

$$I_0 = atm^2 + btn^2 + \tfrac{1}{12}\,(at^3 + bt^3 + qtc^3)$$

$$+ \tfrac{1}{2}qct\left(\left(m - \frac{h'}{2}\right)^2 + \left(\frac{h'}{2} - n\right)^2\right)$$

$$= \frac{th'^2}{4}\,\frac{\tfrac{1}{2}(2a + qc)(2b + qc)^2 + \tfrac{1}{2}(2b + qc)(2a + qc)^2}{(a + b + qc)^2}$$

$$- \tfrac{1}{4}qcth'^2 + \tfrac{1}{12}qc^3t + \tfrac{1}{12}(a + b)\,t^3$$

$$= \frac{th'^2}{4}\,\frac{(2a + qc)(2b + qc)}{a + b + qc} - \tfrac{1}{4}qcth'^2$$

$$+ \frac{qct}{12}(h' - t)^2 + \frac{t^3}{12}(a + b)$$

$$= \frac{th'^2}{4}\,\frac{(2a + qc)\,(2b + qc)}{a + b + qc} - \tfrac{1}{6}qcth'^2 - \tfrac{1}{6}qct^2h' + \tfrac{1}{12}t^3\,(a + b + qc).$$

$$I_0 = A\left(mn + \frac{t^2}{12}\right) - \tfrac{1}{6}qcth'(h' + t).$$

$$I_0 = A\left(mn + \frac{t^2}{12}\right) - \frac{t'}{6}\,hh'c. \tag{67a}$$

$$I_0 = Amn - \tfrac{1}{6}t'h'^3 \text{ nearly.} \tag{67b}$$

Here we have (67*b*) a remarkable expression for the minimum moment of inertia of the double T-section, the degree of approximation clearly shown by comparison with the exact expression (67*a*). The approximate expression may be given in words thus :—

The moment of inertia of a double T-section is sensibly equal to *its area multiplied by the product of the two segments into which the centre of gravity divides the depth between the middle points of its flanges,* from which is to be deducted *twice the moment of inertia of the web considered separately, and as reaching from centre to centre of the flanges.*

Cor.—For a given area the value of I_0 and therefore of the *stiffness* is a maximum when $m = n$, that is, with equal flanges.

By a further reduction we have

$$I_0 = \frac{th'^2}{12}\left(\frac{12ab + 4(a+b)\,qc + q^2c^2}{a+b+qc} - 2t'\right) - \frac{t^3}{12}(a + b + qc + 2qh')$$

$$= \frac{th'^2}{12}\cdot\frac{12ab + 4\,(a+b)\,qc + q^2c^2}{a+b+qc}\text{ nearly.} \tag{68a}$$

By an exactly similar reduction

$$I_0 = \frac{th'^2}{12}\left(\frac{12\alpha\beta + 4\,(\alpha+\beta)\,qh + q^2h^2}{\alpha+\beta+qh} + 2t'\right) - \&c.$$

Or, for the double ⊤-section thickness of Metal constant,

$$I_0 = \frac{th'^2}{12}\,\frac{12ab + 4\,(a+b)\,h + h^2}{a+b+h}, \tag{68b}$$

is a sufficient approximate value for designing struts, even with h for h'.

Economical double ⊤-section of uniform strength to resist bending. In fig. 244 suppose the two skins b and a to simultaneously come to their working or proof strengths f_b and f_a, and putting $\rho = f_b : f_a$ their ratio, then ρ will sensibly be equal to the ratio $f'_b : f'_a$, that is, $n : m$.

Just as for *general stiffness* h or h' must be a suitable fraction of the span, so also for *local stiffness* must also t', the thickness of the web, be a suitable fraction r of h or h'.

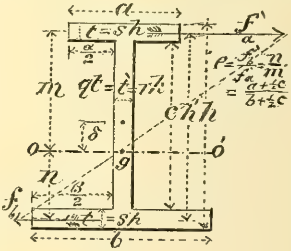


Fig. 244.

The resistance to bending is, by equation (67b),

$$M = \frac{f'_b}{n}\,I_0, \quad\text{or}\quad \frac{M}{f'_b} = Am - \frac{t'h'^3}{6}\,\frac{1}{n}.$$

But $m = \dfrac{h'}{1+\rho}$, and $n = \dfrac{\rho h'}{1+\rho}$, so that

$$\frac{M}{f'_b} = \frac{Ah'}{1+\rho} - \frac{t'h'^2(1+\rho)}{6\rho} = \frac{Ah'}{1+\rho} - \frac{rh'^3(1+\rho)}{6\rho}.$$

Equating zero to the differential coefficient with respect to h',

$$0 = \frac{d}{dh'}\,\frac{M}{f'_b} = \frac{A}{1+\rho} - \frac{rh'^2\,(1+\rho)}{2\rho}.$$

Or the area of the web measured from centre to centre of the flanges, namely,

$$t'h' = \frac{2\rho}{(1+\rho)^2} A \qquad (69)$$

makes the strength a maximum.

$$\frac{M}{f'_b} = \frac{2Ah'}{3(1+\rho)} = \frac{(1+\rho) r h'^3}{3\rho} = \frac{(1+\rho) t' h'^2}{3\rho}. \qquad (70)$$

For homogeneous strength $\rho = 1$, and the section has then equal flanges. For wrought iron $\rho = 4 : 5$ commonly, and is never less than $1 : 2$ for steel or steely-iron. For values of ρ from $\frac{1}{2}$ to 1 the coefficient $2\rho \div (1+\rho)^2$ is $\frac{1}{2}$ sensibly constant. Hence the condition given by (69) that *the double* T*-section of uniform strength shall be the most economical for a given degree of local stiffness is that the area of the web measured from centre to centre of the flanges shall be half of the total area.*

Substituting $\rho = 1$ in (69) and (70) we get for the most economical section of equal flanges for a given degree of local stiffness the two conditions

$$t'h' = \tfrac{1}{2}A \quad \text{and} \quad \frac{M}{f'_b} = \frac{Ah'}{3} = \tfrac{2}{3}t'h'^2. \qquad (71a)$$

The second may be written $M = 4\left(\frac{1}{6}f'_b t' h'^2\right)$ where the expression in brackets is the share of the resistance to bending exerted by the web measured as before from centre to centre of the flanges. Hence the two conditions (71) for designing the economical cross-section with equal flanges for a given degree of local stiffness can readily be remembered thus—*Design the section so that the web measuring from centre to centre of the flanges shall be half the area and take up a fourth of the resistance to bending.*

For the purpose of *direct* design, however, (71*a*) is to be modified by putting

$$f'_b = \frac{h'}{h} f_b\,; \; h' = (h - t)\,; \; t = sh, \quad \text{and} \quad t' = rh,$$

when we have, where f_b is the lesser strength,

$$\tfrac{2}{3} r h'^3 = \tfrac{2}{3} r (1 - s)^3 h^3 = \frac{M}{f_b}. \qquad (71b)$$

For comparison put M_1 in (70), and M_2 in (71a), when

$$\frac{M_1 - M_2}{M_1} = \frac{1-\rho}{2} \quad \text{or} \quad \frac{M_1 - M_2}{M_2} = \frac{1-\rho}{1+\rho} \tag{72}$$

is the fractional excess of strength of the one section over the other where both have the same *area* A, the same depth h', or same *general stiffness*, and the same value of $r = t' \div h'$, or same *local stiffness.* In each section the "web" is half the total area and each is the most economical of its kind. For the ordinary value $\rho = \frac{4}{5}$, the expression (72) is 10 per cent.

To design the economical double T-section of uniform strength for a given degree of local stiffness from the required strength, deduct the percentage indicated by (72), and design the economical section with flanges equal, of the required degree of local stiffness by (71b); then borrow from the flange a a portion, and add it to the flange b, till $(a + \frac{1}{2}h) : (b + \frac{1}{2}h)$ is in the ratio $\rho : 1$.

Care must be taken that the weaker strength f_b in (71b) is used always for the compressive strength of wrought iron and steel. Then too, it must be the weaker or thrust flange that is increased at the expense of the other.

Also the beam is to be placed with the proper flange up, and there must be no possibility of the bending reversing. For beams a saving of 10 per cent. may be effected. Such sections are not suited for struts for which the flanges are best to be equal, the bending being equally like to be one way or the other.

The T-section and angle-iron constrained to bend like a T-section are often used as struts, as they suit constructional purposes.

The angle-iron if *not* braced so as to bend like a T-iron is very weak as a strut, as it tends to open flat and bend over.

These points are best illustrated by the numerical examples to follow.

Double T*-section with equal flanges,* fig. 245.—In equation (67b)

$$I_0 = Amn - \tfrac{1}{6}t'h'^3;$$

we must now put

$$a = b, \quad m = n = \tfrac{1}{2}h', \quad t' = qt,$$

and we have

$$A = t(qh' + 2b), \tag{73a}$$

$$I_0 = \frac{th'^2}{12}(qh' + 6b), \tag{73b}$$

and the radius sq. of gyration

$$i^2 = \frac{I_0}{A} = \frac{h'^2}{12}\,\frac{qh' + 6b}{qh' + 2b}, \tag{73c}$$

where q is the ratio of the thickness of the web to the common thickness of the flanges.

T-*section and angle-iron constrained to bend like it* (fig. 245, IV.).—In equation (68a) put $a = 0$, and now $c = h'$.

$$A = t\,(qc + b). \tag{74a}$$

$$I_0 = \frac{th'^2}{12}\,\frac{q^2c^2 + 4bqc}{qc + b} = \frac{t'h'^2c}{12}\,\frac{qc + 4b}{qc + b} = \frac{t'c^3}{12}\,\frac{qc + 4b}{qc + b}. \tag{74b}$$

$$i^2 = \frac{I_0}{A} = \frac{c^3}{12}\,\frac{qc + 4b}{qc + b}. \tag{74c}$$

Angle-iron unsupported or a crux-section (fig. 245, V.).—Let $t' = t$, for angle-irons are always of one thickness being rolled from a plate of one thickness. The section is readily identified with a rectangle of breadth B and depth H, where

$$H = h\sin\theta = h \div \sqrt{h^2 + b^2}$$

and $$B = t\,(\sec\theta + \operatorname{cosec}\theta) = t\sqrt{h^2 + b^2}\left(\frac{1}{b} + \frac{1}{h}\right),$$

so that putting $I_0 = \frac{1}{12}BH^3$ and reducing, we get the values, of little practical use, given in line *V.* of the accompanying table.

In the Table the struts are constrained to bend only about the dot-and-dash lines.

Line *I.* is box or double channel section.

Line *II.* is double T-section of uniform metal.

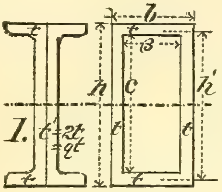
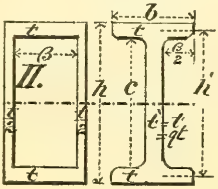
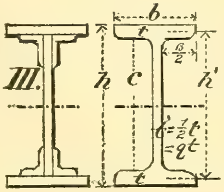
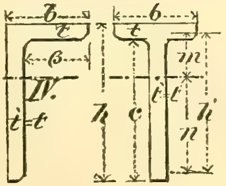
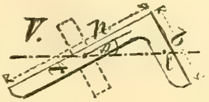
Line *III.* has the web half as thick as the flanges, they are got by putting 2, 1 and $\frac{1}{2}$ for q in equation 73.

Line *IV.* is the uniform metal T or angle-iron constrained to bend like a T-iron. Unity is put for q in equation 74. The approximation is very close, as the terms in t^2 and t^3 vanish together, in the exact expression expanded in powers of t. It is of great practical use, and is not given in the Table, Rankine's *Civil Engineering*, p. 523, where there only appears the value in our,

Line *V.* for *unconstrained* angle iron a section quite unsuited for struts.

TABLE

OF AREAS, MOMENTS OF INERTIA, AND RADII SQUARE OF GYRATION.

SECTIONS.	A	I_0	i^2
	$t(2h'+2b)$;	$\frac{th'^2}{12}(2h'+6b)$,	$\frac{h^2}{12}\cdot\frac{h+3b}{h+b}$.
	$t(h'+2b)$;	$\frac{th'^2}{12}(h'+6b)$;	$\frac{h^2}{12}\cdot\frac{h+6b}{h+2b}$.
	$t(\frac{1}{2}h'+2b)$;	$\frac{th'^2}{12}(\frac{1}{2}h'+6b)$;	$\frac{h^2}{12}\cdot\frac{h+12b}{h+4b}$.
	$t(h'+b)$;	$\frac{th'^3}{12}\cdot\frac{h'+4b}{h+b}$;	$\frac{h^3}{12}\cdot\frac{h+4b}{(h+b)^2}$.
 FIG. 245.	$t(h+b)$;	$\frac{th^2b^2}{12}\cdot\frac{h+b}{h^2+b^2}$;	$\frac{h^2}{12}\cdot\frac{b^2}{h^2+b^2}$.

RANKINE-GORDON FORMULA.

Rankine first proved that Gordon's formula, equation (66), was rational (*see* Rankine, *C. E.*, p. 523). He then substituted i^2 for $\frac{h^2}{12}$, because that is the relation of the radius square of gyration to the depth for the solid rectangular sections upon which the actual experiments were made. He then says the formula is general, only for skeleton sections the proper value of i^2 must be used. To be of practical use the values of i^2 like those in the above table must not contain t, the thickness of the metal otherwise there would result a quadratic equation to determine its value, since t is generally the unknown quantity in this formula.

We prefer to present the formula suitable to a strut with hinged ends, because, as Fidler justly says, in wrought-iron and steel struts, the ends, at best, can only be imperfectly fixed in direction as the other portions of the steel, in a bridge for instance, to which such struts are fixed, are themselves so elastic, that, in yielding to the loads imposed on them, the end of the struts do deflect. Fidler allows for this imperfect fixing by supposing the virtual length of such a strut to be $\frac{6}{10}$ths of its actual length. But every strut imperfectly fixed at the ends can be judged on its merits, and a deduction made from the length never more than $\frac{4}{10}$ths. As the length in the Rankine formula is in inches, it is always wise in making the above deduction to so do it as to leave the virtual length expressed in a *round number of inches.* The reason that l is in inches is because h and i are always in inches.

The Rankine formula then for ends hinged is

$$\text{Iron,}\quad P \text{ tons} = \frac{4 \text{ tons} \times A}{1 + \frac{1}{9000}\frac{l^2}{i^2}}. \tag{75a}$$

$$\text{Mild Steel,}\quad P \text{ tons} = \frac{6\frac{1}{2} \text{ tons} \times A}{1 + \frac{1}{7000}\frac{l^2}{i^2}}. \tag{75b}$$

These, of course, are only good average values.

For a practical knowledge of this formula for designing long struts Fidler's Treatise on Bridge Design should be studied.

The formula for many years was thought to be too general, and, in America, most eminent engineers made formulæ of their own, of like form but varied to suit the different cross-sections, in a way that led to great confusion. Fidler has, in the most skilful way, arranged the Rankine formula, and tabulated results which agree, with great exactness, with the results of the more recent exhaustive and elaborate experiments made on struts with the skeleton sections. His tabulated results agree also with the results of the special formulæ for special sections.

In Britain again many writers put forward a modification of Euler's formula. In our opinion Euler's formula is quite unsuited for struts in engineering steel-work. Euler contemplates a load which will *just not bend* the strut. Now the whole duty of a steel strut in a structure is to resist bending, and bend it will no matter how it is designed, for it will take a *set* to one side or other with its own weight if with nothing else, for the weight of the immense struts in modern structures is very considerable. Also heat will distort a strut; and in riveted work the straining of the adjacent members sends bending strains along a strut, which must be designed with an ample radius of gyration, and it is undesirable to talk about a load which will *just not bend* it.

Fidler has done great service in vindicating the supreme position of the Rankine-Gordon formula as a master formula for *tentative design*, though, of course, experiments on the actual strut of any special section will supersede all formulæ provided the experiments are on a proper scale, and in this matter no perfunctory experiment in special struts *can* compare in importance with the life-history of such a strut in some large structure.

We quote a part of two of the tables given in Fidler's *Practical Treatise on Bridge Construction.* These tables give the coefficient for stiffening long struts, that is, a multiplier for converting the net area of the strut, where quite short, into the gross area to resist both thrust and induced bending. In a good design, this multiplier should not exceed 1·5, that is, the extra volume of steel for stiffening the strut should not be more than 50 per cent. of the net volume. Otherwise the main cross dimension h should be increased to give a bigger radius of gyration.

For section (fig. 245 II.) with $h = 2b = 16t$ and $f_b = 4$ tons per square inch.

Ends fixed, . . Ends hinged, .		5 feet. 3 ,,	10 feet. 6 ,,	20 feet. 12 ,,	30 feet. 18 ,,	40 feet. 24 ,,
P tons.	A_0 sq. in.					
8	2	1·04	1·14	1·45	1·80	2·18
16	4	1·02	1·07	1·25	1·49	1·74
32	8	1·01	1·04	1·14	1·28	1·44
48	12	1·01	1·03	1·10	1·20	1·32
64	16	1·01	1·02	1·08	1·16	1·26
80	20	1·00	1·01	1·06	1·13	1·21

For section (fig. 245 I.) with $h = \frac{4}{3}b = 16t$ and $f_b = 4$ tons per square inch.

Ends fixed, . . Ends hinged, .		5 feet. 3 ,,	10 feet. 6 ,,	20 feet. 12 ,,	30 feet. 18 ,,	40 feet. 24 ,,
P tons.	A_0 sq. in.					
16	4	1·04	1·13	1·42	1·76	2·13
32	8	1·02	1·07	1·24	1·46	1·71
48	12	1·01	1·05	1·17	1·34	1·52
64	16	1·01	1·04	1·13	1·27	1·42
80	20	1·01	1·03	1·11	1·22	1·36
96	24	1·01	1·02	1·09	1·19	1·31

EXAMPLES.

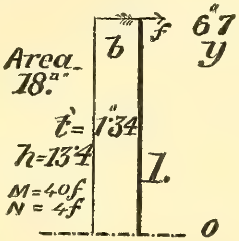
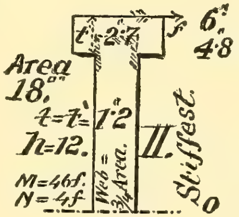
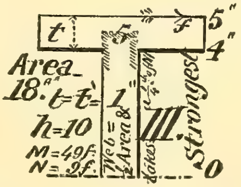
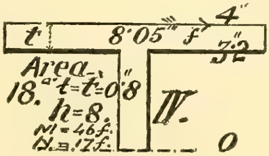
271. Compare the strength to resist bending of four double T cross-sections, all having the common area of 18 square inches, and all having the common local stiffness afforded by the constant thickness of the metal being a *tenth* part of the depth of the section. The four sections are respectively to have depths of 13·4, 12, 10, and 8 inches.

The fig. 246 shows the four sections drawn to scale, and the calculations made by the exact tabular method of Chapter XV., fig. 183.

The section III. has $\frac{M_3}{f} = 49{\cdot}2$ a maximum, and it will be seen that the web is half of the area and is the *strongest* section with area 18 square inches, and the local stiffness due to $t = \frac{1}{10}h$.

TABLE

OF COMPARATIVE STRENGTHS OF FOUR DOUBLE T SECTIONS, WITH THICKNESS OF METAL $\frac{1}{10}$TH OF THE DEPTH.

HALF-SECTIONS.	y	y^3	$\delta y^3 \times \frac{1}{3} \times b$	$M \div f$
 Area 18□" b f 6·7" y t= 1·34" h=13·4 I. M=40f N = 4f 0		300·8 0·0	$300\cdot8 \times \frac{1}{3} \times 1\cdot34 = 134\cdot33$ $\frac{1}{2} I = 134\cdot33$	$\frac{M_1}{f} = \frac{I}{6\cdot7}$ $= 40\cdot1$
 t 2·7 f 6" 4·8" Area 18□" t=t=1·2" h=12. M=46f. N=4f Web = 3/4 Area. II. Stiffest. 0		216·0 110·6 0·0	$105\cdot4 \times \frac{1}{3} \times 2\cdot70 = 94\cdot86$ $110\cdot6 \times \frac{1}{3} \times 1\cdot20 = 44\cdot24$ $\frac{1}{2} I = 139\cdot10$	$\frac{M_2}{f} = \frac{I}{6}$ $= 46\cdot4$
 t 5 f 5" 4" Area 18□" t=t=1" h=10 M=49f. N=.9f. Web = 1/2 Area & 1/4 of flakes III. Strongest. 0		125·0 64·0 0·0	$61\cdot0 \times \frac{1}{3} \times 5\cdot00 = 101\cdot67$ $64\cdot0 \times \frac{1}{3} \times 1\cdot00 = 21\cdot33$ $\frac{1}{2} I = 123\cdot00$	$\frac{M_3}{f} = \frac{I}{5}$ $= 49\cdot2$
 t 8·05" f 4" 3·2" Area 18□" t=t=0·8" h=8. M=46f. N=.17f. IV. 0 FIG. 246.		64·0 32·8 0·0	$31\cdot2 \times \frac{1}{3} \times 8\cdot05 = 83\cdot80$ $32\cdot8 \times \frac{1}{3} \times 0\cdot80 = 8\cdot80$ $\frac{1}{2} I = 92\cdot60$	$\frac{M_4}{f} = \frac{I}{4}$ $= 46\cdot3$

The section II. has the web three-fourths of the area if it be measured from centre to centre of the flanges, and is the *stiffest* section of area 18 square inches, and local stiffness due to $t = \frac{1}{10}h$.

For substituting from (73*a*) into (73*b*), we have $I_0 = \frac{1}{12}h'^2\,(3A - 2th')$ which with A constant and t' a constant r^{th} fraction of h' give the variable factor $\mu = 3Ah'^2 - 2rh'^4$. The differential coefficient with respect to h' equated to zero gives $6Ah' - 8rh'^3 = 0$, or the area of web $t'h' = \frac{3}{4}A$. See fig. 246, II.

272. Design the economical double T section with equal flanges in wrought iron for which $f_b = 4$ tons per square inch. Its strength to be $\mathsf{M} = 196{\cdot}8$ ft.-tons, and to have the local stiffness afforded by the thickness of the web, t' being a *tenth* of its depth h'. The thickness of the flanges to be $t = t'$, that is, "uniform metal."

We must plan the web measured from centre to centre of the flanges to have its area $t'h'$ a *half* of the total area A, and its share in the strength $\frac{1}{6}f'_b t'h'^2$ to be a *fourth* of M. This is done by putting in (71*b*) $\mathsf{M} \div f_b = 49{\cdot}2$, and $s = r = \frac{1}{10}$ when

$$\tfrac{2}{3}\cdot\tfrac{1}{10}\,(\tfrac{9}{10}h)^3 = 49{\cdot}2\,; \quad h^3 = 1013\,; \quad h = 10.$$

So that $t = t' = 1$, $h' = 9$, and $c = 8$, while $t'h' = 9$, and so $A = 18$ square inches. The upper half of this section is shown at fig. 246, III., and the whole section at fig. 245, II.

273. Design the economical double T section of uniform strength for wrought iron with $f_b = 4$ and $f_a = 5$ tons per square inch. For local stiffness the web is to have $t' = \frac{1}{10}h'$, and the thickness of the flanges $t = t'$. The required moment of resistance to bending is $\mathsf{M} = 218{\cdot}6$ inch tons.

In (72) put $\rho = \frac{4}{5}$, and we find that the section will be 10 per cent. stronger than if its flanges were equal. From 218·6 deduct 10 per cent., and it leaves 196·8 ft.-tons. In Ex. 272 we have already designed the section with equal flanges and found the values there quoted. It only remains to borrow from one flange and add to the other, so that, with $a + b$ still equal to 10, we get $a + \frac{1}{2}c : b + \frac{1}{2}c$ equal to $4 : 5$. And

$$a + b = 10\,; \quad \frac{a+4}{b+4} = \frac{4}{5}\,;$$

or $a = 4$ and $b = 6$ inches.

The half-section is seen at fig. 246, III., if an inch of breadth be taken off the upper flange and added to the lower. The section may now be checked by the tabular method at fig. 185, p. 289, when, with the axis at the lower skin of b, the results are $S = 18$, $G_b = 81$, $I_0 = 606$, and $\bar{y} = 4{\cdot}5$, so that h' is exactly divided as 4 to 5, although h is only nearly divided in that ratio; further $I_0 = 242$, and multiplying this by the smaller of the quantities 4 tons $\div\ 4\frac{1}{2}$ in. or 5 tons $\div\ 5\frac{1}{2}$ in., namely the first, we get the exact value of $\mathsf{M} = 215$, almost exactly what was prescribed.

274. Design a double T section for a wrought-iron strut 24 feet long and hinged at the ends, the load being 64 tons. The depth should be $h = 12$ inches for stiffness, and with the breadth of the flanges each 6 inches, and flanges and web of one thickness t as in fig. 245, II., we have

$$i^2 = \frac{h^2}{12}\,\frac{h + 6b}{h + 2b} = 24\,; \quad l = 288 \text{ in.}\,; \quad f = 4 \text{ tons per square inch,}$$

then

$$A = (2b + h)t = 24t \text{ nearly,}$$

being the same degree of approximation as i.

By (75*a*)

$$64 = P = \frac{fA}{1 + \frac{1}{9000}\frac{l^2}{i^2}} = \frac{4 \times 24t}{1 \cdot 384},$$

$t = 1$ inch nearly, but $A' = 24t$ nearly, or $A = 22t = 22$ square inches.

275. Design the same strut, using Fidler's table, p. 417, corresponding to fig. 245, II.

$A_0 = 64 \div 4 = 16$ square inches is the net area. The multiplier is the last in the second last line, that is, 1·26. The gross area $A = 1 \cdot 26 \times 16 = 20$ square inches. As the relative dimensions on the table are

$$h = 2b = 16t, \quad \text{so that} \quad A = (h - 2t)t + 2bt = 30t^2,$$

and therefore

$$t^2 = 20 \div 30 = \cdot 67, \quad \text{or} \quad t = \cdot 816, \quad \text{and} \quad h = 16t = 13 \text{ inches.}$$

This is the nearest that can be done with the table. It is evident that with $h = 12$ inches instead of 13 inches, then A must be greater, say, 22 square inches.

276. A square box steel strut, virtual length 30 feet, is to take a thrust of 80 tons. Design the section. For stiffness $b = h = 16$ inches might be suitable (fig. 245, I.).

$$i^2 = \frac{h^2}{12}\frac{h + 3b}{h + b} = 44\ ; \quad A = 2t(b + h) = 64t\ ; \quad l = 360\ ;$$

$$P = \frac{fA}{1 + \frac{1}{7000}\frac{l^2}{i^2}}, \quad \text{or} \quad 80 = \frac{7 \times 64t}{1 \cdot 42}.$$

As the divisor 1·42 indicates that the excess metal to stiffen the strut is not over 50 per cent. we can feel assured that $h = 16$ was ample enough. Proceeding

$$t = 0 \cdot 25 = \tfrac{1}{4} \text{ inch,} \quad \text{and} \quad A = 16 \text{ square inches.}$$

277. Design a T iron or angle iron constrained to bend like one. The strut is 12 feet long, the load 30 tons. Taking $h = 8$ inches and $b = 4$ inches, then (fig. 245, IV.)

$$A = t(h + b) = 12t\ ; \quad i^2 = \frac{h^3}{12}\frac{h + 4b}{(h + b)^2} = 9\ ;$$

$$P = \frac{4 \times 12t}{1 + \frac{1}{9000}\frac{144^2}{9}}, \quad \text{or} \quad 30 = \frac{48t}{1 \cdot 256}\ ;$$

$$t = \cdot 785, \quad \text{and} \quad A = 9 \cdot 4 \text{ square inches.}$$

278. If the above strut be an angle iron not constrained in any way, design it. Taking $h = b = 10$ inches, say, we have by fig. 245, V.

$$A = 20t, \quad i^2 = 4 \cdot 2 \text{ nearly}; \quad 30 = \frac{4 \times 20t}{1 + \frac{1}{9000}\frac{144^2}{4 \cdot 2}} = \frac{80t}{1 \cdot 55}\ ;$$

and $$t = \cdot 58 = \tfrac{5}{8} \text{ inches,} \quad \text{and} \quad A = 20t = 12 \cdot 5 \text{ square inches.}$$

CHAPTER XX.

THE STEEL ARCHED GIRDER.

FOR *roofs*, the steel arched girder is of great importance. It is used as a roof principal to cover the multiple platforms of great railway stations, a gigantic example being the St. Pancras Railway Station, London. It is struck in arcs of circles with a slight cusp or peak at the crown, span 240 ft., rise 96 ft. The ribs, which are spaced about 30 ft. apart, are each 6 ft. deep, and uniform except a part of the top where the moment of inertia is of double the value elsewhere. The rib is without hinges either at the crown or springings, and a tie-girder holds the ends together. These tie-girders are underneath the platforms which they support. We shall quote from Walmisley on Iron Roofs, 2nd edition, p. 83; the test prescribed in the specification of this roof. "Two main principals of the roof shall be erected at the works upon proper abutments erected for the purpose upon the flooring. The principals shall be erected with purlins complete in every respect, and shall be tested in presence of the engineer or assistant appointed by him. First a weight of 200 tons or 100 tons on each principal shall be evenly distributed. Next 25 tons shall be taken from one side of each principal leaving 50 tons upon the other side, and making the relative loads 25 tons and 50 tons on the two sides of each principal." We would note too, that in Mills' *Railway Construction*, 1898, p. 271, he gives the weight of iron work per square (of 100 square feet) as 2·07 tons for the Central Station Manchester Roof only 30 feet less in span than the St. Pancras. In Mills' book there are four consecutive pages showing arched roofs, most of them like the St. Pancras without hinges. One, however, of the roof of the Anhalt Railway, Berlin, he shows with a hinge at each end, parabolic in outline, uniform in depth, which is greater in proportion to the span 205 feet than that of the St. Pancras. The Anhalt principals suddenly taper at the ends towards the hinges.

Again, arched girder roofs for buildings, often of great span, are generally made in two halves affording three hinges, one at the crown, as well as the two at springing. The third hinge eliminates to a great extent the temperature effects, and facilitates the erection. Besides the two halves can, with economy, be swelled out in the middle, and shaped with a curved lower or inner member, and yet have a polygonal upper member to suit the covering. Two of the finest examples can be seen superimposed on each other in the *World's Columbian Exposition*, reprinted from *Engineering* of April 21st, 1893. They are, one the roof of the Manufacturers' and Liberal Arts Building of the Exposition, and the other the roof of the Machinery Hall, Paris, 1889. The spans have the common value of 368 feet. The Columbian roof has a rise 206 feet, being about 50 per cent. higher than the Paris one, the principals being spaced 50 feet apart. The girders are 8 feet deep at the top, and at least twice as deep as the haunches. Two rival modes were employed in erecting the Columbian roof by two contractors: figures illustrating them are shown. They are called the *Fives Lille* mode and the *Cail* mode.

We mention those examples as the drawings can readily be seen by British students. A complete list of roofs and bridges is given by Howe in his *Treatise on Arches*, Wiley and Son, 1897, an excellent and exhaustive treatise on the subject.

For *bridges*, the steel arched girder seems likely to entirely supersede the suspension bridge. A comprehensive view can be had on one page of *Engineering*, of April 28th, 1899. With the exception of the top one, the spans shown are all between 500 and 600 feet, the rises varying from, judging by the eye, one-fifth to two-thirds of the span. Some are circular, some parabolic, some have no hinges, when the girders have a small depth at crown and increase till they are from twice to thrice as deep at the springings.

Others have a hinge at each springing, and are either uniform with a sudden taper at the hinges or are very deep at the crown and taper gradually to the hinges.

Again, some have abutments, others tie-rods suspended from the arch. Also in some of the examples there shown, the railway runs over the crown on straight girders supported by "Eifel" towers rising from the arch; in other examples these straight girders are suspended or are partly suspended from crowns and partly supported over the haunches.

The figure at the top of the group is the great Niagara Falls bridge, 1898, span 840 feet, rise 150 feet, depth of parallel booms 26 feet. The shape of arched-rib is parabolic, and it rests on steel pins 12 inches in diameter one at each end. Two arched girders support two railway tracks and side walks. The horizontal girders ride over the arches on which they rest by vertical and horizontal spandril bracing. The live load is prescribed in one form as a string of the Ely Tram and Trailer, spaced 200 feet clear apart per track. Such a tram is roughly 40 tons in a 60-foot length.

Disposition of loads.—It is evident that the live load all over the span of an arch produces the greatest thrusts on any section of the rib as a whole. But the live load, say half over, produces bending moments which at some sections greatly increase the thrust on one boom, at the same time decreasing it in the other, it may be to such an extent as to change it to tension. The unsymmetrical position of the live load causes the thrust at some sections not to be truly along the rib but oblique. This oblique thrust is then equivalent to a direct component thrust, and a tangential component which distorts the panel or bay of the rib, which must accordingly be braced by one or two diagonal members.

These effects would be most pronounced for the middle panel or bay with the live load half over the span. And with this position of the live load these effects are sufficiently pronounced at all bays for practical purposes, the span being very great, although with short spans it might be well to put the load three-fourths over for some bays.

From these remarks on bridge arches, and from the specification of the test on the St. Pancras' roof arch, it will be seen that for large span lofty steel arched girders, either for roof or bridge work, it is practically sufficient to consider only two dispositions of the load—(1) A uniform load all over the span, and (2) a uniform load on each half, but greater on one half of the span than on the other.

Lévy has given his solution of this problem in the happiest way. He draws the stress diagram for a unit uniform load on left half of the span only, from which by suitable multipliers the stresses on the members cut by any section can be computed for—(1) The live load as well as the dead load all over the span; (2) the live load on the half in which the section lies, but not on the other half; and (3) the live load on the other half but not on the half of the span in which the section itself lies.

The curve of benders and curve of flatteners.—If an arched rib stand on its ends on a horizontal plane, and if we suppose each end to be a pin furnished with wheels, then if any load be imposed on the back of the arch it will straddle out and be everywhere flatter than it was before, and the flattening moment at each section can be shown by the diagram which, for a straight girder hinged at the ends bearing the same load is called the bending moment diagram. We shall now call that figure the *curve of flatteners.*

If we measure the amount by which the arch has straddled or increased the span and remove the load, and if we now put a tie-rod across between the pins and screw it tight, till they are brought nearer to each other by the same amount that they straddled before, the arch will be like a bow a little more bent at each section than it was at first. That is, there is now at each section a bending moment the amount of which is the pull on the tie-rod multiplied by the height of the middle point of the section above it as a lever. Hence the curve drawn up the middle of the arched rib is itself the *curve of benders* due to the pull on the tie-rod. If the pull on the tie-rod be ascertained in any way, a scale could readily be constructed for this curve of benders.

Suppose that the free girder as at first is loaded with the external load which straddles it out, and that then the tie-rod is put across and screwed till the straddle is destroyed, and the hinges brought to the original span. We have now two diagrams, one the curve of flatteners due to the extern load drawn arbitrarily to any scale, and the other, the middle line up the rib itself, a curve of benders due to the pull on the tie-rod, the scale for which is *unknown,* unless we should happen to know, say by experiment, the pull on the tie-rod. Suppose for a moment that we did know the pull on the tie-rod and so were enabled to construct the scale for it. We propose to call this scale the *normal scale.* On looking now at our curve of flatteners, which is hardly likely by accident to be to this very scale, we could reduce it or draw it over again to this *normal scale,* and in doing so, we could draw it on the chord of the rib as base so as to have both curves, the one of *flatteners* and the other of *benders,* on the chord as a common base, and both to the *normal scale.*

We have already shown that a bending moment diagram is itself a balanced linear rib for the load, so that the two curves thus superimposed on the chord as base and to the same scale, may now be distinguished with propriety as the real rib and the ideal balanced rib.

It will be seen that if the free unstrained rib be first provided with rigid abutments that will not yield appreciably, or with a tie-rod which will not stretch an appreciable amount, the load placed on its back produces simultaneously the flatteners (a curve of which, we can draw to an arbitrary scale independent of everything but the extern loads) and the benders due to the *induced pull* on the tie-rod, of which the central line of the rib is itself the curve to the *normal scale* unknown. This *normal* unknown scale depends only on the induced pull on the tie-rod, that is, on the straddle of the girder without a tie-rod under the extern load. Now this straddle depends on the variations of the cross-sections of the rib from section to section, so that the normal scale is statically indeterminate as a rule.

The exception is the presence of a third hinge, say at the crown. Suppose there had all along been a hinge at the crown, but that it had been securely clamped. The minute it was unclamped the *normal* scale is known, for the ideal rib must pass through this third hinge in order that the flattener and the bender may at that point neutralize each other; hence the mere reduction of the curve of flatteners so that it may pass through the third hinge changes its arbitrary scale at once into the *normal scale* which is then known.

Rankine advised the adoption of a third hinge at the crown of cast-iron arched girders, and says the assumption of such a hinge at the crown in conjunction with hinged springings will always be a good approximation.

Lévy illustrates his graphical construction for an arch with hinged ends by an application to the Oporto bridge over the Douro. In this bridge the rib is crescent-shaped, and is hinged at the springings where it rests on abutments. Span 160, mean rise 42·5, and depth of rib at crown 10 metres. The moments of inertia of the cross-sections from crown out to a hinge vary at intervals as 10, 8, 6, 5, 4. It is described in the Mem. and *Comte Rendu*, de la S. d. I. C. 1878, par T. Seyrig. The line up the middle of the rib is an arc of a circle.

Lévy primarily assumes the rib to be of uniform cross-section, and indicates a correction for the variation.

The stresses on the members of this bridge are calculated by Howe in his treatise already referred to, and his results compared with those of Seyrig. Howe quotes the moving load covering the entire roadway as 277 tonne or 270 tons, which is roughly half a ton per foot, or half a share of a single row of locomotives.

Lévy illustrates his graphical construction for an arch with fixed ends by a rib with the same arc of a circle as the other, but with the section increasing from the crown outwards so that its moment of inertia at each section is proportional to the secant of the slope of the rib to the horizon. Such an arched rib is of *uniform stiffness.* This is proved quite simply by Rankine (*see* "The Sloping Beam," Civil Eng., p. 292).

It is evident that this arch should deepen in section as it goes outward for two reasons. The bending moments at the ends of the straight boom fixed at ends are greatest. Also a broad foot is required to securely fix it to the skewbacks.

Lastly, before illustrating Lévy's construction, the relative weight of steel on the three types are given by Howe in the following proportion :—

(1)	Three hinges, . . .	1.
(2)	Two hinges, . . .	1·2.
(3)	No hinges, ...	1·3.

He says, too, that three hinges are unsuitable for the bridge with moving loads on account of *vertical* vibrations.

The economy of (2) over (3) would, we think, be increased by the extra material in the skewbacks to supply the reacting couples at the ends.

Observe that the type (3) cannot, when circular, subtend much more than 120° at the centre, as the secant of 60° is 2, and beyond 60° it increases enormously. Roof girders of this type (3) are often the complete semicircle with a vertical part in addition at each end. Here only the portion out to 60° on each side of the crown should be treated as the actual elastic rib, the lower part on each side being reckoned as part of the vertical post, and is generally along with the post buttressed or braced to a smaller rib rising on the opposite side of the post.

Other references are *The Engineer*, June 30 and July 14, 1899, for a Danish bridge; also March 30, 1900, for the Rhine bridge at Bonn.

LÉVY'S GRAPHICAL SOLUTION OF THE ARCHED RIB OF UNIFORM SECTION HINGED AT EACH END.

On fig. 247, 2 is shown to scale a segment of a circle, span 240 feet, rise 80 feet, radius 130 feet. It is the lineal real rib up the centre of the iron arch which consists of two parallel booms 10 feet apart. This then is the curve of benders to the unknown *normal scale.* It is divided into 16 equal parts along the curve, and the ordinates are numbered from 0 to 8 to the right hand, and from 0 to 8′ to the left.

The fig. 247, 5 shows the load line $W = 120$ tons drawn to a scale of tons, being one ton per foot on left half of span only. The reactions are $Q = \frac{1}{4} W$ and $P = \frac{3}{4} W$. A pole Q is chosen arbitrarily, and the *curve of flatteners* (fig. 247, 1) drawn to it, and consisting of a straight tail over the unloaded half parallel to the vector Qa. A parabolic arc must be drawn over the loaded half. The vertex is over the point where the shearing force diagram (dotted) crosses the base, that is, at a point dividing the span in the ratio 3 to 8. The curve of flatteners when finished is like a great fish or "Jon Doré," and it is best to construct the points on the parabolic arc corresponding to the ordinates of the figure below ruled up. This is done by producing the straight tail till it makes an intercept on the vertical through the left end (see fig. 249, 1), dividing this vertical intercept proportional to the way the left half of span is divided when vectors from the top point of the straight tail will cut off the proper heights on the ordinates. Number these ordinates from 1 to 8 right, and from 1 to 8′ left (see fig. 90).

The next step is best done with proportional compasses. Double all the ordinates of fig. 247, 2 drawing right and left a series of *fictitious* parallel forces 1 to 8 and 1 to 8′. Set the compasses to the convenient ratio 4 to 1, construct the load line (fig. 247, 3) with 0 equal to a quarter of the middle ordinate of the curve of benders, with 1 equal to a quarter of the sum of its ordinates 1 and 1′, &c., and to a convenient pole draw the link polygon among the horizontal fictitious forces going left, and produce the two end links to make the first intercept on the chord joining the hinges of the real rib.

In the same way lay off the load line (fig. 247, 4) with 0 equal to a quarter of the middle ordinate of the "Jon Dore," with 1 equal to a quarter sum of its ordinates 1 and 1′, &c. With the *same* polar distance as last draw the link polygon among the horizontal fictitious forces going right, and produce the end links to make the second intercept on the chord joining the hinges of the real rib.

On fig. 247 this second intercept is about two-thirds the length of the first. Now we know that these two intercepts should be equal, for their algebraic sum equated to zero means that the moments of all the fictitious forces with respect to the chord joining the hinges is zero.

In this problem three definite integrals are involved, namely,

$$\int M dx = 0, \quad \int x M dx = 0, \quad \int y M dx = 0,$$

where x and y are the coordinates of the real rib, and the limits of the integration being from hinge to hinge.

The first two integrals we have fully discussed as regards the straight beam and illustrated by examples in Chapter XVIII., at the beginning of which are theorems (*a*) and (*b*), corresponding to these first two integrals, and, for an illustration see fig. 230, in which *AK* is drawn that its area may equal that of *ACB*, and that its geometrical moment about the vertical through the centre *O* shall equal that of *ACB*. That is, the joint areas are zero, and the joint geometrical moments are zero.

So with the third integral; *Mdx* is the joint algebraical area of the two corresponding strips of the two curves, one of benders and one of flatters, and by drawing a fictitious force for this partly right and partly left, and giving them the common lever about the chord, namely, the height of the rib itself (only doubled for good definition), we obtain the product $Mdx \times y$ for that finite part of the rib by producing the two links that concur on the fictitious force till they make an intercept on the chord. The total intercept must be zero, and it matters not that we have grouped them in two categories according to whether they tend to flatten or further bend the rib.

The reason that the two intercepts are not equal is that the arbitrarily chosen pole *Q*, to which we drew the "Jon Dore," does not make that figure be to the *normal scale*.

It is now necessary to draw the "Jon Dore" or curve of flatteners over again to a vertical scale increased in the ratio in which the first intercept exceeds the second. It is best by trial and error to set the proportional compasses till one end spans one intercept, and the other spans the other intercept. Take the height of the middle ordinate of the "Jon Dore" on the smaller end, and, with the large end, set it up on the middle ordinate of the real rib. On fig. 247, 2, it will be seen to reach a little above the crown of the real rib; joining this with a straight line to the right hinge we have the tail of the new curve of flatteners, and with the proportional compasses the seven intermediate points of the left half may be set up, or the new parabola may be constructed at those ordinates as at first.

Next from *b* the bottom of the load line fig. 247, 5, a line parallel to the tail of the new curve of flatteners will meet the horizontal from *d*, the junction of the reactions *P* and *Q*, at *O* the new pole. Or *dO* is shorter than *dQ* in the ratio of the intercepts.

This polar distance, which is 130 feet, determines the *normal scale* of the joint curves of flatteners and benders, or as they may now be called the ideal and real ribs. For it is evident that if the right-hand link polygon were redrawn for the new curve of flatteners, the new intercept 2 would now equal the first intercept.

The virtual hinge or point at which there is no bending moment is a little to the right of the crown of the real rib where the tail of the ideal rib crosses it. Had there actually been a hinge at the crown of the rib, the curve of flatteners could have been drawn at once with its tail drawn from the crown to the right end, and the *normal scale* determined at once.

The elevation, to the double scale, of the steel arched girder is shown on fig. 248, 6; it is sketched as if for a roof. It has *radial stiffeners* with double diagonal ties in each square.

Counting the crown square as zero, we will now take out the stress at a radial cross-section through the middle of the third square to the right hand.

This cross-section is shown at A on fig. 247, 2, and A' is the corresponding cross-section in the loaded half of the girder. A portion of the arch reaching from A' to A is shown above, furnished with arrows to indicate the stresses on the upper and under booms and the radial or shearing stress.

The thrust on the ideal rib at the point below A is given by the vector bO of fig. 247, 5, which is resolved into two components a tangential thrust 50 tons, and an outward radial force 19 tons. The 50 tons is divided into 25 tons, each of thrust on the upper and under boom. At A the height of the curve of benders or real rib is greater than that of the curve of flatteners or ideal rib. The intercept at A between the ribs is the residual bender, it measures 5 tons and, when multiplied by the polar distance 130 feet, we have the bender M_A equal to 650 ft.-tons. The booms have a lever of 10 feet to resist it, and accordingly there is a pull of 65 tons on the upper and thrust of 65 tons on the lower boom at A due to this bender. This is the most business-like way of taking out the effect of the bending couple at A.

But there is another and instructive way of looking at it and of scaling it off. Suppose the thrust bO tons acting on the real rib at the point below A, then to shift it parallel to itself so as to act at A instead; we must compound with that force bO, a left-handed couple, the common *force* of which shall be bO tons and whose *arm* is the perpendicular distance between the old and new positions of the force before and after the shift. Now this arm is shorter than the vertical intercept at A. It is equal to the vertical intercept multiplied by the cos θ, if θ be the slope of the tail of the ideal rib to the horizon. So that if we measure the vertical intercept at A in *feet*, we have the arm

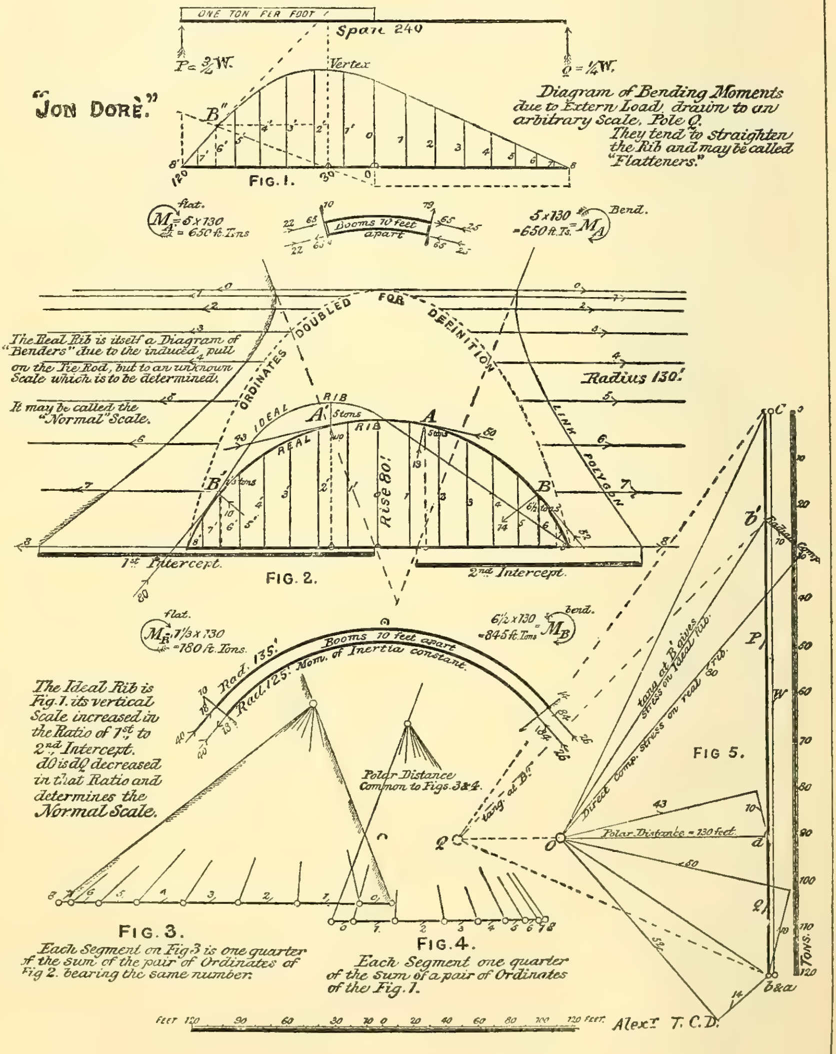

ONE TON PER FOOT
Span 240
P = ¾W.
Q = ¼W.
Vertex
"JON DORÈ."
B''
FIG. 1.
Diagram of Bending Moments due to Extern Load drawn to an arbitrary Scale. Pole Q.
They tend to straighten the Rib and may be called "Flatteners."
flat.
M_A = 5 x 130 = 650 ft. Tons
Booms 10 feet apart
5 x 130 = 650 ft. Ts. = M_A
Bend.
DOUBLED FOR DEFINITION
ORDINATES
LINK POLYGON
The Real Rib is itself a Diagram of "Benders" due to the induced pull on the Tie Rod, but to an unknown Scale which is to be determined.
It may be called the "Normal" Scale.
Radius 130.'
IDEAL RIB
REAL RIB
A'
A
5 tons
B'
1⅓ tons
6½ tons
Rise 80.'
1st Intercept.
2nd Intercept.
FIG. 2.
flat.
M_B = 1⅓ x 130 = 180 ft. Tons.
6½ x 130 = 845 ft. Tons = M_B
bend.
Rad. 135.'
Rad. 125.'
Booms 10 feet apart
Mom. of Inertia constant.
The Ideal Rib is Fig. 1. its vertical Scale increased in the Ratio of 1st to 2nd Intercept. dO is dQ decreased in that Ratio and determines the Normal Scale.
Polar Distance Common to Figs. 3 & 4.
tang. at B''
Q
O
FIG. 3.
Each Segment on Fig 3 is one quarter of the sum of the pair of Ordinates of Fig 2. bearing the same number.
FIG. 4.
Each Segment one quarter of the Sum of a pair of Ordinates of the Fig. 1.
FIG 5.
C
b'
P
W
d
Q
b & a
Radial Comp.
tang. at B' gives stress on Ideal Rib.
Direct Comp. Stress on real rib.
Polar Distance = 130 feet.
TONS.
0 10 20 30 40 50 60 70 80 90 100 110 120
FEET 120 90 60 30 10 0 20 40 60 80 100 120 FEET.
Alexr. T. C. D.

Fig. 247.

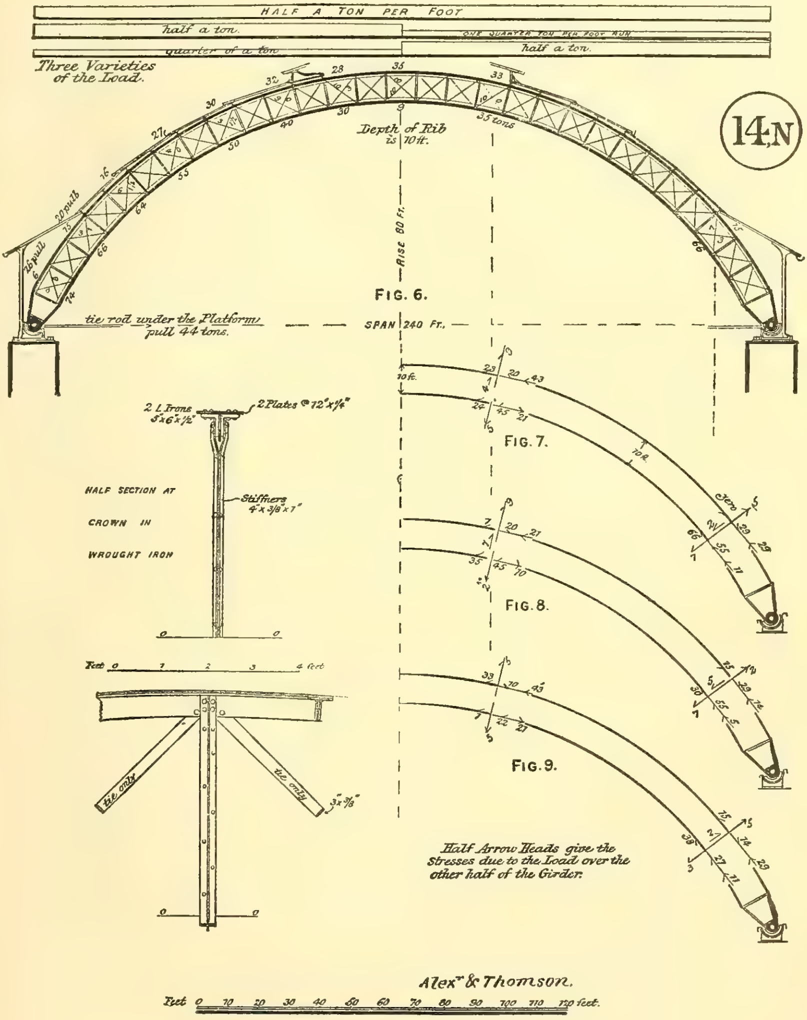

HALF A TON PER FOOT
half a ton.
ONE QUARTER TON PER FOOT RUN
quarter of a ton
half a ton.
Three Varieties of the Load.
14,N
Depth of Rib is 10 ft.
35 tons
Rise 80 Ft.
FIG. 6.
tie rod under the Platform pull 44 tons.
SPAN 240 Ft.
26 pull
20 pull
10 ft.
FIG. 7.
FIG. 8.
FIG. 9.
Zero
2 L Irons 5"x6"x½"
2 Plates @ 12"x¼"
Stiffners 4"x3/8"x7"
HALF SECTION AT CROWN IN WROUGHT IRON
Feet 0 1 2 3 4 feet
tie only
tie only
3"x3/8"
Half Arrow Heads give the Stresses due to the Load over the other half of the Girder.
Alexr. Thomson.
Feet 0 10 20 30 40 50 60 70 80 90 100 110 120 feet.

Fig. 248.

of the shifting couple multiplied by sec θ. But if instead of measuring bO in tons we take instead dO in tons, we have the force of the shifting couple decreased in the exact proportion that the arm was increased. Hence the moment of the shifting couple equals the product of the vertical intercept measured in feet, and of dO measured in tons. But the *product* is in no way altered if we interchange the scales upon which we scale off the two factors. Hence what we measured off before, namely, the intercept at A in tons, and the polar distance dO in feet give, when multiplied together, the moment of the couple which shifts the thrust on the ideal rib up to act at A on the real rib. When the thrust arrives at A it brings with it this left-handed couple, so that the two booms have not only to oppose the thrust, but have also to exert a *right-handed* couple to neutralize the shifting couple.

This is the penalty, as it were, that the real rib pays at A for not coinciding with the ideal one. Again, now that the thrust is at A it is not directly along the rib, and when resolved into a direct thrust along the rib there is also a radial outward or shearing force as well, tending to distort the square at A, and this distorting force is the penalty, as it were, that the real rib pays at A for not being parallel to the ideal one.

Note that dO plays two parts: it is 130 feet as polar distance, but is also 44 tons as the thrust at crown of ideal rib or pull on the tie-rod.

At A' the vertical intercept measures 5 tons, and when multiplied by 130 feet we have $M_{A'} = 650$ foot-tons a residual flattener, giving a thrust of 65 tons on upper boom, and a like pull on the lower. A tangent drawn to the parabola at A happens to be horizontal; it resolves into a direct thrust of 43 tons along the real rib at A', or 22 on each boom, together with an inward tangential force of 10 tons.

The like quantities are constructed for the pair of points B and B' corresponding to the twelfth square from crown. In drawing the tangent to the parabolic part of the real rib at B' it is best to construct the tangent at B'' by ruling it horizontally to meet the vertical through the vertex, setting up a like height above the vertex which gives a point to join B'' to; thus Qb' is drawn parallel to this tangent, and, joining b' to O, we have the tangent to the real rib at B'. When the point B'' lies near the vertex this construction is bad and a better one in such a case is shown for the point S'' on fig. 249, I., namely, to draw a chord from c (the most remote given point on the parabola to the right of S''), to d a point lying at the same distance horizontally from S'' as c does, but on the left side. The tangent at S'' is parallel to dc. The other tangent at K'' is drawn from K'' to f where $Vf = ef$ as already described.

On fig. 248, 6, three varieties of load are shown. Fig. 248, 7 gives the stress at the third square selected from values at A and A' as follows: the pull on the upper boom at A is $65 - 25 = 40$, due to a load of 1 ton per foot on left half of span; but at fig. 248, 7 only half of this or 20 tons is set on the upper boom beside a half barb or arrow-head, because there is only half a ton per foot on the left half of the arch. In the same way we have set 45 as thrust on low boom, and 9 as outward radial stress. These are set beside half barbs to indicate that the stresses are due to the load on left half of arch only.

Now if we suppose the figs. 247, 1 and 2 turned over so as to be seen through the paper at the back, A' will have taken the place of A, so that by halving the stress at A' and writing them on fig. 248, 7 with full barbs, we have the stress due to the load on the right half of the arch. These are summed algebraically and written with a black figure a little to the left of the section (fig. 248, 7).

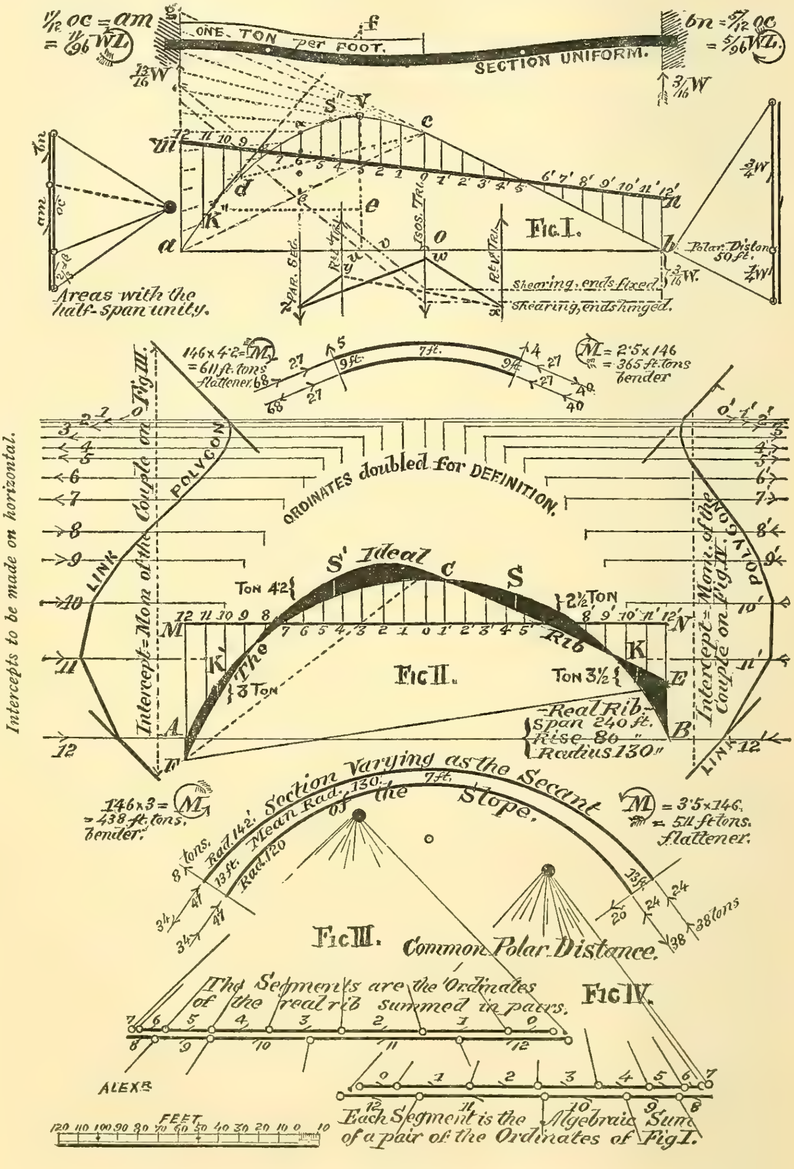
On fig. 248, 8 the quantities for half-barbs are unaltered, as the load on the left half for the second variety is unaltered, but the quantities at the full barbs are halved, as the load on right half is reduced.

On the other hand, fig. 248, 9 shows the stresses for the third variety of load. Here the quantities at the full barbs are unaltered and the others halved.

On the third square to right hand of the arch itself 33 tons is written on the upper boom, being the greatest of the three varieties. On the lower boom it is 35 tons. On one of the diagonals in the third square is written 10 tons: this is a pull. The other diagonal is dotted to indicate that there is no pull on it for any of the three varieties of load.

It will be seen that the gross radial stresses for the three varieties of load are 4, 7, and 0, all *outward*. As the cross-section cuts the diagonal at about 45°, and as 7 tons is only one component of the pull on the diagonal, it has to be multiplied by sec. 45° or by $1\frac{1}{2}$ nearly, giving the pull 10 tons. To make the roof stiff the two diagonals in the third square might both be tightened by a thimble, or one only till they had an initial pull of 2 tons each, giving 12 and 2 as the greatest pulls on the two diagonals respectively.

The figs. 247 to 250 are reduced half lineal size from a set of graphical exercises which we hope to publish soon.



1/12 oc = am = 11/96 WL
ONE TON per FOOT.
SECTION UNIFORM.
bn = 5/12 oc = 5/96 WL
13/16 W
3/16 W
S'' V c
12 11 10 9 8 7 6 5 4 3 2 1 0 1' 2' 3' 4' 5' 6' 7' 8' 9' 10' 11' 12'
m
n
d
K''
e
a
b
o
w
u v y
FIG. I.
Polar Distance 50 ft.
3/4 W
1/4 W
3/16 W
PAR. SEG.
RT. TRI.
ISOS. TRI.
REV. TRI.
Shearing; ends fixed.
Shearing, ends hinged.
Areas with the half-span unity.
am
oc
Intercepts to be made on horizontal.
146 × 4·2 = M = 611 ft. tons flattener.
M = 2·5 × 146 = 365 ft. tons bender
68 27 5 9 ft. 7 ft. 9 ft. 4 27 27 40 40
POLYGON
LINK
Couple on Fig. III.
Intercept = Mom. of the
ORDINATES doubled for DEFINITION.
S' Ideal c S
TON 4·2
2½ TON
M N
A B
K' K E
The Rib
FIG. II.
3 TON
TON 3½
D
Real Rib.
Span 240 ft.
Rise 80 "
Radius 130 "
Intercept = Mom. of the Couple on Fig. IV.
146 × 3 = M = 438 ft. tons. bender.
M = 3·5 × 146 = 511 ft. tons. flattener.
Section Varying as the Secant of the Slope.
Rad. 142'
Mean Rad. 130.
Rad. 120
13 ft.
7 ft.
8 tons.
3½ 47 47 3½
13 ft. 24 24 20 38 38 tons
FIG. III.
Common Polar Distance.
FIG. IV.
The Segments are the Ordinates of the real rib summed in pairs.
7 6 5 4 3 2 1 0
8 9 10 11 12
0 1 2 3 4 5 6 7
12 11 10 9 8
Each Segment is the Algebraic Sum of a pair of the Ordinates of Fig. I.
ALEXR.
FEET
120 110 100 90 80 70 60 50 40 30 20 10 0 10

Fig. 249.

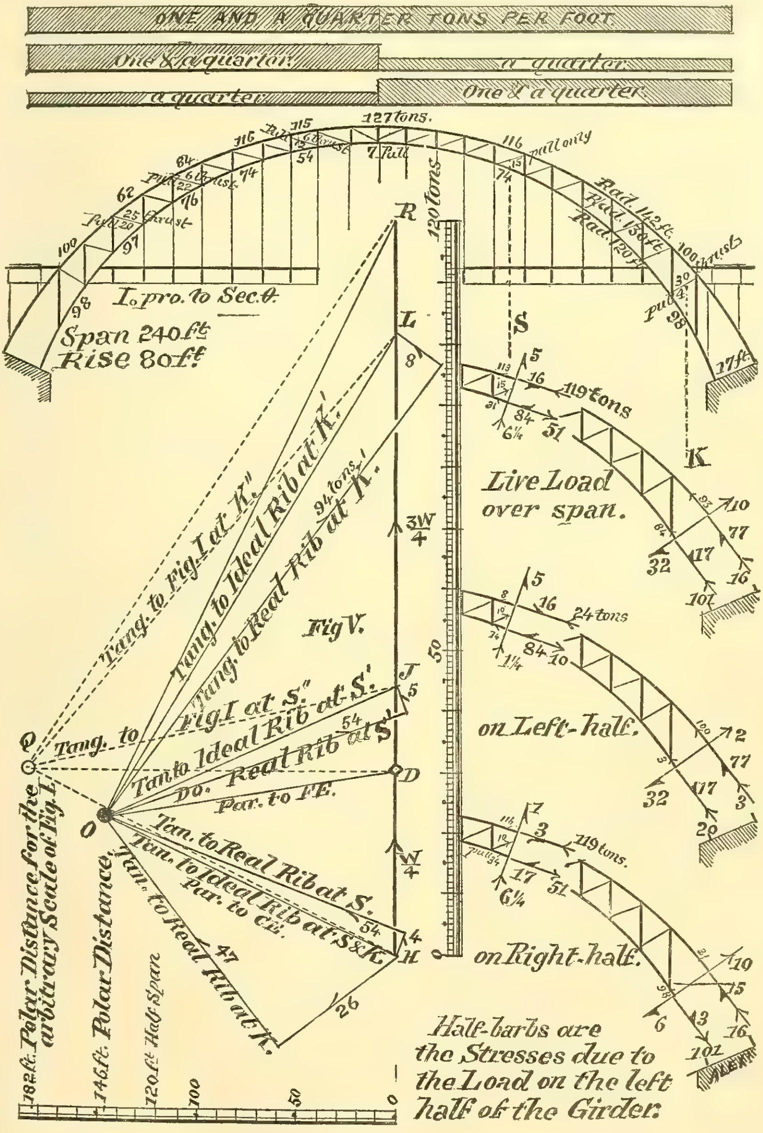

ONE AND A QUARTER TONS PER FOOT.
One & a quarter.
a quarter.
a quarter.
One & a quarter.
127 tons.
120 tons
Pull only
Rad. 142 ft.
Rad. 130 ft.
Rad. 120 ft.
100 thrust
I₀ pro. to Sec.θ.
Span 240 ft.
Rise 80 ft.
R
L
S
K
J
D
H
O
Q
17 ft.
119 tons
Live Load over span.
24 tons
on Left-half.
119 tons.
on Right-half.
Fig V.
3W/4
W/4
Tang. to Fig. I at K''.
Tang. to Ideal Rib at K'.
94 tons.
Tang. to Real Rib at K.
Tang. to Fig. I at S''.
Tan. to Ideal Rib at S'.
Do. Real Rib at S.
Par. to FE.
Tan. to Real Rib at S.
Tan. to Ideal Rib at S & K.
Par. to CE.
Tan. to Real Rib at K.
182 ft. Polar Distance for the arbitrary Scale of Fig. I.
146 ft. Polar Distance.
120 ft. Half-Span
100
50
0
Half-barbs are the Stresses due to the Load on the left half of the Girder.

Fig. 250.

LÉVY'S GRAPHICAL SOLUTION OF THE ARCHED RIB OF UNIFORM STIFFNESS FIXED AT THE ENDS AND WITHOUT HINGE ANYWHERE.

The preceding description must serve in detail, and we shall point out the marked differences in this case.

We have the same central line for the linear-rib. The boom are now 7 feet apart at crown, and 17 feet apart at springing, and the moment of inertia of the cross-section is everywhere proportional to the secant of the slope. The stiffeners are vertical and are prolonged as suspending rods to support the horizontal girders that carry a railway.

The first marked difference in the construction is that the real rib is divided into equal parts along the span or horizontal. There are more parts, being 12 in each half. The curve of flatteners is shown at fig. 249, I. drawn as before to a short load line with a polar distance of 50 feet. Then for theorems (*a*), (*b*), Chapter XVIII., as applied generally at fig. 232 (and particularly at Exs. 242, 243), we must draw a new base *mn* across the curve to include the same area *an* that *aVb* contains, and with its centre of gravity on the same vertical line. This is accomplished by making $am = \frac{11}{12}oc$, and $bn = \frac{5}{12}oc$. The graphical construction is shown to left of fig. 249, I. The half span being reckoned as the horizontal unit, a force is laid down equal to *oc* to represent the area of the isosceles triangle *acb*, followed by a force $\frac{2}{3}a\beta$ to represent the area of the parabolic segment *aac*. The lines of action of those two areas are drawn downward; also two lines of action trisecting the span are drawn upwards to represent the two triangles into which the unknown figure *an* can be divided. A link polygon drawn among these four lines of action is shown, its closing side dotted; a vector from the pole parallel to this divides the load line into two segments which are the areas of the two triangles *amn* and *anb*. But the half span being unity, those segments are *am* and *bn* respectively.

MN is drawn across the real rib so that *AN* may have the same area as *ACB*. To draw *MN* its height *AM* will be a little more than two-thirds of the height of *C*. For if *ACB* had been parabolic it would have been exactly two-thirds. The area *ACB* can be measured with a planimeter, or calculated from three ordinates in a half of it, using Simpson's multipliers.

In drawing the load line, fig. 249, III., 0, 1, ... 7 go left, as long as the ordinates of the real rib in pairs are above *MN*, but 8, 9, ... 12 go right, and only *one-half* of ordinates 12 are to be taken.

This load line should be a closed polygon, but it will only nearly close. This must be averaged so that the two end vectors may coincide. When the link polygon to left of fig. 249, II. is drawn, its two end links are parallel, and make an intercept on the horizontal, the load line being horizontal, the two end links of the right polygon make another intercept on the horizontal.

This second intercept which should be equal to the first, shows that the curve of flatteners is to a scale too small. The proportional compasses may be set so that one end shall span the one intercept and the other end the other. The curve of flatteners is then to be increased and superimposed on the real rib with mn coinciding with MN. To do this from the middle of MN set up oC equal to the increased value of oc. Set downwards NE equal to the increased value of nb, and MF equal to the increased value of ma. Join CE with a straight line and it is the tail of the ideal rib, and the points on the curved half $FS'C$ can have their values set up from the chord FC increased from those of $aS''c$ from its chord ac.

The load line to a bold scale is drawn on fig. 250, V., the poles Q and O determined, and the stresses at four sections S, S', K, K', scaled off as in the last, and then these are manipulated with the proper multipliers as on the hinged rib.

The reason that equi-intervals are taken along the span is, that I is not constant, the integral being $\int (M \div I)\, y dx = 0$; but $I \cos \theta$, the projection on the chord, is constant since I is proportional to the $\sec \theta$, so that in taking finite intervals it is necessary to make them have a constant horizontal projection.

Numbers engraved along the upper and under booms of both of our girders show how the plates must vary, and it is interesting to note the difference in the variation for the two cases.

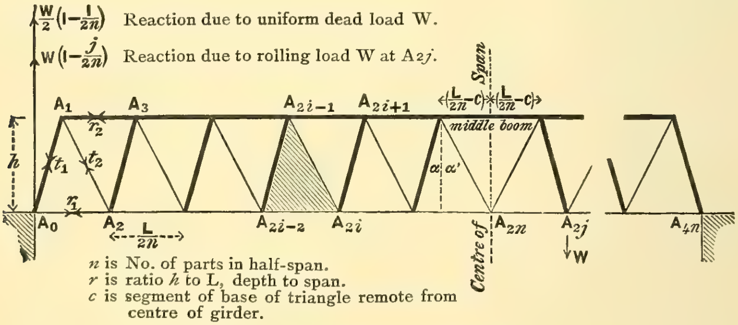
Some booms sustain a pull instead of a thrust for the live load half over, and we note that *eight-tenths* of this *reverse stress* is prescribed on the figure of the Niagara bridge as a quantity to be added to the direct stress.

On Lévy's own figure for the arched linear rib fixed at ends, we think that some error in scale has crept in, as the intercepts between the real and ideal ribs are much greater than those in his example with the ends hinged, which would imply increased bending stresses by the mere fixing of the ends.

CHAPTER XXI.

ANALYSIS OF TRIANGULAR TRUSSING IN GIRDERS WITH HORIZONTAL PARALLEL BOOMS.

THE lower member A_0A_{4n} (fig. 251) is supported at each end and is divided into $2n$ equal parts. Upon the first n of these booms as bases similar triangles are erected, the upper booms joining the vertices of the triangles. This constitutes the left half of the girder with which we have principally to deal. The right half of the girder extending from A_{2n} to A_{4n} is the *image* of the left half, that is, it has the pattern reversed just as an image of the left half would appear in a vertical mirror forming a right cross-section at the centre.

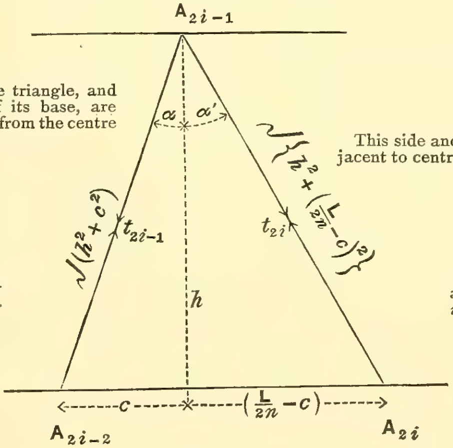

$\frac{W}{2}(1-\frac{1}{2n})$ Reaction due to uniform dead load W.
$W(1-\frac{j}{2n})$ Reaction due to rolling load W at A_{2j}.
Span
A_1
A_3
A_{2i-1}
A_{2i+1}
$(\frac{L}{2n}-c)$
$(\frac{L}{2n}-c)$
r_2
middle boom
h
t_1
t_2
α
α'
r_1
A_0
A_2
$\frac{L}{2n}$
A_{2i-2}
A_{2i}
Centre of
A_{2n}
A_{2j}
W
A_{4n}
n is No. of parts in half-span.
r is ratio h to L, depth to span.
c is segment of base of triangle remote from centre of girder.

TRIANGULAR TRUSSING.—(Fig. 251.)

It will be seen that we are dealing with a girder divided into an even number of bays. This is partly because of the comparison to be drawn with the Fink system of trussing, which only admits of even subdivision, and partly because an odd number of bays is not so desirable from an economical point of view. Neither could the notation be conveniently adapted to both the odd and even subdivision of the span.

The triangular girder with an odd number of equal bays will be taken up separately further on. For the central bay there is no shearing stress with the dead load, so that the material in the diagonal brace is not economically disposed. And again, the pattern will not change at the centre of the girder but at one side of the central bay, unless two diagonals be placed in the central bay, a further departure from economy. But these disadvantages are not felt if the live load is great; that is to say, in small light girders for moving loads.

Returning to fig. 251, it will be seen that there are $2n - 1$ upper booms all equal in length with the exception of the central one. At each lower joint or apex the letter A is placed with an even suffix, and at the upper apexes with an odd suffix.



This side of the triangle, and the segment c of its base, are said to be remote from the centre of girder.

This side and segment are adjacent to centre of girder.

Form of the triangle varies with c.

Position of triangle varies with i.

THE i^{th} TRIANGLE FROM LEFT END OF GIRDER.—(Fig. 252.)

The depth of the girder is h, and the span is $L = A_0A_{4n}$, and the common length of all the booms, with the exception of the upper central one, is $L \div 2n$.

A variable c to regulate the shape or form of the triangle is now introduced in the following manner. Fig. 254 is one of the triangles, singled out from those n triangles forming the left half of the girder. Its base is $A_{2i-2}A_{2i}$, and from its vertex A_{2i-1} a perpendicular h is dropped dividing the base into two segments; c is the segment of the base *remote* from the centre of the span (the shorter in the diagram), while $L \div 2n - c$ is the segment *adjacent* to the centre. The surd $\sqrt{(h^2 + c^2)}$ gives the length of the diagonal $A_{2i-1}A_{2i-2}$, which, in the same way, is said to be *remote* from the centre of the span, and a similar surd gives the length of the diagonal *adjacent* to the centre. For all symmetrical loads, such as the dead weight of the girder itself, the diagonals remote from the centre are struts, and those adjacent to it are ties, so that diagonal struts and diagonal ties are convenient short names for those members, which however become rather misleading when moving loads are to be considered. It will now be seen that the upper central boom is twice the length of the segment adjacent to the centre. To the variable i are to be assigned the integer values 1, 2, 3, &c., ... n, when the triangle (fig. 252) will become in succession the triangle at the left support, the second, the third, &c., each and all of the triangles forming the left half of the girder.

Dealing only with the left half of the span, we have

Lower apexes, ...	$A_0 A_2 A_4 \ldots A_{2i-2} A_{2i} \ldots A_{2n}$.
Stress on upper boom opposite,	$r_2 \; r_4 \ldots r_{2i-2} \; r_{2i} \ldots r_{2n}$.
Upper apexes, ...	$A_1 A_3 \ldots\ldots A_{2i-1} \ldots\ldots A_{2n-1}$.
Stress on lower boom opposite,	$r_1 \; r_2 \ldots\ldots r_{2i-1} \ldots\ldots r_{2n-1}$.

The stresses on diagonals remote from the centre are $t_1 t_3 t_5 \ldots t_{2i-1} \ldots t_{2n-1}$, and on the adjacent diagonals they are $t_2 t_4 t_6 \ldots t_{2i} \ldots t_{2n}$.

The bending moment at a section through any apex is represented by the letter M with the same subscript which the A at that apex bears, and the shearing force, constant for any interval of the span, is represented by the letter F with two subscripts separated by a comma, which subscripts are those borne by the A's at the two apexes bounding that interval.

If we confine ourselves to the representative triangle (fig. 252), we have generally

Bending moment at upper apex, .	M_{2i-1}.
Stress on a lower boom, . .	$r_{2i-1} = M_{2i-1} \div h$.
Bending moment at under apex, .	M_{2i}.
Stress on an upper boom, . .	$r_{2i} = M_{2i} \div h$.
Shearing force in the i^{th} bay, . .	$F_{2i-2,\,2i}$.
Stress on a diagonal remote from centre, ...	$\dfrac{\sqrt{(h^2+c^2)}}{h} F_{2i-2,\,2i}$.
Stress on a diagonal adjacent to centre, ...	$\dfrac{\sqrt{\left\{h^2+\left(\dfrac{L}{2n}-c\right)^2\right\}}}{h} F_{2i-2,\,2i}$.

The values are given irrespective of sign, the first two are obtained by dividing the bending moment at the section through the opposite apex by the depth of the girder. The upper booms are always struts and the lower ties, the girder being simply supported at its ends and not being continuous over its supports. The second pair of quantities are obtained by multiplying the shearing stress in the bay by the secant of the slope of the diagonal brace to the vertical; as already pointed out, the first of the pair of members is usually a strut, and the second a tie; taking all loads into account, the stresses on them due to moving loads may be thrust or tension.

Here it may be remarked that the girder might be turned over when all the pieces as struts and ties would interchange. It is also to be remarked here that all loads, even the dead loads, are to be looked upon as something apart altogether from the girder itself, and that the loads are all *reduced* to the joints of the lower boom. For this reason F is taken as constant from A_{2i-2} to A_{2i}. The weight of the girder itself reckoned uniform might be more accurately apportioned among all the joints, the top ones as well as the under ones, when, in consequence, $F_{2i-2, 2i-1}$ and $F_{2i-1, 2i}$ would differ by the amount concentrated at the upper joint A_{2i-1}. The weight of the girder is however only a part of the dead load and the last to be determined, and, in order to draw a comparison with other systems of trussing, it is better to confine the loads to the lower joints only.

Dead Load.—Let the dead load be w tons per inch of span uniformly distributed on the lower boom, then the total load is $W = wL$ tons where all lineal dimensions as L and h are to be taken in inches. This load when reduced to the joints of the lower boom gives a share $W \div 2n$, concentrated at each of the lower joints except the two end ones which have only a half share each, and those are directly over the abutments, so that the reaction of the left abutment on the left end of the truss is

$$\frac{W}{2} - \frac{wL}{4n} = \frac{W}{2}\left(1 - \frac{1}{2n}\right).$$

The shearing forces are

$$F_{0,1} = F_{1,2} = \frac{W}{2}\frac{2n-1}{2n},$$

$$F_{2,3} = F_{3,4} = \frac{W}{2}\frac{2n-1}{2n} - \frac{W}{2n},$$

$$F_{4,5} = F_{5,6} = \frac{W}{2}\frac{2n-1}{2n} - 2\cdot\frac{W}{2n},$$

. . . .

$$F_{2i-2,\,2i-1} = F_{2i-1,\,2i} = \frac{W}{2}\frac{2n-1}{2n} - (i-1)\frac{W}{2n} = \frac{W}{2}\left(\frac{2n+1}{2n} - \frac{i}{n}\right).$$

The bending moments are

$$M_1 = \frac{W}{2}\frac{2n-1}{2n}c,$$

$$M_3 = \frac{W}{2}\frac{2n-1}{2n}\left(\frac{L}{2n}+c\right) - \frac{W}{2n}c,$$

$$M_5 = \frac{W}{2}\frac{2n-1}{2n}\left(2\cdot\frac{L}{2n}+c\right) - \frac{W}{2n}\left(\frac{L}{2n}+2c\right),$$

$$M_9 = \frac{W}{2}\frac{2n-1}{2n}\left(4\cdot\frac{L}{2n}+c\right) - \frac{W}{2n}\left(\overline{1+2+3}\,\frac{L}{2n}+4c\right),$$

. . . .

$$M_{2i-1} = \frac{W}{2}\frac{2n-1}{2n}\left\{(i-1)\frac{L}{2n}+c\right\} - \frac{W}{2n}\left\{\frac{(i-2)(i-1)}{2}\frac{L}{2n}+(i-1)c\right\}.$$

The numerical coefficient of the second last term being the sum of the natural numbers from 1 to $(i-2)$. Then by reduction,

$$M_{2i-1} = \frac{WL}{4n}(i-1)\left(\frac{2n+1}{2n} - \frac{i}{2n}\right) + \frac{Wc}{2}\left(\frac{2n+1}{2n} - \frac{i}{n}\right),$$

$$M_2 = \frac{W}{2}\frac{2n-1}{2n}\frac{L}{2n},$$

$$M_4 = \frac{W}{2}\frac{2n-1}{2n}\cdot 2\cdot\frac{L}{2n} - \frac{W}{2n}\cdot\frac{L}{2n},$$

$$M_8 = \frac{W}{2}\frac{2n-1}{2n}\cdot 4\cdot\frac{L}{2n} - \frac{W}{2n}(1+2+3)\frac{L}{2n},$$

. . . .

$$M_{2i} = \frac{W}{2}\frac{2n-1}{2n}\cdot i\cdot\frac{L}{2n} - \frac{W}{2n}\frac{(i-1)\,i}{2}\frac{L}{2n}.$$

The numerical coefficient of the last term being the sum of the natural numbers from 1 to $(i-1)$. Then by reduction,

$$M_{2i} = \frac{WL}{4n} i \left(1 - \frac{i}{2n}\right).$$

Substituting these values in the preceding expressions for the stresses on the booms and diagonals, and arranging in ascending powers of i, we have

$$r_{2i-1} = \frac{WL}{4nh}\left(-\frac{2n+1}{2n} + \frac{n+1}{n} i - \frac{i^2}{2n}\right) + \frac{Wc}{2h}\left(\frac{2n+1}{2n} - \frac{i}{n}\right),$$

$$r_{2i} = \frac{WL}{4nh}\left(i - \frac{i^2}{2n}\right),$$

$$t_{2i} = \frac{W}{2h}\left(\frac{2n+1}{2n} - \frac{i}{n}\right)\sqrt{\left\{h^2 + \left(\frac{L}{2n} - c\right)^2\right\}},$$

$$t_{2i-1} = \frac{W}{2h}\left(\frac{2n+1}{2n} - \frac{i}{n}\right)\sqrt{(h^2 + c^2)}.$$

The strength of the material is f tons per square inch. It is this unit which demands that the lineal dimensions shall be in inches. It is reckoned to be of the same value with respect to tension and thrust. It will be shown afterwards how a correction can be made if the material be not homogeneous in strength.

Dividing r_{2i-1} by f we have, in square inches, the theoretical area of the cross-section of the i^{th} lower boom $A_{2i-2}A_{2i}$, reckoning from the left abutment, that is, of the boom which is the base of the i^{th} or specimen triangle (fig. 252). Multiplying by $L \div 2n$, its length, we have a general expression, containing the integer variable i, for the volume in cubic inches of that boom. To i in that general expression is to be assigned the values 1, 2, . . . n and the whole summed up, giving then the volume of the left half of the lower boom. Doubling so as to include the right half we have the theoretical volume of the lower boom in cubic inches to resist the dead load

$$\begin{aligned}
V_{\substack{\text{under booms}\\ \text{dead load}}} &= 2\sum_{i=1}^{i=n}\left[\frac{L}{2n}\,\frac{r_{2i-1}}{f}\right] \\
&= \frac{WL^2}{4n^2fh}\sum_1^n\left[-\frac{2n+1}{2n}+\frac{n+1}{n}\,i-\frac{i^2}{2n}\right] \\
&\qquad\qquad + \frac{WLc}{2nfh}\sum_1^n\left[\frac{2n+1}{2n}-\frac{i}{n}\right] \\
&= \frac{WL^2}{4n^2fh}\left\{-\frac{2n+1}{2}+\frac{n+1}{n}\,\frac{n(n+1)}{2}-\frac{n(n+1)(2n+1)}{12n}\right\} \\
&\qquad\qquad + \frac{WLc}{2nfh}\left\{\frac{2n+1}{2}-\frac{n(n+1)}{2n}\right\} \\
&= \frac{WL^2}{48n^2fh}(n-1)(4n+1)+\frac{WLc}{4fh}.
\end{aligned}$$

In summing, terms not containing i are simply to be multiplied by n; those containing i involve the summation of the natural numbers from 1 to n, while the term containing i^2 involves the summation of the squares of the natural numbers from 1 to n.

The theoretical volume of the upper booms from the left end to the centre of the span is obtained in the same way, only in summing for values of i from 1 to n we include a portion of the upper central boom extending a distance c past the centre of the girder. Hence a deduction equal to c multiplied by the cross-section of the middle upper boom is to be subtracted, before doubling to include the right half of the girder.

$$\begin{aligned}
V_{\substack{\text{upper booms}\\ \text{dead load}}} &= 2\sum_{i=1}^{i=n}\left[\frac{L}{2n}\,\frac{r_{2i}}{f}\right]-2c\,\frac{r_{2n}}{f} \\
&= \frac{WL^2}{4n^2fh}\sum_1^n\left[i-\frac{i^2}{2n}\right]-\frac{WLc}{2nfh}\left[i-\frac{i^2}{2n}\right]_{i=n} \\
&= \frac{WL^2}{4n^2fh}\left\{\frac{n(n+1)}{2}-\frac{n(n+1)(2n+1)}{12n}\right\}-\frac{WLc}{2nfh}\cdot\frac{n}{2} \\
&= \frac{WL^2}{48n^2fh}(n+1)(4n-1)-\frac{WLc}{4fh}.
\end{aligned}$$

It is remarkable that the sum of the volumes of all the booms both upper and under is independent of c, that is, of the shape or form of the triangle. It amounts to

$$\frac{WL^2}{6fh}\left(1-\frac{1}{4n^2}\right),$$

a quantity continually increasing with n, but sensibly constant for values of n above 10. It has its greatest value when n is infinite, that is, for the ideal plate-girder with areas of flanges everywhere proportional to the bending moment. It is verified by considering that particular case in which the load is uniformly distributed, the maximum bending moment one-eighth of the product of the load and the span, and the average bending moment two-thirds of that maximum. Dividing that average bending moment by h and by f, and multiplying by $2L$ the joint length of the flanges, we get $\frac{WL^2}{6fh}$.

For the diagonals *remote* from the centre of the girder,

$$\begin{aligned} V_{\substack{\text{remote diagonal}\\ \text{dead load}}} &= 2\sum_{i=1}^{i=n}\left[\frac{t_{2i-1}}{f}\sqrt{h^2+c^2}\right] \\ &= \frac{W}{fh}(h^2+c^2)\sum_1^n\left[\frac{2n+1}{2n}-\frac{i}{n}\right] \\ &= \frac{W}{fh}(h^2+c^2)\left(\frac{2n+1}{2}-\frac{n(n+1)}{2n}\right) \\ &= \frac{Wn}{2fh}(h^2+c^2). \end{aligned}$$

And for the *adjacent* diagonals similarly,

$$V_{\substack{\text{adjacent diagonal}\\ \text{dead load}}} = \frac{Wn}{2fh}\left\{h^2+\left(\frac{L}{2n}-c\right)^2\right\}.$$

Adding these four quantities, we find the theoretical volume of the truss in cubic inches, to resist the dead load of W tons uniformly spread on the lower boom to be, after reduction,

$$V_{\substack{\text{triangle truss}\\ \text{dead load}}} = \frac{W}{f}\left\{nh+\frac{L^2(n+1)(4n-1)-24n^3c\left(\frac{L}{2n}-c\right)}{24n^2h}\right\}. \quad (1)$$

Consider now the effect upon this volume of varying only the shape or form of the triangle. The only variable is c, and it will be seen that the volume is a minimum when the product $c\left(\dfrac{L}{2n} - c\right)$ is a maximum which occurs when the factors are equal, that is, when the perpendicular from the vertex upon the base bisects it. Hence the isosceles, or, as it is sometimes called, the Warren truss (fig. 253), is the most economical form of triangular trussing to resist a dead load. Substitute, then, in (1), c = half a bay $= L \div 4n$, and for convenience put $h = rL$, where r is the fractional depth of the span, or ratio of depth to span, and we have, after reduction,

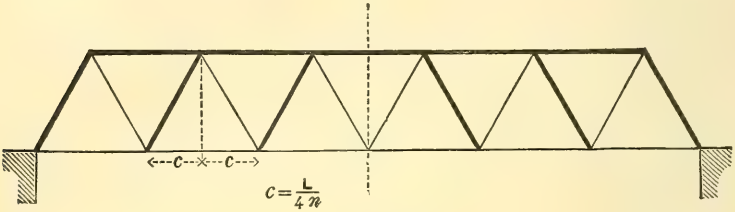


Fig. 253.

$$V_{\substack{\text{isosceles}\\ \text{dead load}}} = \frac{WL}{f}\left\{nr + \frac{8n^2 + 3n - 2}{48n^2}\cdot\frac{1}{r}\right\}. \qquad (2)$$

A form of bracing that rivals the isosceles is the rectangular of which there are two types; in one the diagonals remote from the centre (shortly the struts) are vertical; in the other, those adjacent to the centre (shortly the ties) are vertical (figs. 254, 255). Their volumes are obtained from equation (1) by substituting $c = 0$, and $c = L \div 2n$ respectively, and what is remarkable both have the *same* value, so that one expression serves for the volume of either. It is

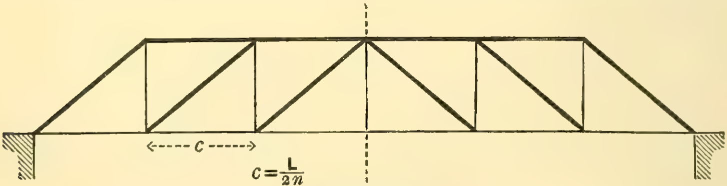


Fig. 254.

$$V_{\substack{\text{rectangular}\\ \text{dead load}}} = \frac{WL}{f}\left\{nr + \frac{(n+1)(4n-1)}{24n^2}\cdot\frac{1}{r}\right\}. \qquad (3)$$

To resist a dead load the Warren truss is considerably more economical than the rectangular truss when the number of bays is small. Thus if both have a depth one-tenth of the span and a common span, and if there be only two bays in the span, then the Warren has an economy of 33 per cent. as may be seen from the tabulated results which follow, or directly by substituting $n = 1$ and $r = \frac{1}{10}^{th}$ into equations (2) and (3). When, however, $n = 4$ or 8, between which it will usually lie in bridge work, the percentage of economy is reduced to 7 and 3. Now the rectangular joints will probably recover this economy.

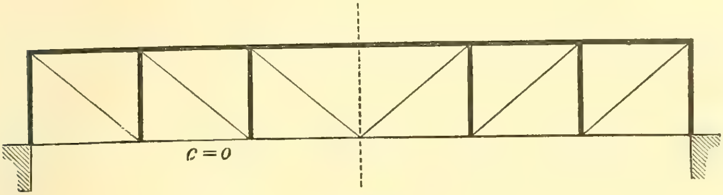
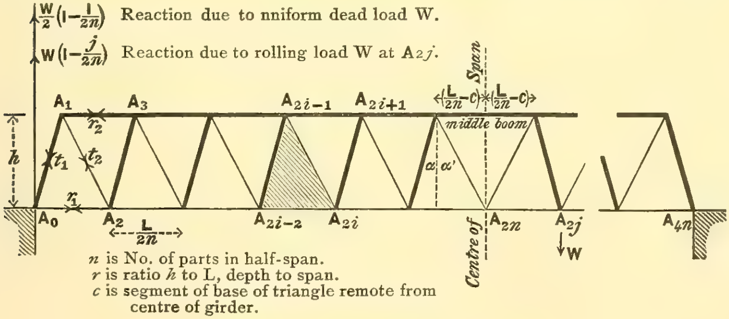


Fig. 255.

The economical values of n, and especially of r, the ratio of depth to span, will be discussed after the live loads have been dealt with.

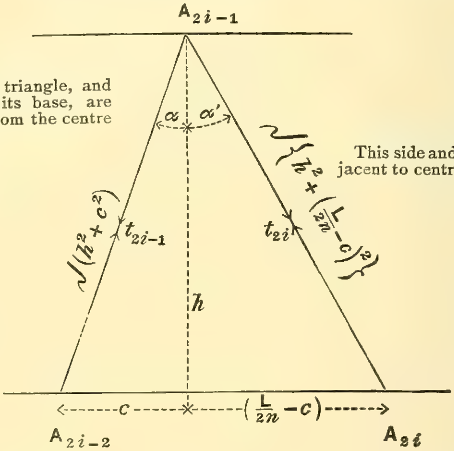
Live load.—The live load is reduced to an *equivalent rolling load*, that is, a load rolling on one wheel and slowly making a transit of the girder and producing a maximum bending moment at the centre, and a maximum shearing force at the end, of the same magnitudes as the maximum values of those quantities produced by the actual live load. Thus $R = \frac{1}{2}wL$ tons is the equivalent rolling load to a distributed uniform load of w tons


$\frac{W}{2}(1-\frac{1}{2n})$ Reaction due to nniform dead load W.
$W(1-\frac{j}{2n})$ Reaction due to rolling load W at A_{2j}.
A_1
A_3
A_{2i-1}
A_{2i+1}
$(\frac{L}{2n}-c)$
$(\frac{L}{2n}-c)$
Centre of Span
r_2
t_1
t_2
r_1
h
middle boom
α
α'
A_0
A_2
$\frac{L}{2n}$
A_{2i-2}
A_{2i}
A_{2n}
A_{2j}
W
A_{4n}
n is No. of parts in half-span.
r is ratio h to L, depth to span.
c is segment of base of triangle remote from centre of girder.

Triangular Trussing.—(Fig. 251 *bis.*)

per inch, which covers the girder during the transit, but if the transits be made quickly a percentage is to be added varying from 20 to 100, the greater for the shorter span and swifter load. In the same way a moving load distributed on wheels in any proportion can be reduced to an equivalent rolling load.

Let R roll on the lower member of the truss (fig. 251). At any instant it is at the apex A_{2j}, and by assigning to j the integral values 1 to $2n$ the transit is completed, R passing in jumps from apex to apex. It is not necessary to consider positions of R intermediate between two apexes. For if the position of R be more commanding at one apex than at the next, any intermediate position throws a fraction to each of these apexes, so that only a fraction of R is in the more commanding position. By *commanding* is meant effective in the matter of producing stress on any bar of the girder under consideration for the time being.



This side of the triangle, and the segment c of its base, are said to be remote from the centre of girder.

This side and segment are adjacent to centre of girder.

Form of the triangle varies with c.

Position of triangle varies with i.

THE i^{th} TRIANGLE FROM LEFT END OF GIRDER.—(Fig. 252 *bis.*)

The supporting forces at the left and right abutments respectively are

$$P = R\left(1 - \frac{j}{2n}\right) \quad \text{and} \quad Q = R\,\frac{j}{2n}.$$

The shearing force on any bay is a maximum, first when the load rolls on from the right end of the girder and stops at the right end of that bay, and again when the load rolls on from the left end of the girder and stops at the left end of the bay. For the i^{th} bay reckoned from the left abutment, that is, for a cross-section through the i^{th}, or representative triangle (fig. 252), the two maxima values of the shearing force are $P_{j=i}$ and $Q_{j=i-1}$.

But since we only consider bays on the left half of the girder the greatest value of i is n. So that

$$i < n + \tfrac{1}{2}, \quad 2i < 2n + 1, \quad 2i - 1 < 2n, \quad i - 1 < 2n - i;$$

therefore $\dfrac{i-1}{n} < \left(1 - \dfrac{i}{2n}\right)$, that is, $Q_{j=(i-1)} < P_{j=i}$.

Hence $F_{2i-1, 2i} = P_{j=i} = R\left(1 - \dfrac{i}{2n}\right)$ a max.

$$V_{\substack{\text{diagonal remote}\\\text{rolling load}}} = 2\frac{(h^2+c^2)}{fh}\sum_{i=1}^{i=n} R\left(1 - \frac{i}{2n}\right)$$

$$= \frac{2R(h^2+c^2)}{fh}\left(n - \frac{n(n+1)}{4n}\right)$$

$$= \frac{R}{f}\left\{\frac{3n-1}{2}h + \frac{(3n-1)c^2}{2h}\right\}.$$

For the other diagonals replace c by the other segment of the base, and

$$V_{\substack{\text{diagonal adjacent}\\\text{rolling load}}} = \frac{R}{f}\left\{\frac{3n-1}{2}h + \frac{3n-1}{2h}\left(\frac{L}{2n} - c\right)^2\right\}.$$

The stress on an upper boom A_{2i-1}, A_{2i+1} is calculated by dividing h out of M_{2i}. Now M_{2i} is evidently greatest when R comes to the apex A_{2i}, at which the bending moment is being calculated, that is, when $j = i$. Then

$$[M_{2i}]_{j=i} = \frac{R}{L}\frac{i \,.\, L}{2n} \,.\, \frac{(2n-i)L}{2n} \text{ a max.} = \frac{RL}{2n}i\left(1 - \frac{i}{2n}\right).$$

$$V_{\substack{\text{upper booms}\\\text{rolling load}}} = 2\frac{L}{2n}\,.\,\frac{1}{fh}\sum_{i=1}^{i=n} M_{2i}]_{j=i} - 2c\,.\,\frac{1}{fh}[M_{2i}]_{j=i=n}$$

$$= \frac{RL}{2n^2fh}\sum_{1}^{n}\left[i - \frac{i^2}{2n}\right] - \frac{RLc}{2fh}$$

$$= \frac{RL}{2n^2fh}\left\{\frac{n(n+1)}{2} - \frac{n(n+1)(2n+1)}{12n}\right\} - \frac{RLc}{2fh}$$

$$= \frac{RL^2(n+1)(4n-1)}{24n^2fh} - \frac{RLc}{2fh}.$$

In summing from 1 to n, the left half of the central upper boom is included as if it were of the common length $L \div 2n$, but it is c shorter than this: hence the deduction.

In dealing with the lower boom $A_{2i-2}A_{2i}$, we take the bending moment M_{2i-1} at a section through the apex A_{2i-1} opposite the boom. But since R only comes to the joints of the lower booms, as already explained, the closest that R can come to the section is either A_{2i-2} or A_{2i}, one or other. Each of these is the most commanding position of R on its own side of the section. Hence we have two rival maximum bending moments, namely,

$$[M_{2i-1}]_{j=i} = P_{j=i} \times \left\{(i-1)\frac{L}{2n} + c\right\}$$

$$= R\left(1 - \frac{i}{2n}\right)\left\{(i-1)\frac{L}{2n} + c\right\} \text{ a max.}$$

And $$[M_{2i-1}]_{j=(i-1)} = Q_{j=(i-1)} \times \left\{(2n-i+1)\frac{L}{2n} - c\right\}$$

$$= R\,\frac{i-1}{2n}\left\{(2n-i+1)\frac{L}{2n} - c\right\} \text{ a max.}$$

Putting shortly M and M' for these two maxima we get, by reduction,

$$M' = M - R\left(c\,\frac{2n-1}{2n} - \frac{L(i-1)}{4n^2}\right). \tag{4}$$

So that $$M > = < M',$$

according as $$c\,\frac{2n-1}{2n} > = < L\frac{i-1}{4n^2},$$

according as $$c > = < \frac{L}{4n}\,\frac{i-1}{n-\frac{1}{2}}. \tag{5}$$

Hence the maximum stress on i^{th} lower boom A_{2i-2}, A_{2i} is either

$$r_{2i-1} = \frac{(M_{2i-1})_{j=i}}{h}, \quad \text{or} \quad r'_{2i-1} = \frac{(M_{2i-1})_{j=i-1}}{h}$$

accordingly.

In the most general case, the lower booms of the left half of the girder arrange themselves into two categories, the first reaching from the left abutment to the lower apex A_{2q}, and the second category extending from A_{2q} to A_{2n}, the central lower apex. For any boom in the first category, the load R must be at its end adjacent to the centre of the girder, and for a boom in the second category R must be at its end remote from the centre. Or, in symbols, for integer values of i from 1 to q, let r_{2i-1} be greater than r'_{2i-1}, while for the remaining integer values of i from $q+1$ to n let r_{2i-1} be not greater than r'_{2i-1}. The inequality (5) separates into two branches with

$$c > \frac{L}{4n}\left[\frac{i-1}{n-\frac{1}{2}}\right]_{i=1}^{i=q}, \quad \text{but} \quad c \leqq \frac{L}{4n}\left[\frac{i-1}{n-\frac{1}{2}}\right]_{i=q+1}^{i=n}. \qquad (6)$$

$$V_{\substack{\text{lower boom}\\ \text{rolling load}}} = 2\frac{L}{2nf}\left\{\sum_{i=1}^{i=q} r_{2i-1} + \sum_{i=q+1}^{i=n} r'_{2i-1}\right\}$$

$$= \frac{L}{nf}\left\{\sum_{i=1}^{i=n} r_{2i-1} + \frac{R}{h}\sum_{i=q+1}^{i=n}\left[\frac{L(i-1)}{4n^2} - \frac{c(2n-1)}{2n}\right]\right\}.$$

Expressing r' in terms of r, and a remainder from equation (4). Arranging next in powers of i, and then summing

$$V_{\substack{\text{lower boom}\\ \text{rolling load}}} = \frac{RL}{nfh}\left\{\sum_{1}^{n}\left(1-\frac{i}{2n}\right)\left(c+\frac{L(i-1)}{2n}\right)\right.$$

$$\left. + \sum_{q+1}^{n}\left(\frac{L(i-1)}{4n^2} - \frac{c(2n-1)}{2n}\right)\right\}$$

$$= \frac{RL}{nfh}\sum_{1}^{n}\left[\frac{2nc-L}{2n} + \frac{(2n+1)L-2nc}{4n^2}i - \frac{L}{4n^2}i^2\right]$$

$$+ \frac{RL}{nfh}\sum_{q+1}^{n}\left[\frac{L}{4n^2}i - \frac{L+4n^2c-2nc}{4n^2}\right]$$

$$= \frac{RL}{nfh}\left\{\frac{2nc-L}{2} + \frac{(2n+1)L-2nc}{4n^2}\cdot\frac{n(n+1)}{2}\right.$$

$$\left. - \frac{L}{4n^2}\,\frac{n(n+1)(2n+1)}{6}\right\}$$

$$+ \frac{RL}{nfh}\left\{\frac{L}{4n^2}\,\frac{(n-q)(n+q+1)}{2} - \frac{L+4n^2c-2nc}{4n^2}(n-q)\right\}$$

$$= \frac{RL^2}{12hn^2f}(n-1)(2n-1) + \frac{RLc}{4hnf}(3n-1)$$

$$+ \frac{WL}{2nfh}(n-q)\left(\frac{L(n+q-1)}{4n^2} - \frac{c(2n-1)}{n}\right).$$

Total theoretical volume of triangular trussing to resist a rolling load R is

$$V_r = \frac{R}{f}\left\{(3n-1)h + \frac{(n+1)(4n-1)L^2 - 3n(5n-1)Lc + 12n^2(3n-1)c^2}{12n^2h}\right\}$$
$$+ \frac{RL}{2nfh}(n-q)\left(\frac{L(n+q-1)}{4n^2} - \frac{c(2n-1)}{n}\right). \quad (7)$$

To find the economical form of triangle to resist the rolling load, assume the quantities in the expression (7) to be all constant except c, and equate to zero its first differential coefficient with respect to c, and we have

$$0 = \frac{d}{dc}V_r$$
$$= -3(5n-1)L + 24nc(3n-1) - 6(n-q)\frac{2n-1}{n}.L.$$

Or $\frac{d}{dc}V_r = 0$, $\frac{d^2}{dc^2}V_r = 24n(3n-1) =$ a positive quantity, and V_r is therefore a minimum when

$$c = \frac{L}{4n}\left\{\frac{3}{2} - \frac{q}{n}\cdot\frac{2n-1}{3n-1}\right\}. \quad (8)$$

But $$c > \frac{L}{4n}\left\{\frac{q-1}{n-\frac{1}{2}}\right\}, \quad (9)$$

obtained by substituting $i = q$, the greatest integer value, which satisfies the left hand of the inequality (6). Combining equation (8) with the inequality (9) we have

$$\frac{q-1}{n-\frac{1}{2}} < \frac{3}{2} - \frac{q}{n}\,\frac{2n-1}{3n-1},$$

or $$q < \frac{n(3n-1)(6n+1)}{2(10n^2-6n+1)}.$$

Now since q is the largest integer which satisfies the inequality, it may be written

$$q = \left[\frac{n(3n-1)(6n+1)}{2(10n^2-6n+1)}\right] \text{ integer part only.} \quad (10)$$

From the equations (8) and (10) we can now find the economical shape of the triangle in triangular bracing to resist the rolling load R. Unlike the economical shape to resist the dead load which was isosceles in every girder, the shape to resist the rolling load varies according to the number of equal bays or subdivisions in the span. Assigning to n the integer values 1, 2, &c., and solving the simultaneous equations (10) and (8), we get the results in the table on next page, in which are placed in the

First Column.—The number of equal parts into which the girder is divided by the triangular trussing.

Second Column.—The number of lower booms on the semi-span.

Third Column.—The number of lower booms on semi-span consecutively from the end which fall into the first category. That is, which individually require the rolling load at the end *adjacent* to the centre of the girder for the stress to be a maximum.

Fourth Column.—The length of one of the two segments of the lower boom made by the perpendicular dropped on it from the apex above it. In particular the length of the segment remote from the centre of the girder, which, curiously enough, is always the smaller. It is expressed as a fraction of $L \div 2n$, the length of a boom.

Fifth Column.—The deviation of the foot of that perpendicular from the middle point of the boom, expressed as a percentage of the length of the boom. This percentage is a measure of how far the form of the triangle departs from the isosceles form.

For the 2, 4, 6, and 8 equal parts in the span $q = n$ when the triangle is of the economical shape to resist the rolling load, the lower booms are all in the first category requiring the load at their ends adjacent to the centre of the girder for maximum stresses. The first requires the triangle to be isosceles, for the other three it departs more and more from the isosceles form. For 10 parts in the span the economical shape is again almost isosceles. The greatest departure from the isosceles form is in the girder with 8 equal bays in the span. If the base of one of the triangles be divided into 44 equal parts, the *remote* or shorter segment made by a perpendicular from the vertex occupies 19 parts and the other 25, so that the foot of the perpendicular is distant from the centre of the base 3 of those parts. It may be called a deviation of 7 per cent.

Even in the most marked case, then, the isosceles triangle is *sensibly* the most economical form. As there is always a dead load as well as a rolling load, we may conclude that for the joint load the isosceles bracing is still more *sensibly* the economical shape.

$2n$	n	q	c	Deviation from isosceles form.
2	1	1	$\frac{1}{2}\left(\frac{L}{2n}\right)$	0
4	2	2	$\frac{9}{20}\,\frac{L}{2n}$	5 per cent.
6	3	3	$\frac{7}{16}\,\frac{L}{2n}$	6¼ ,,
8	4	4	$\frac{19}{44}\,\frac{L}{2n}$	7 ,,
10	5	4	$\frac{69}{140}\,\frac{L}{2n}$	¾ ,,
20	10	9	$\frac{66}{145}\,\frac{L}{2n}$	4½ ,,
30	15	13	$\frac{613}{1320}\,\frac{L}{2n}$	4 ,,
40	20	18	$\frac{267}{590}\,\frac{L}{2n}$	4¾ ,,

For isosceles bracing, c = half a bay $= L \div 4n$. Substituting this into the expression (9) we have q the greatest integer which shall satisfy

$$\frac{q-1}{n-\frac{1}{2}} < 1, \quad \text{or} \quad q = n.$$

So that all the lower booms are in the first category no matter what value n may have, that is, they require for maximum stress that the load shall be at the end of the boom adjacent to the centre of the span. Substituting their values into equation (7), and also putting, as before, $h = rL$, we have

$$V_{\substack{\text{isosceles}\\ \text{rolling load}}} = \frac{RL}{f}\left\{(3n-1)\,r + \frac{8n^2+3n-2}{24n^2}\cdot\frac{1}{r}\right\}. \tag{11}$$

For rectangular bracing, the diagonals adjacent to centre of girder being vertical, c = a bay $= L \div 2n$, and $q = n$. For rectangular bracing, diagonals remote from centre being vertical, $c = 0$, and so from (6) q is an integer less than unity, that is, q is zero, so that all the lower booms are in the *second category* requiring the live load to be at the end *remote* from the centre of the span. Substituting into equation (7), the expression in each case reduce to a common value, so that

$$V_{\substack{\text{rectangular}\\ \text{rolling load}}} = \frac{RL}{f}\left\{(3n-1)r + \frac{8n^2+9n-5}{24n^2}\cdot\frac{1}{r}\right\}. \qquad (12)$$

In the expressions for the theoretical volumes of trusses given in equations (1), (2), (3), (7), (11), and (12), if all the quantities be constant except n, the volume will be seen to vary directly with n. Practical considerations fix the values of n between very narrow limits, the chief being the necessity for *stiffness* which is of paramount importance, especially with moving loads. The stiffness of a girder is of two kinds—stiffness of the girder as a whole and local stiffness.

Now these two affect the question of economy in opposite ways, for increasing the depth of the girder increases the stiffness of the girder as a whole, but makes the struts so long that to be effective they must be greatly stiffened individually. Considering these jointly, a depth of girder from one-eighth to one-tenth of the span is most suitable for iron and steel girders. Where the booms are timber which demands a large scantling for strength, and so affording at the same time local stiffness, the depths may be greater in proportion to the span. In the American early practice with timber, depths of even one-fourth the span were used. In the British early practice, in passing from cast-iron to solid plate girders, depths as low as one-fourteenth the span were used. The modern practice in both countries is now from one-eighth to one-tenth of the span.

Economical Depth.—Having discussed the economical shape and subdivision, it now remains to consider the variable r in the six expressions above. They are all of the form $\dfrac{WL}{f} \times u$

where
$$u = \alpha r + \beta\frac{1}{r}, \qquad (13)$$

and what is more, the theoretical volume of all trusses without redundant bars reduces to this form. There is a value of r which necessarily makes u a minimum because with very large values of r, u is sensibly αr, *increasing* with r, while with small values of r, u is $\beta \div r$ *increasing* as r decreases. Put

$$0 = \frac{du}{dr} = \alpha - \beta\frac{1}{r^2}, \quad \text{or} \quad r = \sqrt{\frac{\beta}{\alpha}}, \tag{14}$$

making $$u = 2\sqrt{\alpha \, . \, \beta} \text{ a minimum.} \tag{15}$$

TABLES OF THEORETICAL VOLUMES OF TRUSSES.

The quantities u are given in the first table, both for isosceles or Warren bracing, and for right-angled bracing to resist a rolling load W, and a uniform dead load W in alternate columns. The five rows are for 2, 4, 8, 16, and 32 subdivisions or bays on the span, and in each pane of the table there appears four values of u corresponding to the four depths expressed as fractions of the span, viz., $r = \frac{1}{15}, \frac{1}{10}, \frac{1}{8}$, or $\frac{1}{4}$, respectively. In each 'pane' there appears also a heavy-faced number, being the minimum value of u by equation (15), while the corresponding value of r by equation (14) appears with an asterisk (*) at it. These heavy-faced numbers are put in their proper sequence among the plain-faced ones, so that one can see at a glance whether this most economical depth is feasible, that is, whether it falls within those demanded by stiffness. This, of course, decides whether the economy of material can be realised or not. For instance, on the first bottom 'pane,' p. 466, $^*r = \cdot 103$, while on the second, $^*r = \cdot 085$, so that jointly for live and dead load the economical depth lies between $r = \frac{1}{10}$ and $r = \frac{1}{8}$, just as demanded by stiffness. On the other hand, consider the last two bottom 'panes' of the second table p. 467, $^*r = \cdot 415$ and $^*r = \cdot 300$, so that the economical depth for the Bollmann trusses is quite out of the question, and for the Fink truss too it can hardly be realized if we may use such a word.

If the span be divided into an *odd* number of parts $(2m + 1)$, we find the theoretical volumes to be as under.

Warren Truss.—Dead and rolling load.

$$V_d = \frac{WL}{f}\frac{m(m+1)}{2m+1}\left(2r + \frac{8m+7}{6(2m+1)^2}\frac{1}{r}\right). \tag{2*}$$

$$V_r = \frac{WL}{f}\frac{m}{2m+1}\left(2(3m+2)r + \frac{(m+1)(8m+7)}{3(2m+1)^2} \, . \, \frac{1}{r}\right). \tag{11*}$$

Rectangular Trusses.—Dead and rolling loads.

$$V_d = \frac{WL}{f}\frac{m(m+1)}{2m+1}\left(2r + \frac{4m+5}{3(2m+1)^2}\cdot\frac{1}{r}\right). \qquad (3^*)$$

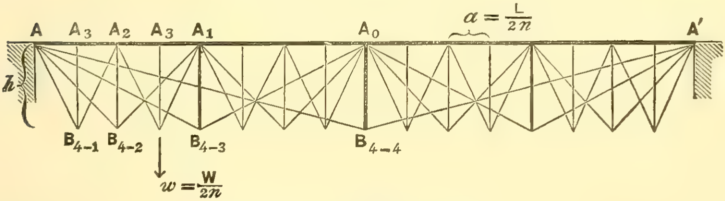
$$V_r = \frac{WL}{f}\frac{m}{2m+1}\left(2(3m+2)r + \frac{8m^2+21m+10}{3(2m+1)^2}\cdot\frac{1}{r}\right). \qquad (12^*)$$

For (2*) and (3*) there is no shearing stress in the *odd* middle bay. In (12*) it is assumed that there is only *one diagonal* in the middle bay. If there are a pair acting only one at a time, say both ties, and incapable of acting as struts, then $(8m^2 + 21m + 10)$ is to be replaced by $(8m^2 + 21m + 13)$.

We have not tabulated these quantities, as values sufficiently correct, can be found by proportional parts from those which are tabulated for the even division of the span.

The Fink Truss.

Here the beam AA' (fig. 256) is divided into a number of equal parts by a series of repeated bisections. There is thus



n is No. of bisections.
n is No. of varieties of Trusses.
2^n is No. of parts in span.

Fink Truss.—(Fig. 256).

1 primary truss, 2 secondary trusses, 2^2 tertiary trusses, &c., so that the number of trusses of the i^{th} variety is 2^{n-i} where n is the number of bisections. Hence summing 2^{n-i} for integer values of i from n to 1, we have

$$1 + 2 + 2^2 \ldots 2^{n-1} = 2^n - 1,$$

the total number of apexes of all orders, and therefore also 2^n is the number of equal parts into which the boom is divided.

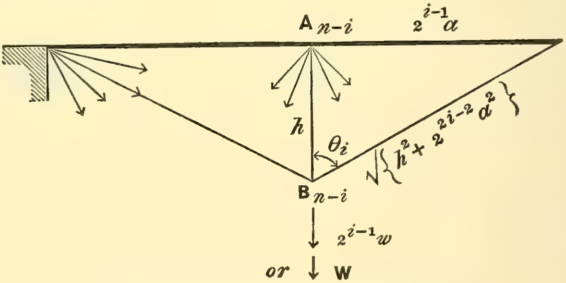
The Dead Load.—If W be the dead load uniformly spread on the boom AA', then the load directly over each vertical strut is

$$w = \frac{W}{2^n}, \quad \text{and} \quad a = \frac{L}{2^n}$$

is the common length of each segment.

Consider an apex of the order B_{n-i}, the stress on the vertical strut is w the load on top of it, and this is all for $i = 1$; but for $i = 2$ there is besides the downward component pull of a pair of tie rods attached to the top of the strut, being between them another w, or in all $2w$. Now a pair of the steepest ties transmit loads with a vertical component w per pair, the next steepest a vertical load of $2w$ per pair, and the third steepest, a vertical component load 2^2w per pair, &c.

The load on the strut A_{n-i}, B_{n-i} (fig. 257) then is $2^{i-1}w$. The pull on each tie from B_{n-i} is $\frac{1}{2} \times 2^{i-1}w \sec\theta_i$.



THE i^{th} VARIETY OF TRUSS.—(Fig. 257).

The thrust on all segments of the boom is constant. At the end A a tie rod from one of each variety of apex comes in. The horizontal component load on the tie from B_{n-i} is $\frac{1}{2}\,2^{i-1}w \tan\theta_i$. Substituting for $\tan\theta_i$, and summing for all values of i from 1 to n so as to include one of each variety, we have

$$\text{thrust on boom} = \frac{wa}{2h}\sum_1^n 2^{2(i-1)} = \frac{wa}{2h}\left(1 + 2^2 + 2^4 \ldots 2^{2(n-1)}\right)$$

$$= \frac{wa}{2h}\,\frac{2^{2n}-1}{3} = \frac{2^n w \,.\, 2^n a}{h}\,.\,\frac{1-(\frac{1}{2})^{2n}}{6}$$

$$= \frac{WL}{h}\,\frac{1-(\frac{1}{2})^{2n}}{6}.$$

Dividing this by f gives the constant sectional area of the boom, and further multiplying by its length, we have

$$\text{volume of boom} = \frac{W}{f}\frac{1-(\tfrac{1}{2})^{2n}}{6}\frac{L^2}{h}. \tag{A}$$

The volume of the strut $A_{n-i}B_{n-i}$ is $2^{i-1}wh \div f$, and multiplying by 2^{n-i} gives us $2^{n-1}\,wh \div f$ as the volume of all the struts of the same variety. But the expression being independent of i shows that *one* strut of the primary truss has the same volume as the *two* struts of the secondary trusses, as the *four* struts of the tertiary trusses, &c. We have to simply multiply the last expression by n, the number of varieties, and

$$\text{volume of struts} = \frac{(2^n w)}{2}\frac{nh}{f} = \frac{W}{2}\frac{nh}{f}. \tag{B}$$

Again, the volume of the pairs of ties from all apexes B_{n-i} is

$$2 \times \text{number} \times \frac{\text{stress}}{f} \times \text{length} = 2 \times 2^{n-i} \times \frac{\tfrac{1}{2}\,2^{i-1}w\sec\theta_i}{f} \times h\sec\theta_i$$

$$= 2^{n-1}\frac{wh}{f}\sec^2\theta_i = 2^{n-1}\frac{wh}{f}\frac{h^2+2^{2i-2}a^2}{h^2}.$$

Summing for values of i from 1 to n,

$$\text{volume of ties} = 2^{n-1}\frac{nwh}{f} + 2^{n-1}\frac{wa^2}{hf}\sum_1^n 2^{2(i-1)}$$

$$= \frac{2^n w}{f}\frac{nh}{2} + \frac{2^n w}{f}\frac{a^2}{2h}(1+2^2+2^4\ldots+2^{2n-2})$$

$$= \frac{W}{f}\frac{nh}{2} + \frac{W}{f}\frac{a^2}{2h}\frac{2^{2n}-1}{3}$$

$$= \frac{W}{f}\left(\frac{nh}{2} + \frac{1-(\tfrac{1}{2})^{2n}}{6}\frac{L^2}{h}\right). \tag{C}$$

Summing (A), (B), and (C), and putting $L = rh$, then the volume of the Fink truss to resist a dead load is

$$V_D = \frac{WL}{f}\left(nr + \frac{1-(\tfrac{1}{2})^{2n}}{3}\cdot\frac{1}{r}\right). \tag{16}$$

The Rolling Load.—When W the rolling load is at the apex A_{n-i}, the load on the strut $A_{n-i}B_{n-i}$ is W, which also is the vertical component load on the pair of ties supporting it at B_{n-i}. The stresses in all pieces from the *other* apexes of the same order, or of higher orders, is zero; while those on struts from apexes of lower order are fractions of W. Multiplying W, the greatest load on a strut, by h its length, and dividing by f, we have its volume, and as they are all alike, we must multiply by $2^n - 1$ their number.

$$\text{Volume of struts} = (2^n - 1)\,h\,\frac{W}{f}. \tag{D}$$

The greatest pull on a tie from B_{n-i} is $\frac{1}{2}W \sec \theta_i$, the area of this tie $W \sec \theta_i \div 2f$, and its length $h \sec \theta_i$, and there are twice as many as there are apexes of order B_{n-i}, that is, there are $2 \times 2^{n-i}$. The volume then is

$$\frac{W}{f}\,2^{n-i}\sec^2\theta_i = \frac{W}{f}\,2^{n-i}\,\frac{h^2 + 2^{2i-2}a^2}{h},$$

or

$$\frac{W}{f}\left(2^{n-i}h + 2^{n+i-2}\,\frac{a^2}{h}\right),$$

which has to be summed for values of i from 1 to n, and

$$\begin{aligned}\text{volume of ties} &= \frac{Wh}{f}(1 + 2 + 2^2 \ldots + 2^{n-1}) \\ &\quad + \frac{Wa^2}{fh}(2^{n-1} + 2^n \ldots + 2^{2n-2}) \\ &= \frac{W}{f}\left\{(2^n - 1)\,h + (2^n - 1)(\tfrac{1}{2})^{n+1}\,\frac{2^{2n}a^2}{h}\right\} \\ &= \frac{W}{f}\left((2^n - 1)\,h + (2^n - 1)\,(\tfrac{1}{2})^{n+1}\,\frac{L^2}{h}\right).\end{aligned} \tag{E}$$

The greatest thrust on the boom occurs when W is at A_0 the centre of the span. It is $\frac{1}{2}W \times \frac{1}{2}L \div h$, dividing by f and multiplying by L,

$$\text{volume of boom} = \frac{W}{f}\,\frac{1}{4}\,\frac{L^2}{h}. \tag{F}$$

Summing (D), (E), and (F), and putting $L = rh$, the volume of the Fink truss to resist a rolling load W is

$$V_R = \frac{WL}{f}\left(2\left(2^n - 1\right)r + \frac{3 - (\frac{1}{2})^{n-1}}{4} \cdot \frac{1}{r}\right). \tag{17}$$

If the dead load be *below* the truss, each vertical strut is relieved of a load w, but is not altogether relieved of load as Lévy assumes. Diminishing then the volume of each strut by $wh \div f$, and multiplying by $2^n - 1$ the number of struts, then we have the proper deduction to be made from V_D to be

$$\frac{2^n - 1}{2^n} \frac{Wh}{f},$$

which leaves the volume of the Fink truss to resist a dead load *below* the truss

$$V_d = \frac{WL}{f}\left\{\left(n - 1 + (\tfrac{1}{2})^n\right) r + \frac{1 - (\frac{1}{2})^{2n}}{3} \cdot \frac{1}{r}\right\}. \tag{16*}$$

And again for the rolling load below the Fink truss, we must not leave out the whole volume of the vertical struts as Lévy inadvertently does. For the rolling load W below the truss, the greatest stress on a strut occurs when W is at the foot of the next adjoining strut. The amount is then

$$W\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} \cdots \frac{1}{2^{i-1}}\right) = W\left(1 - \left(\frac{1}{2}\right)^{i-1}\right).$$

Hence the volume of all such struts is

$$2^{n-i}\, W\left(1 - \left(\frac{1}{2}\right)^{i-1}\right)\frac{h}{f} = \frac{Wh}{f}\, 2^n \left\{\left(\frac{1}{2}\right)^i - 2\left(\frac{1}{2}\right)^{2i}\right\}.$$

Summing for all the n varieties

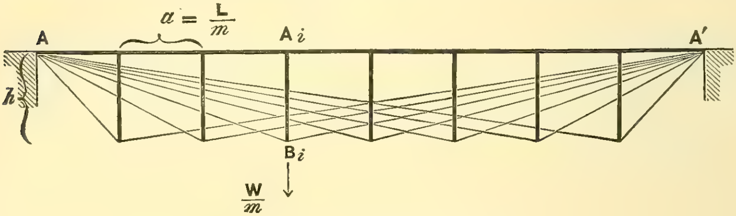
$$\text{volume of struts} = \frac{Wh}{f}\, 2^n \left\{\sum_1^n \left(\frac{1}{2}\right)^i - 2\sum_1^n \left(\frac{1}{2}\right)^{2i}\right\}$$

$$= \frac{Wh}{f}\left\{\frac{2^n + (\frac{1}{2})^{n-1}}{3} - 1\right\}. \tag{D*}$$

Summing (D*), (E), and (F), and putting $L = rh$, we find the volume of the Fink truss to resist a load W rolling below it to be

$$V_r = \frac{Wh}{f}\left\{\left(\frac{2^{n+2}+(\frac{1}{2})^{n-1}}{3} - 2\right)r + \frac{3-(\frac{1}{2})^{n-1}}{4}\cdot\frac{1}{r}\right\}. \qquad (17^*)$$

We have not thought it necessary to tabulate the values for (16*) and (17*), as the Fink and Bollman trusses are only given here for comparison, and for an exercise in analysis. The trusses themselves are out of date and far from economical for the rolling load. The economy for dead load, too, is only apparent, as it could only be realised with a depth of truss which is impracticable, as it would involve a stiffening of long struts that would eat up the economy.

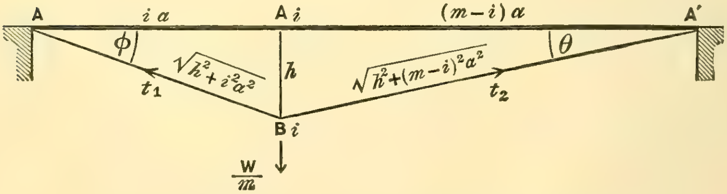


$m = 2n$ is No. of parts in span.

Bollman Truss.—(Fig. 258.)

The Bollman Truss.

In this system of trussing (fig. 258) the upper boom AA' is divided into m equal parts; at each joint is a vertical strut whose foot is tied back to the two abutments.



The i^{th} Truss from Left Abutment.—(Fig. 259.)

Dead load.—Consider the i^{th} truss from the left abutment (fig. 259).

$$\text{volume of struts} = \text{number} \times \text{common stress} \times \frac{\text{length}}{f}$$

$$= (m-1)\frac{W}{m}\frac{h}{f} = \frac{m-1}{m} \cdot \frac{W}{f} \cdot h\,; \qquad \text{(G)}$$

$$\sin AB_iA' = \sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi,$$

and using the surds in fig. 259, which are derived by the forty-seventh proposition of Euclid I., we have

$$\sin(\theta+\phi) = \frac{mha}{\sqrt{h^2+i^2a^2}\,\sqrt{h^2+(m-i)^2a^2}}\cdot$$

For equilibrium at B_i we have

$$t_1 : t_2 : \frac{W}{m} :: \sin A_iB_iA' : \sin A_iB_iA : \sin AB_iA'$$

$$:: \cos\theta : \cos\phi : \sin(\theta+\phi).$$

Using the surds in the fig. 259, we have

$$t_1 = \frac{W}{m}\frac{(m-i)\sqrt{h^2+i^2a^2}}{mh},$$

and multiplying by the length AB_i and dividing by f we have the volume of

$$AB_i = \frac{W}{m^2hf}(m-i)(h^2+i^2a^2),$$

and also the volume of

$$A'B_i = \frac{Wi}{m^2hf}(h^2+(m-i)^2a^2).$$

Their sum is the volume of the pair of ties from the apex B_i and is

$$\frac{W}{mhf}(h^2+i(m-i)a^2).$$

To sum the first term, which is independent of i, it must be multiplied by $(m-1)$ the number of apexes; the other two terms are to be summed for values of i from 1 to $(m-1)$. Putting L for am finally

$$\text{vol. of ties} = \frac{W}{mhf}\left((m-1)h^2 + ma^2\sum_1^{m-1} i - a^2\sum_1^{m-1} i^2\right)$$

$$= \frac{W}{mhf}\left((m-1)h^2 + ma^2\frac{(m-1)m}{2} - a^2\frac{(m-1)m(2m-1)}{6}\right)$$

$$= \frac{W}{f}\,\frac{m-1}{m}\left(h + \frac{m+1}{6m}\,\frac{L^2}{h}\right). \qquad \text{(H)}$$

Part of the thrust on AA' is the horizontal component of t_1, which is

$$t_1 \cos\phi = \frac{W}{m}\cdot\frac{m-i}{mh}\cdot ia = \frac{W}{mh}\,i\,(m-i)\,\frac{a^2}{L}.$$

Summing for values of i from 1 to $(m-1)$ includes the horizontal pull of one rod from each apex, and

$$\text{thrust on boom} = \frac{W}{mh}\,\frac{a^2}{L}\left(m\sum_1^{m-1} i - \sum_1^{m-1} i^2\right)$$

$$= \frac{W}{mh}\,\frac{a^2}{L}\left(\frac{m^2(m-1)}{2} - \frac{(m-1)\,m\,(2m-1)}{6}\right)$$

$$= \frac{WL}{6h}\,\frac{m^2-1}{m^2}.$$

Multiplying by $L \div f$, we get

$$\text{volume of boom} = \frac{W}{f}\,\frac{m^2-1}{6m^2}\,\frac{L^2}{h}. \qquad \text{(I)}$$

Summing (G), (H), and (I), and putting $m = 2n$ so that n may be the number of parts in *half span* as in the other trusses, also putting $L = rh$, we have volume of Bollman truss to resist a dead load W over the truss to be

$$V_D = \frac{WL}{f}\cdot\frac{2n-1}{n}\left(r + \frac{2n+1}{12n}\cdot\frac{1}{r}\right). \qquad (18)$$

Live load.—Let the live load W come to the point A_i, and it will be seen that the load is now m times what it was for the dead load. Now as W comes to every point in succession, the volume of the struts and of the ties will be obtained by multiplying the expressions (G) and (H) by m, when we have

$$\text{volume of strut} = (m-1)\frac{W}{f}h; \qquad \text{(J)}$$

$$\text{volume of ties} = (m-1)\frac{W}{f}\left(h + \frac{m+1}{6m}\frac{L^2}{h}\right). \qquad \text{(K)}$$

With the boom it is otherwise. When W is at B_i, we get the thrust on the boom by multiplying the former value of the horizontal component of t_1 by m which gives

$$\frac{W}{h}\,i\,(m-i)\,\frac{a^2}{L}.$$

but this is thrust on the boom only as long as W remains at A_i, but $i(m-i)$ is greatest when the factors are equal, so that the maximum thrust on the boom occurs when $i = m - i$ or $i = \frac{1}{2}m$, that is, when W is at the middle point of the boom. Substituting this we have

$$\text{max. thrust on boom} = \frac{W}{h}\,\frac{m^2}{4}\,\frac{a^2}{L}.$$

Dividing this by f to get the sectional area, and multiplying by L, we get

$$\text{volume of boom} = \frac{W}{fh}\,\frac{m^2a^2}{4} = \frac{W}{fh}\,\frac{L^2}{4}. \qquad \text{(L)}$$

Summing (J), (K), and (L), and putting $m = 2n$, and $h = rL$, we have the volume of the Bollman truss to resist a load rolling over it to be

$$V_R = \frac{WL}{f}\left(2(2n-1)\,r + \frac{4n^2+3n-1}{12n}\cdot\frac{1}{r}\right). \qquad (19)$$

If the load be below the truss instead of above it, the volume of the struts (G) and (J) can be left out, as there is no load on them theoretically, so that V_d and V_r are obtained from (18) and (19) by *halving* the first term.

The Tables described on page 456 now follow.

In the span.	Ratio of depth to span, $r = \frac{h}{L}$	ISOSCELES TRUSS.			RIGHT-ANGLED TRUSS.	
		Parts in half-span.	UNIFORM LOAD. $nr + \frac{8n^2 + 3n - 2}{48n^2}\frac{1}{r}$	ROLLING LOAD. $(3n-1)r + \frac{8n^2 + 3n - 2}{24n^2}\frac{1}{r}$	UNIFORM LOAD. $nr + \frac{(n+1)(4n-1)}{24n^2}\frac{1}{r}$	ROLLING LOAD. $(3n-1)r + \frac{8n^2 + 9n - 5}{24n^2}\frac{1}{r}$
2 equal parts.		$n = 1$	$r + \frac{3}{16}\frac{1}{r}$ *r = ·433	$2r + \frac{3}{8}\frac{1}{r}$ *r = ·433	$r + \frac{1}{4}\frac{1}{r}$ *r = ·500	$2r + \frac{1}{2}\frac{1}{r}$ *r = ·500
	$\frac{1}{15}$		2·879	5·758	3·817	7·633
	$\frac{1}{10}$		1·975	3·950	2·600	5·200
	$\frac{1}{8}$		1·625	3·250	2·125	4·250
	$\frac{1}{4}$		1·000	2·000	1·250	2·500
			0·866	**1·732**	**1·000**	**2·000**
4 equal parts.		$n = 2$	$2r + \frac{3}{16}\frac{1}{r}$ *r = ·306	$5r + \frac{3}{8}\frac{1}{r}$ *r = ·274	$2r + \frac{7}{32}\frac{1}{r}$ *r = ·331	$5r + \frac{15}{32}\frac{1}{r}$ *r = ·306
	$\frac{1}{15}$		2·946	5·958	3·415	7·365
	$\frac{1}{10}$		2·075	4·250	2·388	5·188
	$\frac{1}{8}$		1·750	3·625	2·000	4·375
	$\frac{1}{4}$		1·250	2·750	1·375	3·125
			1·225	**2·739**	**1·323**	**3·062**
8 equal parts.		$n = 4$	$4r + \frac{23}{128}\frac{1}{r}$ *r = ·212	$11r + \frac{23}{64}\frac{1}{r}$ *r = ·181	$4r + \frac{25}{128}\frac{1}{r}$ *r = ·221	$11r + \frac{53}{128}\frac{1}{r}$ *r = ·194
	$\frac{1}{15}$		2·961	6·124	3·196	6·944
	$\frac{1}{10}$		2·197	4·694	2·353	5·241
	$\frac{1}{8}$		1·937	4·250	2·063	4·688
			1·696	**3·977**	**1·768**	**4·268**
	$\frac{1}{4}$		1·719	4·188	1·781	4·406
16 equal parts.		$n = 8$	$8r + \frac{89}{512}\frac{1}{r}$ *r = ·147	$23r + \frac{89}{256}\frac{1}{r}$ *r = ·123	$8r + \frac{93}{512}\frac{1}{r}$ *r = ·151	$23r + \frac{193}{512}\frac{1}{r}$ *r = ·128
	$\frac{1}{15}$		3·141	6·748	3·258	7·188
	$\frac{1}{10}$		2·538	5·777	2·616	6·070
				5·655		
	$\frac{1}{8}$		2·391	5·656	2·453	5·891
			2·359		**2·411**	**5·889**
	$\frac{1}{4}$		2·695	7·141	2·727	7·258
32 equal parts.		$n = 16$	$16r + \frac{349}{2048}\frac{1}{r}$ *r = ·103	$47r + \frac{349}{1024}\frac{1}{r}$ *r = ·085	$16r + \frac{357}{2048}\frac{1}{r}$ *r = ·104	$47r + \frac{729}{2048}\frac{1}{r}$ *r = ·087
	$\frac{1}{15}$		3·624	8·246	3·682	8·458
				8·005		**8·180**
	$\frac{1}{10}$		3·304	8·108	3·343	8·250
			3·302		**3·341**	
	$\frac{1}{8}$		3·363	8·590	3·395	8·715
	$\frac{1}{4}$		4·682	13·113	4·697	13·170

[See page 456.]

* The value of r for minimum volume

n is the number of bisections.	FINK TRUSS. UNIFORM LOAD. $nr+\frac{1-(\frac{1}{2})^{2n}}{3}\frac{1}{r}$	FINK TRUSS. ROLLING LOAD. $2(2^n-1)r+\frac{3-(\frac{1}{2})^{n-1}}{4}\cdot\frac{1}{r}$	Parts in half-span.	BOLLMAN TRUSS. UNIFORM LOAD. $\frac{2n-1}{n}\left(r+\frac{2n+1}{12n}\frac{1}{r}\right)$	BOLLMAN TRUSS. ROLLING LOAD. $2(2n-1)r+\frac{4n^2+3n-1}{12n}\frac{1}{r}$	Ratio of depth to span, $r=\frac{h}{L}$
$2^n = 2$ or $n = 1$	$r+\frac{1}{4}\,\frac{1}{r}$ $*r = \cdot500$	$2r+\frac{1}{2}\,\frac{1}{r}$ $*r = \cdot500$	$n = 1$	$r+\frac{1}{4}\,\frac{1}{r}$ $*r = \cdot500$	$2r+\frac{1}{2}\,\frac{1}{r}$ $*r = \cdot500$	
	3·817	7·633		3·817	7·633	$\frac{1}{15}$
	2·600	5·200		2·600	5·200	$\frac{1}{10}$
	2·125	4·250		2·125	4·250	$\frac{1}{8}$
	1·250	2·500		1·250	2·500	$\frac{1}{4}$
	1·000	**2·000**		**1·000**	**2·000**	
$2^n = 4$ or $n = 2$	$2r+\frac{5}{16}\,\frac{1}{r}$ $*r = \cdot395$	$6r+\frac{5}{8}\,\frac{1}{r}$ $*r = \cdot323$	$n = 2$	$\frac{3}{2}\left(r+\frac{5}{24}\,\frac{1}{r}\right)$ $*r = \cdot456$	$6r+\frac{7}{8}\,\frac{1}{r}$ $*r = \cdot382$	
	4·821	9·775		4·788	13·525	$\frac{1}{15}$
	3·325	6·850		3·275	9·350	$\frac{1}{10}$
	2·750	5·750		2·688	7·750	$\frac{1}{8}$
	1·750	4·000		1·625	5·000	$\frac{1}{4}$
	1·581	**3·873**		**1·369**	**4·583**	
$2^n = 8$ or $n = 3$	$3r+\frac{21}{64}\,\frac{1}{r}$ $*r = \cdot331$	$14r+\frac{11}{16}\,\frac{1}{r}$ $*r = \cdot222$	$n = 4$	$\frac{7}{4}\left(r+\frac{3}{16}\,\frac{1}{r}\right)$ $*r = \cdot433$	$14r+\frac{25}{16}\,\frac{1}{r}$ $*r = \cdot334$	
	5·122	11·246		5·039	24·371	$\frac{1}{15}$
	3·582	8·275		3·456	17·025	$\frac{1}{10}$
	3·000	7·250		2·844	14·250	$\frac{1}{8}$
		6·205				
	2·063	6·250		1·750	9·750	$\frac{1}{4}$
	1·984			**1·516**	**9·354**	
$2^n = 16$ or $n = 4$	$4r+\frac{85}{256}\,\frac{1}{r}$ $*r = \cdot288$	$30r+\frac{23}{32}\,\frac{1}{r}$ $*r = \cdot155$	$n = 8$	$\frac{15}{8}\left(r+\frac{17}{96}\,\frac{1}{r}\right)$ $*r = \cdot421$	$30r+\frac{93}{32}\,\frac{1}{r}$ $*r = \cdot311$	
	5·247	12·781		5·105	45·594	$\frac{1}{15}$
	3·720	10·188		3·508	32·063	$\frac{1}{10}$
	3·156	9·500		2·891	27·000	$\frac{1}{8}$
		9·287				
	2·328	10·375		1·797	19·125	$\frac{1}{4}$
	2·305			**1·578**	**18·675**	
$2^n = 32$ or $n = 5$	$5r+\frac{341}{1024}\,\frac{1}{r}$ $*r = \cdot258$	$62r+\frac{47}{64}\,\frac{1}{r}$ $*r = \cdot109$	$n = 16$	$\frac{31}{16}\left(r+\frac{11}{64}\,\frac{1}{r}\right)$ $*r = \cdot415$	$62r+\frac{357}{64}\,\frac{1}{r}$ $*r = \cdot300$	
	5·328	15·149		5·124	87·805	$\frac{1}{15}$
	3·830	13·544		3·524	61·981	$\frac{1}{10}$
		13·495				
	3·289	13·625		2·906	52·375	$\frac{1}{8}$
	2·582	18·438		1·816	37·813	$\frac{1}{4}$
	2·581			**1·607**	**37·190**	

The heavy figures give the minimum volume.

EXAMPLES.

279. A Warren girder has a span of 160 feet and a depth of 16 feet, and is subdivided into 16 equal parts of 10 feet each. Find the theoretical volume of wrought iron for which $f = 4$ tons per square inch, that the girder may resist a rolling load of 80 tons. Here

$$n = 8, \quad r = \tfrac{1}{8}, \quad R = 80 \text{ tons}, \quad f = 4 \text{ tons per sq. in.} \quad \text{and} \quad L = 1920 \text{ inches.}$$

From Lévy's table, second column, fourth line, we get 5·656, and

$$V_1 = 5{\cdot}656 \, \frac{RL}{f} = 217190 \text{ cubic inches.}$$

280. Determine a girder of similar type to bear the dead weight of the last, there being 8000 cubic inches of iron to the ton, that is

$$W_1 = 217190 \div 8000 = 27{\cdot}2 \text{ tons},$$

and also
$$W' = 160 \times \tfrac{3}{8} = 60 \text{ tons},$$

a share of the deck or platform which is three-eighths of a ton per foot.

$$V_2 = 2{\cdot}391 \, \frac{W_1 L}{f} = 28167 \text{ cubic inches}, \quad W_2 = 3{\cdot}52 \text{ tons},$$

$$V_3 = 2{\cdot}391 \, \frac{W' L}{f} = 68866 \text{ cubic inches}, \quad W_3 = 8{\cdot}61 \text{ tons.}$$

281. Find now the theoretical weight of the girder to resist the rolling load R, its own weight, and a share of the deck or platform

$$W = W_1 + W_2 + W_3 = 39{\cdot}33 \text{ tons nearly.}$$

But no provision has been made to resist $W_2 = 3{\cdot}52$ tons. For this

$$v = 2{\cdot}391 \times W_2 L \div f = 4040 \text{ cubic inches},$$

or its weight is half a ton, so that $W = 40$ tons. Adding 20 per cent. to stiffen the long struts and allow for rivet heads, &c., the probable weight is 48 tons.

282. Find the weight of the girder if it were braced on the Fink system. The tabular numbers are now 9·500 for the rolling load, and 3·156 for the dead load.

$$W_1 = 45{\cdot}6, \quad W_2 = 8{\cdot}6, \quad W_3 = 11{\cdot}4, \quad \text{and} \quad w = 0{\cdot}67,$$

$$W = W_1 + W_2 + W_3 + w = 66{\cdot}3 \text{ tons.}$$

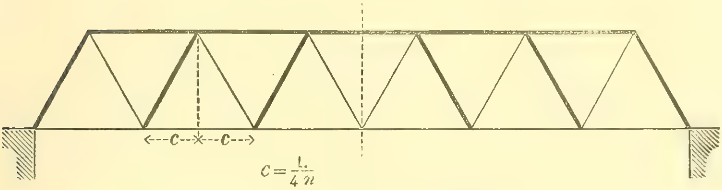
Adding 20 per cent. we have 80 tons.

283. Find the theoretical volume of the isosceles truss shown on fig. 260.

Here $n = q = 3$ and $r = \tfrac{1}{7}$. As $n = 3$ is not given in the table, these values of n and r are to be substituted into the general formula at the top of columns one and two, giving 1·709 for the dead load, and 3·703 for the rolling load, as printed on fig. 260, which see.

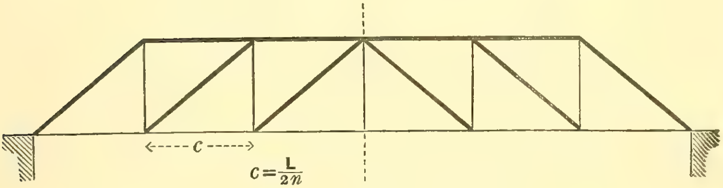
284. A girder is braced as shown in fig. 251. Span $L = 168$ feet, $h = 12$ feet, and $n = 6$. The struts are shorter than the ties somewhat in the manner of the Post truss. The shape of each triangle is such that a perpendicular from the vertex divides the base into a short segment $c = 5$ feet remote from the centre, and a longer one of 9 feet adjacent to it. It will be seen that the struts are 13 feet long, while the ties are 15 feet long. Find the theoretical volume of the truss to resist a rolling load $R = 48$ tons.

Isosceles and Rectangular Bracing.



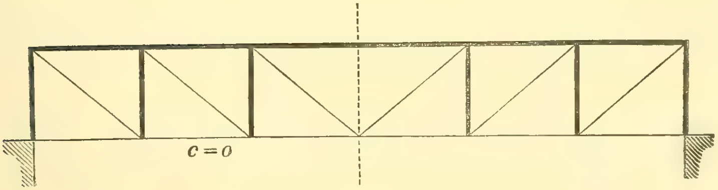
$n = 3$. $q = 3$. $V = 1.709 \frac{WL}{f}$ cub. in.

$r = h \div L = \frac{1}{7}$. $V = 3.703 \frac{WL}{f}$,,



$n = 3$. $q = 3$. $V = 1.854 \frac{WL}{f}$ cub. in.

$r = \frac{1}{7}$. $V = 4.189 \frac{WL}{f}$,,



$n = 3$. $q = o$. $r = \frac{1}{7}$.

Fig. 260.

From the pair of inequalities (6)

$$5 > \frac{168}{24}\,\frac{q-1}{6-\frac{1}{2}}, \quad \text{while} \quad 5 \leqq \frac{168}{24}\,\frac{q+1-1}{6-\frac{1}{2}},$$

or

$$4{\cdot}21 > q, \quad \text{while} \quad 3{\cdot}93 \leqq q.$$

Hence q is the integer less than 4·21 but not less than 3·93, that is, q is 4. Substituting into the general equation (7) the theoretical volume to resist the rolling load is

$$V_r = 148520 + 448 = 148968 \text{ cubic inches.}$$

285. Solve the same question directly by calculating the volume of each piece. Note that the length of each piece is a whole number.

For the diagonals first. The shearing forces in each of the first six triangles (fig. 251), when the 48 tons is placed at the end of its base nearest the centre, are

44, 40, 36, 32, 28, 24 tons,

their sum is 204, the secant of the slope of a strut is 13 ÷ 12, so that 204 × 13 ÷ 12 is the sum of the thrusts on all the struts which may be reckoned as all in one bundle; this must further be multiplied by 13 feet, their common length, and divided by $f = 4$. The secant of slope of ties is 15 ÷ 12,

$$\tfrac{1}{2} \text{ volume of all diagonals} = \frac{204}{4}\,\frac{13^2 + 15^2}{12} = 1674{\cdot}5 \text{ cub. units.}$$

The bending moments on the first six upper booms with the 48 tons at the joint below it in each case are

$$44 \times 14, \quad 40 \times 28, \quad 36 \times 42, \quad 32 \times 56, \quad 28 \times 70, \quad 24 \times 84.$$

The sum 9016 when divided by 12 feet the depth of girder, gives the united thrust on those six booms which may be reckoned in a bundle, multiplying by 14 feet, the length, and dividing out $f = 4$ tons per square inch, we have

$$\text{volume of 6 upper booms} = \frac{9016}{12}\cdot\frac{14}{4} = 2629\tfrac{2}{3} \text{ cubic units.}$$

But the central boom is $c = 5$ feet shorter than the others in the bundle, so we have to deduct its excess volume (24 × 84) ÷ 12, multiplied by 5 ÷ 4, in all 210 cubic units. Compare the two terms $2629\frac{2}{3} - 210$ with those in the general expression, p. 449, from which they could be derived at once by substitution, and to illustrate which is the object of this exercise. So far we have

$$\tfrac{1}{2} \text{ volume upper boom} = 2419\tfrac{2}{3} \text{ cubic units, 1 foot} \times \text{1 square inch.}$$

For the first six lower booms the bending moment on each is calculated twice. The upper row of values are for the 48 tons placed at that end of each boom in succession which is nearer the left abutment. The second row is for the same load placed at the other end.

$$0 \times 163, \quad 4 \times 149, \quad 8 \times 135, \quad 12 \times 121, \quad 16 \times 107, \quad 20 \times 93.$$

$$44 \times 5, \quad 40 \times 19, \quad 36 \times 33, \quad 32 \times 47, \quad 28 \times 61, \quad 24 \times 75.$$

or

0, 716, 1080, 1452, 1712, 1860.

220, 760, 1188, 1504, 1708, 1800.

Now $q = 4$ means that for the first four the lower line gives a greater maximum than the upper. For convenience, however, the lower line is taken all along in the first instance. Sum of lower line 7180 foot-tons; dividing by $h = 12$ feet gives the pull on the whole six reckoned as one bundle, further dividing by $f = 4$ tons per square inch gives sectional area of bundle in square inches, and then multiplying by 14 feet gives its volume in cubic units 1 foot by 1 square inch, so that

$$\tfrac{1}{2} \text{ volume low booms} = \frac{7180}{12} \cdot \frac{14}{4} = 2094\tfrac{1}{6} \text{ cubic units.}$$

Summing and doubling we have $12376\frac{2}{3}$ cubic units; multiplying by 12 gives 148520 cubic inches. But for the two lower booms next the centre the upper bending moments are 4 and 60 greater than the lower; their sum $64 \div 12$ gives the extra pull on these two; multiplying by 14 feet and dividing by $f = 4$ tons per square inch, the extra volume required is $18\frac{2}{3}$ cubic units, or 224 cubic inches. Doubling it is 448 cubic inches. Compare with the answer to last exercise.

$$V_r = 148520 + 448 = 148968 \text{ cubic inches.}$$

286. Find the volume of the same girder to resist a uniform load of four-sevenths of a ton per foot. That is, $W = 96$ tons or 8 tons concentrated at each lower joint of fig. 251. The student should work this example by taking out the volumes in detail as on last example, and check his result with that obtained by substituting

$$W = 96, \quad f = 4, \quad L = 12 \times 168, \quad r = \tfrac{1}{14}, \quad c = 12 \times 5 \text{ in equation (1).}$$

Ans. 148480 cubic inches.

CHAPTER XXII.

THE SCIENTIFIC DESIGN OF MASONRY ARCHES.

In this chapter we have chiefly in view the design of important Masonry Railway Bridges and Viaducts, having arches of long span, with as small a rise as is consistent with moderately heavy abutments, and in which the surcharge, over the top of the key-stone, is no more than sufficient for the proper formation of the bed of the railway, which may be taken at about eighteen inches. In metaphor, they may be called greyhound arches, taking a long bound with but little rise, and having not an ounce of needless fat, that is dead load, but subject to heavy live loads consisting of a train of locomotives. This train, not yet on a particular arch but only approaching it, is upon the instant on one-half only of that arch, and the next it is all over it, and the next again it is on the second half only, then away altogether on to the adjacent arches. To successfully resist such incessant, swift heavy traffic, the arch-ring, or at least some central portion of it, has need of the suppleness of the back of the greyhound.

An arch-ring well built of wedges of stone, or *voussoirs* as they are called, and more especially one built of brick masonry in concentric rings, as is the most modern practice, has ample elasticity to accommodate itself to all the vagaries of the live load, provided it be not overmuch constrained by a too rigid superstructure. The two joints at equal distances from the key-stone, which map out this central elastic part of the arch-ring, are practically the joints of rupture; and a horizontal line, through the upper edges of these joints, is the level of heavy backing. The backing above this level should consist of light spandril walls bonded into the arch-ring by through headers, or simply riding on the arch-ring. Their principal function is to give a horizontal passive reaction on the arch-ring when it bulges out due to the live load on the other half. They stand upon heavy spandril walls below them, and stretch horizontally such a distance, that the friction on their base shall at least be equal to the excess of the horizontal thrust of the loaded half arch over that of the unloaded half. Thus it will be seen that the passive horizontal reaction of the light spandrils is called out alternately on each side of the arch during a transit of the live load, and is zero while the live load is either entirely absent or completely covers the arch.

The light spandrils are best built of rubble masonry, or of brickwork with plenty of mortar in the vertical joints, to allow the arch-ring sufficient play while the passive horizontal reaction is being called out. The spandrils serve also to lighten the superstructure and reduce its average weight per cubic foot, as the spaces can be left void and the walls covered across.

The heavy backing consists of thicker walls with vertical-dressed joints, and having the stones or bricks packed close horizontally, so as to yield but little when the centre is struck. The *voussoirs*, below the point of rupture, have their backs dressed in a flight of steps, so as to take the horizontal thrust truly. The heavy spandrils must stretch horizontally a sufficient distance to give friction on their base equal to from two-thirds to one-half of the horizontal thrust of the arch at the crown. The earth filled in behind the spandrils can be relied upon to give promptly the remaining third or half, according as it is filled loose, as in an embankment, or is rammed in layers between the masonry and the face of the excavation, as in some cuttings in old consolidated earth or rock.

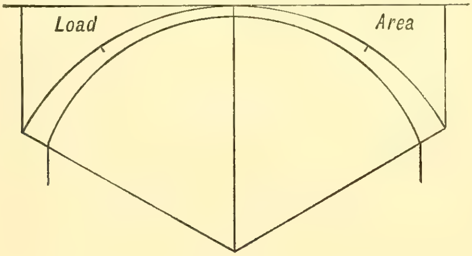


Fig. 261.

Observe, before the earth is filled, that the heavy backing can only take a large fraction of the horizontal load. But the centre of an arch is always struck when the superstructure is only partly built, else the superstructure would crack, due to the after-subsidence, so that only a part of the horizontal load is required. Of course the earth must be filled behind the spandrils before the remainder of the superstructure is built.

Two rival designs present themselves for long-span light-surcharged masonry arches—the Segmental Arch and the Semi-elliptic Arch.

In the one, the *soffit* is a segment of a circle, with a rise about a quarter of the span. The joints of rupture should be at the springings, which demands that the total headroom, from springing level to level of rails, should bear to the span the ratio, almost constant, of one to three. In this design, too, it will be found that the arch-ring must thicken outwards from the crown to the springing, so that a skewback shall be from once and a half to twice as thick as the keystone. This may be disguised on the front of the bridge by a uniform moulding, of the thickness of the keystone, cut upon the face of the pend-ring of stones.

In the other design, the *soffit* is a false semi-ellipse struck from five centres. At the ends of a quadrant the radii are those of the ellipse, being the square of one semi-diameter divided by the first power of the other. Between them a centre is interpolated, as on fig. 262 (see Rank., *C.E.*, p. 420), at the intersection of two arcs, one from each of the main centres, with a radius equal to the difference of that main radius and the rise. It will require three templates to work in these three varieties of groups of *voussoirs*. In this design the rise of the arch may be about a fifth of the span, and the ring in this case is of uniform thickness. As the thrust at crown will not exceed twice that at springing, the ring can accommodate the line of stress in its middle third.

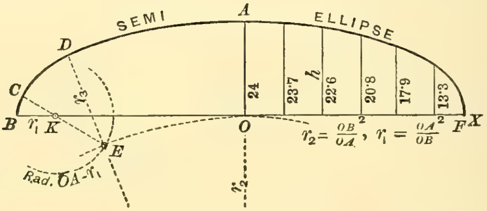


Fig. 262.

It is well to observe that this *soffit*, although called elliptic by courtesy, approximates in an equal degree to one of the elastic curves, called by Rankine the Hydrostatic Arch, or to that elastic curve modified and then called by him the Geostatic Arch. (Rank., *C.E.*, pp. 208, 419; 212, 420).

We have already given a practical definition of the *joint of rupture*, and, as Rankine says (*ibid.*, p. 428), the more rigid part of the ring, below it, is really part of the pier or abutment. It will appear as we proceed that this practical definition corresponds to Rankine's *point of rupture for a linear rib* loaded in any way. By locating these joints as far out from the crown as is consistent with equilibrium two advantages are gained: one, economy of heavy backing; the other, a long elastic field.

LINE OF STRESS.

In calculations connected with the arch, it is convenient to consider one foot of the ring in the direction of its axis, just as if the bridge were only a foot broad, and the same symbol t gives the thickness of the ring in feet and the area in square feet of the *joint* on which two adjacent *voussoirs* abut. It is assumed that their mutual pressure on each other is either uniformly distributed over that joint, or varies uniformly from a maximum value at the one edge to a minimum at the other. Such a stress is indicated on a drawing by a sheaf of arrows varying in length, and with their feathers ranged in a straight slope, or more simply by a trapezium mapping them out.

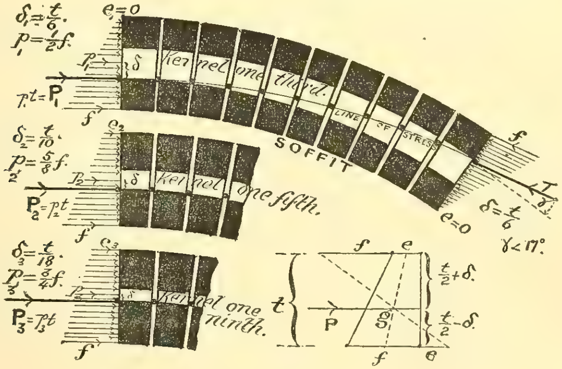


Fig. 263.

Three varieties of this distribution of the stress on the left cheek of the keystone are shown upon fig. 263. Each has the same maximum value f at the lower edge of the joint, and the minimum values are $e_1 = 0$, $e_2 = \frac{1}{4}f$, and $e_3 = \frac{1}{2}f$ at the upper edges, respectively, while the average values at the middle point of the joints are, respectively, p_1, p_2, and p_3, equal to 4, 5, and 6 eighths of the maximum. The whole stress may be supposed concentrated at a point called the *centre of stress*. To find the centre of stress it is only necessary to construct the centre of gravity of the wedge whose face is the trapezium of which e and f are the parallel sides. A construction shown on figs. 191, 263 is—Produce e to the left a distance equal to f, and f to the right a distance equal to e, and the line joining the extremities cuts the line joining the middle points of e and f in the point g. If now δ stands for the *deviation* of the centre of stress from the middle of the joint, we have, by similar triangles, $\frac{1}{2}t + \delta : \frac{1}{2}t - \delta :: \frac{1}{2}e + f : e + \frac{1}{2}f :: p + \frac{1}{2}f : 2p - \frac{1}{2}f$. Since the average value $p = \frac{1}{2}f + \frac{1}{2}e$. Then the ratio—

$$\left.\begin{aligned} \frac{\text{Maximum stress}}{\text{Average stress}} &= \frac{f}{p} = 1 + \frac{6\delta}{t}, \\ \text{or} \qquad \delta &= \left(\frac{f}{p} - 1\right)\frac{t}{6}. \end{aligned}\right\} \qquad (1)$$

So then, on fig. 263, we have $\delta_1 = \frac{t}{6}$, $\delta_2 = \frac{t}{10}$, and $\delta_3 = \frac{t}{18}$, and $e_1 = 0$, $e_2 = \frac{1}{4}f$, and $e_3 = \frac{1}{2}f$.

Let the ring on fig. 263 be that of a small model, the blocks resting on a centre and separate from each other, with square pins to preserve the spacing. The ring being loaded in a suitable manner, the pins are to be moved up and down in the joints till it is found that the centre can be lowered, leaving the ring in equilibrium. Then the curve joining the pins is the *line of stress* for that manner of loading. The equilibrium is unstable: any slight disturbance would make each pair of blocks to rotate about the pin between them till either their upper or lower edges came in contact. Note, too, that the thrust between some of the blocks being *oblique* to the joint, there is a tendency for the pin between them to roll over, if it were too small, or had its corners rounded. In the one case, the equilibrium is said to be destroyed by *bending*, which is only stopped by the edges coming in contact, while in the other, the equilibrium is destroyed by *sliding*, which is only stayed by the roughness and squareness of the pins.

EQUILIBRIUM AND STABILITY.

For every alteration or re-arrangement of the load, the pins or fulcrums would require to be shifted into new positions to maintain equilibrium, so that the line of stress changes its form to suit the load. A small extra load laid on the centre of the part of the arch shown on fig. 263 would slightly *flatten* the arch, and tend to make the central joints shut at their upper edges, so that the pins there would have to be moved up, and the new line of stress would then be a *sharper* curve.

The same thing happens in the masonry arch where the mortar fills the spaces between the joints. Every slight alteration of the load alters the distribution of the stress over the joints in the elastic untrammelled part of the arch. The centres of stress move up and down in the joints, just as we moved the pins in the model, and the line of stress freely changes its shape and accommodates itself to the change of load. And wherever the ring is itself *flattened and depressed*, there the line of stress becomes *sharper* in curvature, and *rises into a higher position* in the ring. Also, the *friction* prevents the stones sliding where the obliquity of the stress is only slight, say less than 17°.

To render the equilibrium *secure* it is only necessary to have deep joints, and to limit the displacement of the centre of stress to some central field of the joints. This displacement should not be greater than a sixth of the thickness of the joint, for then the minimum stress at one edge is zero (see fig. 263); to force the stress further would mean opening the joints, so that as the live load crossed and re-crossed, the joints would 'work,' the mortar crumble and drop out, and the whole be destroyed.

The conditions of statical and dynamical equilibrium of an arch-ring practically reduce to this—That the line of stress shall always be in the middle third of the arch-ring, and the joints be normal to the soffit; it being then practically impossible to draw a curve, in the middle third or '*kernel*' of the ring, which shall cross any of those joints at an angle greater than 17°.

STRENGTH.

We will take the ultimate resistance of sandstone to crushing at $f = 576{,}000$ lbs. per square foot, and its weight at 140 lbs. per cubic foot. And the least factor of safety shall be *ten*. The f shown in fig. 263, indicating the working intensity of the thrust at the keystone, is not to exceed 57,600 lbs. per square foot, if the ring be sandstone. For strong brick we take $f = 154{,}000$ lbs. per square foot, and $w = 112$ lbs. per cubic foot. Now as the loads are to be reckoned in cubic feet or columns of the actual substance of the bridges, then, speaking directly as the strength, but inversely as the mass, strong brick is *one-third* as efficient as sandstone, while granite is *twice* as efficient. For granite, $f = 1{,}350{,}000$ and $w = 164$.

For large bridges the question of strength is entirely dominated by that of stability; but for small arches where the *surcharge is relatively larger*, the question of strength, on the other hand, dominates that of stability, and it becomes convenient to confine the line of stress to closer limits, and we then take as a 'kernel,' a middle *fifth* of the ring, and as a 'kernel,' a middle *ninth* (see fig. 263). In this last, the line of stress may be looked upon as *practically* up the centre of the ring, furnishing no reliable clue to the proper thickness of the ring. But even in this case, the maximum and average stress on the keystone differ by 25 per cent. In designing arches (usually short in span) with heavy surcharges, one is really designing a bent strut, a very difficult matter in itself. The line of stress is practically up the centre, but with a probable variation of 25 per cent. The difficulty in the one case is, that the factor of safety to be used is left entirely to the judgment, whereas, in the long span lightly surcharged arches, the stability demands so deep an arch-ring, that the factor of safety against direct crushing is beyond what prudence would otherwise dictate.

BALANCED LINEAR RIB OR CHAIN.

The Linear Rib and Chain are ideal structures, usually curved or polygonal. They are designed to suit or *resist* a given external set of loads, with the intention of afterwards clothing the sides or arcs in timber, metal, or masonry. They are then designs for actual engineering structures. The pins in the model (fig. 263) map out such a linear rib, and its clothing is masonry. Rib and chain are to distinguish whether the members of the structure are in compression or tension. The rib is of far greater practical importance, but its equilibrium is unstable, until a measure of security is given to it by the material clothing. It is hard for the mind to conceive of the linear rib without its clothing; and still harder to write about its equilibrium, clearly and accurately, in words. On the other hand, the linear chain may be regarded as having a slight amount of material, of too little depth to interfere with its flexibility, and of little weight compared to the external loads. It may also be supposed to be furnished with hinges at short intervals, and so be perfectly free to assume the proper shape for equilibrium, whatever may be the external loads, and to return again to that shape if disturbed. Consider the equilibrium of the pin of one hinge. The two links which it joins must be pulling the pin in opposite directions with equal force. This can only be the case if the mutual interaction of the two links, through the medium of the pin, be along the tangent to the chain at that point. We must assume a hinge at every point, and so—*The general condition of equilibrium is, that the pull or thrust along a linear chain or rib, at each point, is along the tangent.* Having designed a linear chain, for given loads, it is only necessary to invert it, or suppose the loads reversed, and we have the corresponding linear rib. The funicular polygon of older writers, and the link polygon (see pp. 100, 103, 144, 147) of graphic statics, are examples of linear structures. All the ribs, we are about to treat of, shall be horizontal at the central highest point or *crown*, and shall *spring* from two points on the same level.

The suspension bridge is the only practical example of a linear chain. In its primitive form, with vertical rods, and free from diagonal bracing or stiffening girders, part of the load was nearly uniform along the chain, and another part uniform along the span or platform. It formed a good example of a balanced chain under *vertical loads alone.* The two theorems, that the *linear chain* is in form a *catenary* when loaded uniformly along the chain, and a *parabola* when loaded uniformly along the span, seemed to have an important application to the suspension bridge, and much has been written about them under that misapprehension. It is impossible to have either of these loads alone, in a useful structure. Still, the two theorems inform us that the chain of the bridge is, in form, something between the catenary and the parabola. As *both* of these curves are sharpest at the vertex, they show that this also is the character of the bridge-chain. But the sharpness at the centre of the bridge-chain is not very marked, because the dip is always a mere fraction of the span, so that the corresponding arcs near the vertex of both catenary and parabola are sensibly circular. In the modern suspension bridge, what with oblique rods, stiffening girders and bracing, the load is no longer wholly vertical, and the chain may have a variety of forms.

These two theorems, although of little practical use, are exceedingly convenient as a start-off in the study of the balanced semicircular rib or chain. An elegant proof that the linear chain hangs in *the catenary* when loaded uniformly along the chain, and that that curve is sharpest at its lowest point, may be found in Williamson's *Integral Calculus*.

The parabolic chain, balanced under a load uniform along the span, is figured in Rankine's *Civil Engineering* at page 188. The proof is simple and instructive, and may be given here by supposing fig. 264 modified for that purpose. Let the quadrant ABS be the edge of a thin steel rib or curved ribbon, one foot broad normal to the paper, and supported by a horizontal thrust T_0 at the crown hinge A, and a tangential thrust T_1 at the springing hinge S. The only load on it is to be the load of p lbs. per foot, spread uniformly on the horizontal platform ab. This load is to be transmitted down to the rib by vertical struts without weight themselves. The platform ab is also one foot broad normal to the paper, and is to be quite flexible, so as to allow the load to be transmitted to the rib without constraint. The parallelogram standing on the platform ab, and having the height p, is called the *vertical-load-area*. Consider the equilibrium of a portion of the rib from the crown hinge A, out to any other hinge B, and suppose it to be rigid, having all hinges between A and B clamped. The corresponding part of the vertical-load-area, which is shaded on fig. 264, may now be supposed to be concentrated into one force P, acting down through the centre of gravity of that shaded area. AB is balanced by the three forces T_0, T, and P. They must therefore meet at one point.

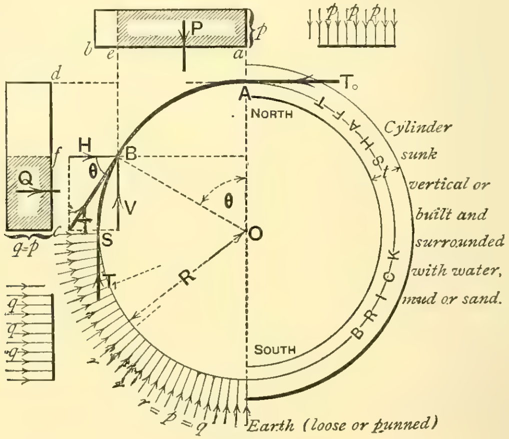


Fig. 264.

The general condition of equilibrium requires T to act along the tangent at B. If we lower the shaded area till a coincides with the crown A, it becomes apparent that the tangent at B bisects its base ae. The rib is parabolic, for that is the only curve whose tangent at any point behaves in this manner. Now, as B is any point on the rib, it can be in succession each joint from the crown outwards, so every joint can be unclamped, still preserving equilibrium.

Hence, the particular condition of equilibrium of a linear rib under a vertical-load-area alone is—*That the tangent at any point shall meet the crown tangent on the vertical through the centre of gravity of the portion of the load-area from the crown out to that point.*

CONJUGATE-LOAD-AREAS.

We have already discussed a linear rib to resist a vertical-load-area alone. The vertical load was spread on a horizontal platform, equal in length to the span of the rib, and one foot broad normal to the paper. The shape of the area was a parallelogram of height p. The rib itself was parabolic, and so peaked at the crown A. On fig. 265 this rib is shown: ABS is a quadrant of it. It is no longer parabolic, but has been deliberately pulled out horizontally at each pair of points, except the springing hinges, until it is a semicircle. It will no longer be in equilibrium, the vertical load will tend to flatten it at the crown, and spread it out further at each pair of points. To prevent it spreading horizontally, we may suppose a pair of vertical platforms, of which cd is one, each in length equal to a radius, and a foot broad normal to the paper, and furnished with loads gravitating horizontally inwards. One of these loads is indicated on fig. 265 by a parallelogram of height $q = p$, standing on the platform cd as its base. It is called the conjugate horizontal-load-area. If we consider the whole circular rib, it is almost axiomatic, that the two conjugate-load-areas should be alike, from symmetry. The proof is as follows:—Let the quadrant ABS be rigid, with hinges only at its ends A and S, then for horizontal component equilibrium, we have the thrust at the crown T_0 equal to the whole of the horizontal-load-area on cd. Again, let AB be rigid, with hinges at its ends, and resolve T, the thrust along the tangent at B, into its vertical and horizontal components V and H. Then, for horizontal component equilibrium we have T_0 equal to H, together with the unshaded part of the horizontal-load-area. It follows then that H must be equal to Q, the shaded part of that area. Also, V is equal to P, the shaded parts of the vertical-load-area. If θ be the slope of the rib at B to the horizontal, then V must bear to H the ratio of $\sin \theta$ to $\cos \theta$, that T may be along the tangent at B, as required by the general condition of equilibrium. But it is evident that P and Q are also in this ratio, as the bases ae and cf are $\sin \theta$ and $\cos \theta$, respectively, when the radius is taken as unity. Hence, with q constant and equal to p, we have the condition of equilibrium satisfied at each point B of the quadrant.

For a right circular or other quadrantal linear rib, that is, a quadrant horizontal at the crown and vertical at the springing, which has to resist a pair of conjugate loads, one vertical and the other horizontal, the particular condition of equilibrium is—*That the part of the vertical-load-area from the crown out to any point, and the part of the horizontal-load-area from the springing up to that point, shall bear the same ratio to each other as that of the sine and cosine of the slope of the rib, at that point, to the horizon.*

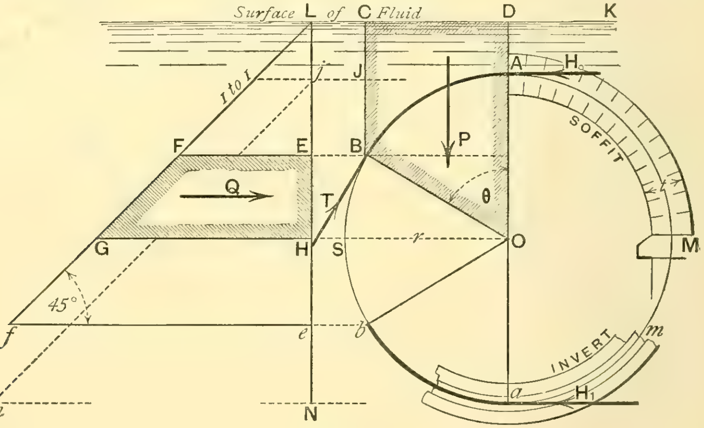


Fig. 265.

On fig. 265 is shown a manner of loading the linear quadrant ABS of a circle with another pair of conjugate loads which balance it. The horizontal conjugate load, being the simpler, may be described first. It is what is called a *uniformly varying load*. The horizontal-load-area Gj stands on the base Hj and is mapped out by the 45°, or 1 to 1 sloping line drawn from L, where the bases of the two areas meet. The vertical load consists of two parts. The first part is the substance between the horizontal base *DCL* and *ABS* the rib itself. The second part is a load distributed *uniformly along the rib.* The rate of loading along the rib is such that the amount on an arc *AB* would fill up the area *OAB* with the same substance as the first part is made of. On the arc *AB* the first part of the vertical load is a slab *ABCD*, one foot wide normal to the paper. The second part might be a uniform ring of *voussoirs*, pinned to the arc *AB* at close intervals by their middle points, as on fig. 263. If the ring of *voussoirs* be a foot wide normal to the paper, and be made of the same substance as the slab *ABCD* (fig. 265), it is clear that the thickness of the *voussoirs* (t, on fig. 263) must be *half the radius*, that their volume might complete the slab *OBCD*. It will now be seen that P, equal to the shaded area *OBCD*, is the vertical load from the crown out to B, while the shaded area *EFGH* equals Q, the horizontal load from the springing up to B. The two shaded areas have their parallel sides equal, each to each, $BC = EF$, and $OD = HG$. Their areas are proportional to the distances between the parallel sides, respectively. Hence $P : Q :: CD : HE :: \sin\theta : \cos\theta$, and the rib is balanced.

It is to be observed that only the upper part *ABCD* of the vertical-load-area truly represents the distribution of the first part of the vertical load. The area *OAB* merely gives the amount of the second part of the vertical load on *AB*, but in no way represents its distribution; the arrow P should be further to the left on fig. 267, instead of going through the centre of buoyancy. Just as in Graphic Statics a line is used in the force-polygon to represent a force in the limited degree of magnitude only, so also is the area *OBCD* used here.

SUPERPOSITION OF LOADS.

It is evident that if a rib be balanced under two systems of loading separately, it will balance under the systems conjointly. The whole load will then be represented by a pair of conjugate-load-areas described by geometrically adding or subtracting the two given pairs of areas. The proof of the balancing of the circular quadrant *ABS* for the manner of loading shown on fig. 265 is quite independent of the height of the horizontal platform *DCL* above A, the crown of the rib. Suppose it to be lowered till D coincides with A, then *AJ*, the horizontal platform in its new position, cuts off a parallelogram *JD* from the vertical-load-area. At the same time the new 45° sloping boundary *jn* cuts off a parallelogram *jG*, of exactly the same area, from the horizontal conjugate-load-area. This is just the same thing as if we had removed the load shown on fig. 264. So that equilibrium of a circular quadrant *ABS* for the load on fig. 264 follows as a corollary once the equilibrium for the load fig. 265 has been established. In the same way the load on fig. 264 may be added to that on fig. 265. The rectangle *P* will sit on the top of *CD*, while the rectangle *Q* may either be placed against *HE* on its right side, or *Q* may be *distorted* till its sides slope at 45° and placed against *GF*. See Distorting-table (fig. 151).

The practical importance of the load shown on fig. 264 is now evident, and the assumption of weightless struts transmitting the load to the rib is got rid of. The vertical load *P* is the share of the live load, which falls to a slice of a bridge one foot broad, due to two rows of locomotives covering a thirty-foot broad platform. The height of *P* is $p = 1{\cdot}5$ feet when reduced to a column of the same material as the superstructure of the bridge. If w be weight per cubic foot of the superstructure, then $p = 1{\cdot}5\, w$. A good average value of w is about one cwt. The other rectangle *Q* is the passive *additional horizontal reaction* with which the backing must promptly oppose the extra effort of the arch to spread when the live load comes upon it.

On the other hand, fig. 265 is a picture of the dead load on the bridge. The level of rails is *DCL*, while *ABS* is the line of stress up the middle of the *voussoirs*. These two lines form the upper and lower boundaries of the *great bulk* of the dead load. It is convenient to have names for such boundaries. They are called the *extrados* and the *intrados* of the load. The other portion of the vertical load, spread uniformly along the line of stress, is the weight of the half of the arch ring itself lying below *ABS*, and the *excess weight* of the half lying above *ABS*, the material of the ring being always heavier than that of the superstructure by about 50 per cent. The conjugate-area *jHG* maps out from the *very crown* down to the springing, the passive horizontal resistance with which the backing must oppose the tendency of the linear rib to spread due to the dead weight. As far as the masonry arch is concerned, fig. 265 is, as yet, a very imperfect picture of the dead load, the load spread along the rib being out of due proportion. We saw that it would require the thickness of the arch-ring to be *half* a radius, whereas the actual thickness of the key-stone is only about *one-fifteenth* of the radius. It is only for the convenience of being able to draw the horizontal-load-area with the 45° set-square, that we assumed this enormous load along the ring, the intention being to afterwards remove the greater part of it, making at the same time the corresponding correction upon the horizontal-load-area. It is better to remove the *whole* of the load *assumed* along the ring, as a small proportion can readily be restored again to suit the practical requirement.

The next step will be to find the shape of the horizontal-load-area for the circular quadrant *ABS* bearing a uniform load along the rib of an intensity half its radius. We will calculate its breadths at ten equidistant points of the platform *jH*, map out its shape with three straight lines or *batters* sufficiently correct for practical purposes. The area is then to be drawn standing on the left side of *jH*, so as to subtract it from the area already drawn. The treble-batter then furnishes, *once and for all*, one boundary of the conjugate horizontal-load-area for the linear quadrant *ABS* bearing the vertical load between itself and the straight line *DCL*. The other boundary of the horizontal-load-area is the 1 to 1 batter *LFG*, drawn with the 45° set-square; this boundary, *and this one only*, changes its position, as you vary the position of the rails *DCL* to different heights above the crown, or as you add or take off the live load.

FLUID LOAD UNIFORM OR VARYING POTENTIAL.

The pair of conjugate loads shown on fig. 264 can be produced simultaneously by placing the circular ring horizontally in water, the water being excluded from its inside. The platforms now form two pairs of barricades running north to south and east to west. It is evident the water will attack the barricades in the manner shown on the figure. The stresses p and q are now both passive. The active stress is the column of water from the surface down to level at which the ring of the cylinder lies. The height of this active column is called the potential, and the load shown on fig. 264 is shortly called a *fluid load* of *uniform potential.*

The stress as depicted on fig. 264 is called uniplanar stress, because the equilibrium of bodies subjected to it is in no way affected by the *potential stress* which is normal to the paper. The whole stress at this depth in the fluid is represented by a sphere.

From this it will be seen that the resultant stress on the back of the rib due to the two conjugate equal loads p and q, fig. 264, is a normal stress $r = p = q$ as indicated on the S. W. quadrant. For if we remove the four barricades, we know that is how the water will attack the back of the rib.

The formal proof is given in Chapter III., page 41.

In the same way the load on fig. 265 is described as a *fluid load of varying potential* together with a uniform load along the rib of an intensity, such that the load on any arc equals the weight of the fluid which would fill up the hollow subtended by the arc at the centre.

THRUST AT THE CROWN OF A RIB.

To get the thrust along the rib at any point B of the circular rib, loaded with the normal load of constant intensity r, as shown on fig. 264, we have $T^2 = V^2 + H^2 = p^2 \,.\, ae^2 + q^2 \,.\, cf^2 = r^2R^2$ for $p = q = r$, and the radius R is the hypotenuse of the right-angled triangle with sides equal to ae and cf. The thrust at any point is the product of the radius and the intensity of the external normal stress on the back of the rib at the point. At the crown A the constant value of the thrust is readily found by inspection of the figure, for $T_0 = q \,.\, cd = rR$. Now this would be the thrust at the crown, although the rib only extended for a little arc of the circle on each side of it, and would still be the thrust at the crown, if the rib, after that arc, continued on, of any form or shape whatever. At any point of a rib, if we are sure the external load is *normal*, and if we know the radius of a circle which *fits* the rib for a little arc there, we can, by taking their product, get the thrust along the rib. Now the ribs we are considering are all horizontal at their crown, and suffer only a vertical load there, so we have this rule :—*The thrust* T_0 *at the crown of the rib is the product* $p_0\rho_0$ *of the depth of the load at the crown and the radius of curvature there.*

If we apply the rule for the load shown on fig. 265, and remember that the part of the vertical-load-area inside the rib is due to *voussoirs* of thickness $\frac{1}{2}r$, we get $H_0 = (DA + \frac{1}{2}r)r$, where the weight of a cubic foot of the material is taken as unity. This is exactly the value of the horizontal conjugate-load-area jHG, which has to be balanced by H_0, and so the rule is verified.

BURIED ARCHES, SHAFTS AND SEWERS.

An application to the design of the above, chiefly a question of strength, is given in the examples.

EXAMPLES.

287. A brick shaft or well, inside diameter 6 feet, is sunk vertically in water which is kept on the outside. In a one-foot length, see the east half of fig. 264, at a depth of 32 feet, $t = 18''$ being made up of four rings of common bricks. Find the factor of safety against crushing.

$R = 3{\cdot}75$ ft. and $q = 32w$, where w is 62 lbs., the weight of a cubic foot of water. $T_0 = q \,.\, dc = 32w \,.\, R = 7440$ lbs., and as this is uniformly distributed over 1·5 sq. feet of brick, we have then for the intensity of the crushing load 4960 lbs. per sq. foot. The crushing strength of common brick is only half of what we gave for strong brick, that is 77,000 lbs. per sq. foot. The factor of safety against direct crushing is $77{,}000 \div 4960 = 15$, which is a prudent value on the face of so treacherous a load as an *actual* fluid load.

288. If the same shaft be loaded with earth spread all round it in horizontal layers, the earth being filled quite loosely, find the factor of safety against crushing at the same ring.

The potential is doubled as earth is twice as heavy as water. But the passive horizontal stresses $p = q$ are no longer equal to the potential, but are now only one-third of it. Hence $p = q$ is now two-thirds of what it was before. Hence the factor of safety, 50 per cent. greater, is 23.

289. If the earth be rammed in 9″ layers as it is filled outside the shaft, or *punned* as it is called, find the factor of safety now.

The punning can only make the passive horizontal conjugate stresses $p = q$, fig. 264, equal to the potential, so the earth may now be assumed to be *pressing like a fluid*, but like an imaginary fluid twice as heavy as water. Hence the factor of safety is now of half the value in Ex. 287, namely 8.

Notwithstanding the fact that the factor of safety in this example is only 8, while in Ex. 288 it is 23, yet the absolute element of safety is just as satisfactory, for the punned earth is a *steady load*. A part of the brick lining might even be removed, and the consolidated earth would hold itself up for a short time, with the assistance, it might need be, of some shoring. On the other hand, if a piece of the brick lining were removed, the loose earth load would cave in, the water load or a quicksand load rush in. Again the loose earth load may, by its own weight, gradually be consolidated at the north and south ends more than at the east and west ends, and compel the line of stress to become elliptical to such an extent that the stress at A will be varyingly distributed as on the highest keystone shown on fig. 263, when the maximum stress on the brick will be double the average. If this should happen, the factor of safety 23, calculated in Ex. 288, is only an *apparent factor of safety*, the real factor of safety being one-half or 11. Again, with the loose earth load and water load, the circular line of stress is likely to be not quite up the centre of the brick ring, when the stress at A will be distributed as on the middle or lower keystones shown on fig. 263. In which cases the real factors of safety would be only five-eighths and six-eighths of 23. The water load, too, would gain an advantage to cause the shaft to collapse, if it were not quite truly circular, and may sap the joints if the joints yield at all from bad workmanship. For such reasons, with a *steady load*, the factor of safety need only be about *half as much* as for an uncertain load.

Practically, then, the answers to Exs. 287, 288, and 289 are 15, 11, and 8.

290. If the shaft be bored and lined, discuss the factor of safety.

If the shaft be *bored* in old consolidated earth, and then be lined with brick, the three rings would suffice at the depth of 32 feet.

At first, the load on the lining is nominal, but increases gradually as cracks run out into the earth, and suddenly when slips occur. In this way, the potential comes into play, and crowds the earth in behind the shell. At depths beyond 32 feet, the cracks due to this small bore hole would not run out sufficiently far to bring the whole potential into play, so that the three rings of brick might practically suffice down to 64 feet, double the depth. The mere arithmetical calculation would infer that the factor of safety was 4, the half of 8, though, in all probability, it is still 8. Thus it is that with *buried arches*, the factor of safety may seem to be as low as 4 : see Rankine's *Civil Engineering*, p. 437.

291. The upper right quadrant of fig. 267 shows, to a scale of 10 feet to one inch, half of a semicircular arch ring, 14 feet in span. The ring is 2 feet thick, and made of *voussoirs* which are $2\frac{1}{2}$ times as heavy as water. Show that the line of stress AM up the centre of the *voussoirs* is balanced under a water load up to *any* level DK, together with the weight of the *voussoirs* themselves.

If we suppose the water to reach from DK down to AM, the excess weight of the upper half of the ring is $1\frac{1}{2}$ times that of water. That is, a share of the weight of that upper half of the ring is already reckoned; it is, as it were, buoyed up. The lower half of the ring is, in weight, $2\frac{1}{2}$ times that of water. That is the excess weight of the 2 foot thick ring, as a whole, is twice that of water. That is the same as if the *voussoirs* were 4 feet thick, and of the same weight as water. But 4 feet is *half the radius* of AM, and this is exactly the required load along the circular rib which is required in conjunction with the fluid load of varying potential to give absolute equilibrium. Such an arch-ring requires slight spandrils to balance it where the centre is struck and before the water load is filled in.

292. It is well to observe that the inverted circular quadrant abS (fig. 265) is likewise balanced by the fluid load and load along the rib. Let b be directly below B. On the arc ab the vertical load is the buoyancy of the area $CDOabBC$ acting upwards, from which there falls, to be deducted, the area Oab for the weight along the arc. This leaves the buoyancy of the trapezium Db to be balanced by the downward vertical component thrust at b. If we consider the quadrant aS, we have H_1 balancing the whole trapezium GN; but if we consider only the arc ab, we have H_1 balancing the lower part fN, together with the horizontal component thrust at b. This last must be equal to the upper part Ge. We have the two components of the thrust at b given by the trapeziums, Db and Ge, with parallel sides equal each to each and areas proportional to CD and He, or to sine and cosine of θ. Hence the thrust at b is along the tangent there, so that there may be a hinge at b, and b is any point on the quadrant, so the rib is balanced.

293. A culvert siphon or sewer, of 7 feet inside diameter, with two circled rings of brick, is laid horizontally along the bottom of a sheet of water. Show that the equilibrium is perfect when the inside is empty, if the bricks be $2\frac{1}{2}$ times as dense as water.

The section is shown on fig. 265, to a scale of five feet to an inch, $r = 4$ and $t = 1$, both in feet. The problem is that already discussed in Ex. 291, only with the dimensions halved and the circle completed. The equilibrium, as a whole, must be considered. If the cylinder be one foot long, and if w be the weight of a cubic foot of water, the weight of the empty ring is $2\pi rt \times 2{\cdot}5w$, and its buoyancy is $\pi(r + \frac{1}{2}t)^2 w$, which are in the ratio 20 to $20\frac{1}{4}$, so that the empty cylinder would float upwards. Increasing the thickness of the ring by a couple of inches the proportion would now be 23 to 21, and the cylinder would sit on its bed, or if the brickwork were a trifle more than twice and a half the density of water, the same end would be accomplished.

294. If the axis of the cylinder be at a depth of 20 feet, find the thrust at the highest and lowest points A and a, fig. 265, for a ring of the cylinder one foot long.

$$H_0 = jG = \tfrac{1}{2}LH^2 - \tfrac{1}{2}Lj^2 = \tfrac{1}{2}(20^2 - 16^2)\,w = 72w = 72 \times 62 \text{ lbs.} = 4464 \text{ lbs.},$$

$$H_1 = GN = \tfrac{1}{2}LN^2 - \tfrac{1}{2}LH^2 = \tfrac{1}{2}(24^2 - 20^2)\,w = 88w = 88 \times 62 \text{ lbs.} = 5456 \text{ lbs.},$$

and this is spread on one square foot, so that, for ordinary brick of crushing strength 77,000 lbs. per sq. foot, the factor of safety is 14.

295. An empty circular cylinder, 7 feet inside diameter, with a shell 12 to 13 inches thick of brickwork $2\frac{1}{2}$ times as dense as water, will, if completely submerged, remain at rest in *neutral* buoyancy at any depth and in any position. If the axis be vertical, the surface of stress is the circular cylinder up the middle of the brickwork (fig. 264). But what is most remarkable is that this is still the surface of stress if the axis be horizontal (fig. 265). It follows, by induction, that it is the surface of stress for all positions, so that there is no tendency for the tube to collapse under the external water pressure, though it float in any position whatever, completely submerged.

296. If the cylinder be made of a metal very much denser than water, it would be a *thin shell,* and there would be no necessity to distinguish between the inside and outside diameters. Thus a platinum circular shell need only have a thickness one-twentysecond part of half its radius, so as to displace, when empty, its own weight of water. Such a shell might be 8·8 inches in diameter, and 0·1 inches thick. Let it be submerged with its axis horizontal, the water being kept out by face plates at the ends of the same density themselves as water. There is no tendency for the cylinder to collapse even if it were made in staves like a barrel. The staves might be hinged together, and the whole covered with a thin coat of water-tight material, the hinges to have a little stiffness to give slight stability to the equilibrium.

297. If this hinged cylinder were split into two semicircular troughs, one of them would float with the water up to its lip, and remain semicircular in form in perfect equilibrium if only the hinges have a little stiffness to give stability. In the same way two vertical plates, submerged in and excluding water from between them, might be arched over and inverted under with the two semicircular troughs.*

The practical importance of Exs. 293, 294, and 295, where the shell by its very thickness in proportion to its radius, and by the very density of the brickwork, *corrects* the tendency of the extern fluid-load to distort its circular form, will appear when the conditions, in which culverts, sewers, and inverts are usually built, are considered. The circular form recommends itself in situations where great strength is required, the simplicity of this form, too, lends itself to good workmanship. The surrounding load is due to earth, loose or more or less consolidated and in a transitional unsettled state. With the gathering of water sapping the earth close to the brickwork, the external load becomes more and more *like a fluid load*; and it is well that the very increased thickness of shell required in such exposed situations, to give extra *strength,* should at the same time render the *equilibrium* more secure.

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* This theorem of the equilibrium of the thin horizontal empty circular cylinder, displacing its own weight of fluid, was given by the authors in a letter to *Nature,* of 18th February, 1897. From some private correspondence with scientific men interested in it, it appears as if the theorem were new to hydrostatics. From criticism of the method of conjugate-load-areas employed, it seems as if this elegant method, especially lending itself to graphical construction, were little known or understood. The theorem can be proved by the strict but laborious methods of the integral calculus. See letter on Polygonal Shells, *Nature,* 1st May, 1902.

Stereostatic Rib.

This is the name by which Rankine calls the most general case of a rib balanced by two conjugate loads: see his *Applied Mechanics*, p. 198. Three quantities are involved, the shape of the rib and the shapes of the two component conjugate-load-areas equivalent to the actual distribution of the uniplanar load, these two may be either oblique or rectangular. The theorem is that it is sufficient and necessary to give, either implicitly or explicitly, two of these shapes to find the third. In most cases the solution ends in integrals which can only be approximated to very roughly and with great labour.

We are confining ourselves to the particular case where the ribs are *complete*, that is they are horizontal at the crown, and vertical at the springings. The conjugate loads are vertical and horizontal, and the two quadrants are symmetrical. For one quadrant the load-areas stand on *finite bases*, the base or platform of the vertical-load-area being the *half-span* of the rib and the *rise* of the rib being the base or platform of the conjugate horizontal-load-area. We have already solved two important examples (figs. 264 and 265) by the geometry of the areas. One was the *direct problem*: given the shapes of the rib and the vertical-load-area to find the shape of the horizontal-load-area. In the other, the shape of the rib is given and the shape of the horizontal-load-area assumed, and a suitable vertical load built up in two portions: this is the inverse problem. We established a rule for building up these areas, namely, that the shaded portions of the areas should be in the same proportion as the sine and cosine of the slope of the rib.

The platform for the vertical load is always finite, even in the most general case, being the half-span of the rib; the whole area to be built on it, too, is also finite, but for some distributions of the load, the height of the area will, at some points, be indefinitely great. For instance, the portion of the vertical load on fig. 265 shaded inside the rib would, if distributed in an equivalent way on the horizontal platform, reach up to infinite heights over the springing points. The whole horizontal-load-area is finite in the general case; for it equals the thrust at the crown of the rib, which again depends on the product of the radius at crown, and intensity of vertical load there, both of which are always finite. But it is evident that the base, or vertical platform on which the horizontal-load-area stands, will reach down indefinitely if the rib only approximate to the vertical, without ever being absolutely so, as you travel out from the crown.

Notwithstanding these difficulties in the way of actually drawing the two areas in some cases, the mind can conceive of them as drawn. Let, in the most general case, the two finite areas be drawn, and part of each shaded as shown on fig. 264, where the rib is any shape symmetrical about the vertical through A, and P and Q have any shapes for which the rib balances. If B shift a little further out from the crown, θ increases by $d\theta$, P increases by dP, a little parallelogram of height p standing on a base dx, while Q grows smaller by dQ, a little parallelogram of breadth q standing out horizontally from the base dy. The inter-relation of these increments is the quite general solution of *the Stereostatic Arch.*

Given the rectangular equation to the curve, with the axis of X horizontal, and that of Y vertical. Given also the vertical-load-area so that P is known, and how it changes as we shift from point to point. Let q be the unknown breadth of the horizontal-load-area at the level of B. Let B shift out a little and the decrement of Q is the small parallelogram $dQ = q \,.\, dy$. Also $Q : P :: \cos\theta : \sin\theta$ or $Q = P \cot\theta$. So that

$$q = \frac{dQ}{dy} = \frac{d}{dy}(P \cot\theta). \qquad (2)$$

HORIZONTAL-LOAD-AREA FOR SEMICIRCULAR RIB, LOADED UNIFORMLY ALONG THE RIB.

A quadrant ACB is shown on fig. 266. It is convenient to take the intensity of the uniform load along the rib as half a radius. We may suppose it to consist of a ring of material of unit density, with a uniform thickness $\frac{1}{2}r$, so that the area of the part of the ring loading any arc AC is equal to the area OAC, subtended by the arc of the rib at its centre. By the rule for the thrust at crown, we have $H_0 = \frac{1}{2}r \,.\, r = \frac{1}{2}r^2$. This must be the area of the total horizontal-load-area $abkd$, standing on the platform $ad = r$. On fig. 264, if the intensity of the load spread on the horizontal platform were $p = \frac{1}{2}R$, then $q = \frac{1}{2}R$, and the total horizontal-load-area would be $\frac{1}{2}R^2$, the same in each case. In this new case, however, the boundary bhk no longer maps out a parallelogram.

Suppose, at first, the rib is balanced under the uniform load along the rib alone. Its shape, the catenary, would be peaked at the crown A. If it were deliberately pulled into a semicircle, each pair of points on one level, except the springing pair, have been pulled further apart, giving the load an advantage to spread the rib. The horizontal load must, at each level, *press inwards*, to prevent the spreading, most violently at the springing level, and decreasing gradually till it is least at the crown level. To find fv, the breadth of the horizontal-load-area at the level of any point C on the rib, we have for the vertical-load-area from the crown A outwards to C, a ring of breadth $\frac{1}{2}r$, and whose length is $s = AC$ the arc. With i for the slope of the curve at C.

$$P = \frac{r}{2} s = \frac{r}{2} ri = \frac{r^2}{2} i.$$

$$P \cot i = \frac{r^2}{2} i \cot i ; \ \frac{d}{di}(P \cot i) = \frac{r^2}{2}(\cot i - i \operatorname{cosec}^2 i).$$

$$y = fd = r \cos i ; \ \frac{dy}{di} = -r \sin i ; \quad \text{and} \quad \frac{di}{dy} = \frac{-1}{r \sin i}.$$

So that irrespective of sign the value of fv is

$$\begin{aligned} q &= \frac{d}{dy}(P \cot i) = \frac{d}{di}(P \cot i)\frac{di}{dy} \\ &= \frac{r^2}{2}(\cot i - i \operatorname{cosec}^2 i)\frac{1}{r \sin i} \\ &= r \frac{\sin i \cos i - i}{2 \sin^3 i}. \end{aligned}$$

This can be modified into a form more convenient for calculation, as it is engraved on fig. 266,

$$q = fv = \frac{2i - \sin 2i}{3 \sin i - \sin 3i} . r. \tag{3}$$

To find $q_0 = ba = \frac{1}{3}r$, the breadth of the area at the level of the crown A, we must substitute $i = 0$, when the coefficient of r takes the indeterminate form of the ratio of zero to zero. The numerator and denominator are to be differentiated separately and $i = 0$ substituted, which has to be repeated three times when the ratio becomes determinate. For $i = 0$, we have

$$L^t \frac{2i - \sin 2i}{3 \sin i - \sin 3i} = L^t \frac{2 - 2 \cos 2i}{3 \cos i - 3 \cos 3i} = L^t \frac{4 \sin 2i}{-3 \sin i + 9 \sin 3i}$$

$$= L^t \frac{8 \cos 2i}{-3 \cos i + 27 \cos 3i} = \frac{1}{3},$$

or this may be found by substituting the series for the sines.

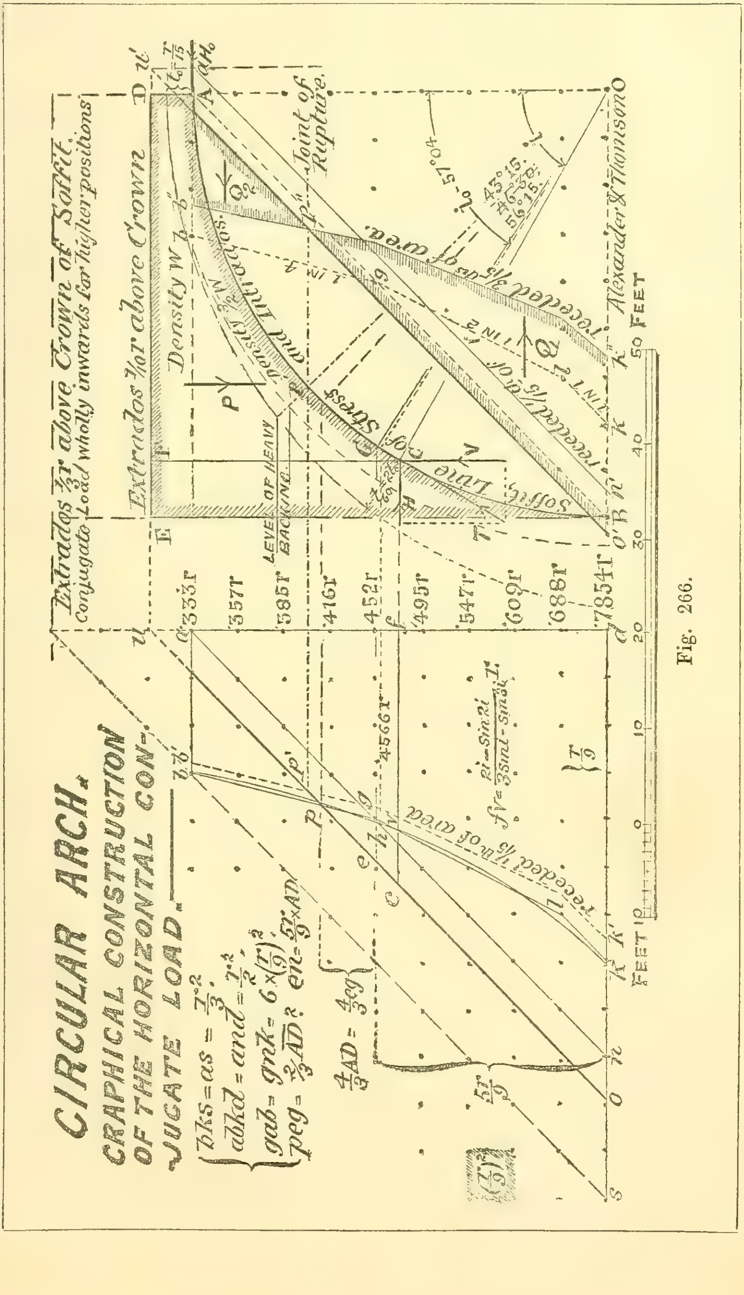
CIRCULAR ARCH.
GRAPHICAL CONSTRUCTION OF THE HORIZONTAL CONJUGATE LOAD.
bks = as = r²/3,
abkd = and = r²/2,
gab = gnk = 6 × (r/9)²,
peg = 2/3 AD²; en = 5r/9 × AD
4/3 AD = 4/3 eg
5r/9
(r/9)²
r/9
fv = (2i - sin 2i)/(3 sin i - sin 3i) · r
Extrados 2/3 r above Crown of Soffit.
Conjugate Load wholly inwards for higher positions
Extrados r/6 or above Crown
Density W 3/2
Density 3/2 W
Level of heavy backing
Joint of Rupture.
Stress
and Intrados.
Line of
Soffit
receded 1/3 of area.
receded 1/5 of area
receded 1/5 of area
57°04′
45°15′
56°15′
Alexander & Thomson
·333r
·357r
·385r
·416r
·452r
·4566r
·495r
·547r
·609r
·688r
·7854r
Feet 10 0 10 20 30 40 50 Feet
P
Q
Q/2
A
B
C
D
E
F
G
H
O
T
V

Fig. 266.

The value of $q_1 = kd = \frac{\pi}{4} r$, the breadth of the area at the springing level B is found by substituting $i = \frac{\pi}{2}$. Since kd acts normal at B, and is the total external load on the back of the rib there, by the rule, the vertical thrust along the rib at B must equal the product of kd and the radius. But the thrust at B is the area of the quadrant of the ring-area whose thickness is $\frac{1}{2}r$, that is, an area $\frac{1}{2}r \times \frac{\pi}{2} r$, and from this the value of kd is found by dividing out the radius.

The breadths of the horizontal-load-area at ten equidistant points of ad are found by substituting in the expression for fv, ten values of the angle i, whose cosines are 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 *ninths* of r the radius, respectively. They are marked on fig. 266. They are—

$\cdot 33\dot{3}r$, $\cdot 357r$, $\cdot 385r$, $\cdot 416r$, $\cdot 452r$, $\cdot 495r$, $\cdot 547r$,
$\cdot 33\dot{3}r$, $\cdot 36\dot{1}r$, $\cdot 38\dot{8}r$, $\cdot 41\dot{6}r$, $\cdot 44\dot{4}r$, $\cdot 5\dot{0}0r$, $\cdot 55\dot{5}r$,
$\cdot 609r$, $\cdot 688r$, and $\cdot 785r$.
$\cdot 61\dot{1}r$, $\cdot 66\dot{6}r$, and $\cdot 7\dot{7}7r$.

The first row gives points on the boundary, which is a gentle curve, the second row are the breadths to a boundary mapping out the area with three straight lines bg, gl, and lk. The first extends downwards for 4 ninths of the radius, giving a breadth of 3 and 4 ninths of the radius at b and g, the second extends downwards for 4 ninths of the radius, giving a breadth of 6 ninths at l, and the last extends downwards the remaining ninth of the radius, and gives the final breadth of 7 ninths at k. This approximate boundary begins with the proper breadth one-third of the radius at the top, and encloses the exact area $\frac{1}{2}r^2$, which may be found by counting the number of squares on the paper, square-ruled with the sides *a ninth of the radius.* Thus bg encloses 14, gl encloses 20, and lk encloses $6\frac{1}{2}$ squares, in all, $40\frac{1}{2}$ squares; but each square is $r^2 \div 81$, so the whole area included is $\frac{1}{2}r^2$. It is called shortly the *treble-batter-boundary bglk*, and may be seen clearly near the centre of fig. 266, drawn in dotted lines with the *batters* written upon them. Beginning at b on the crown level, bg is drawn with a *batter of 1 in 4* for the first 4 ninths, then gl with a *batter of 1 in 2* for the next 4 ninths, and lastly, lk is drawn with a *batter of 1 in 1* for the remaining ninth of the radius.

HORIZONTAL-LOAD-AREA FOR SEMICIRCULAR RIB, LOADED WITH THE AREA BETWEEN ITSELF AND A HORIZONTAL STRAIGHT LINE OVER IT.

Let ACB be the circular quadrant, and DE the horizontal extrados (fig. 266). Suppose, for a little, that besides the shaded vertical-load-area bounded thus, there were also, as on fig. 265, a uniform load along the rib of intensity *half a radius*. Then the 45° slope, or 1 to 1 batter uo, gives the horizontal-load-area for the joint loads. Now remove, on fig. 266, the load along the rib, and at the same time remove the corresponding horizontal-load-area $abglkd$, it is to be drawn on the same base ad, and to be supposed to gravitate horizontally *outwards* from the rib, while the area already drawn gravitates horizontally *inwards* towards the rib. Hence, for the given vertical-load-area between the rib and the straight extrados, the horizontal-load-area lies between the 1 to 1 batter uo, and the curve approximated to by the treble-batter $bglk$. In form it is like a figure eight, as these two boundaries generally cross each other at a point p. The part below p gravitates horizontally *inwards*, and is to be reckoned as *positive*; the part above p gravitates horizontally *outwards*, and is *negative*. They are to have the same density as the given vertical-load-area.

Allowance for the Excess-weight of a Masonry-ring of Uniform thickness.—It is Rankine's practice in the masonry arch to assume, as a first approximation, that the line of stress is along the soffit. In the meantime we follow this practice in fig. 266, so that the quadrant of the circle is at once the soffit of the masonry-ring, the intrados of the vertical-load-area, and the linear-rib or line of stress. It will be seen that the weight of the masonry-ring has been included in the vertical-load-area, just as if it were of the same density w as the superstructure. It is usually fifty per cent. denser, and its uniform thickness, see $t_0 = \frac{1}{15}r$, on the figure, may be taken as a fifteenth of the radius of the soffit, and although its area lies wholly above the soffit it may be assumed to act along it in the meantime. We have then to consider an additional vertical load uniformly spread along the rib, and mapped out by two concentric circles at a distance apart $\frac{1}{15}r$, and of density $\frac{1}{2}w$. But this is the same as if the two concentric circles were at a distance apart *one-fifteenth of a half-radius*, and the area of the full density w. This is exactly one-fifteenth of the load which for convenience we assumed along the rib, and the removal of which introduced the treble-batter *bglk*. It is only necessary, then, to restore a fifteenth part of the area *abglkd* by pricking back $bb' = \frac{1}{15}ba$ and $kk' = \frac{1}{15}kd$, and similarly at g and l, and drawing the dotted treble-batter instead. Or shortly, with the line of stress assumed to be along the soffit, an allowance is to be made for a masonry-ring of an excess density of 50 per cent., and of a uniform thickness a fifteenth of the radius by *receding the treble-batter boundary* one-fifteenth.

Allowance for the Excess-weight of a Masonry-ring, Thickening outwards from Keystone to Skewbacks.—In the best French and English practice, the skewbacks of the segmental circular arch are from *once and a half* to *twice* as thick as the keystone. It depends in part on the size of the segment, and it will answer all purposes to suppose the masonry-ring included between two slightly excentric circles, with their centres on one vertical line, with their crowns at a distance apart t_0 and the pair of points 60° out from the crown $2t_0$ apart nearly. They are shown on fig. 266, by the soffit and the upper dotted circle. The lower dotted circle is concentric with the soffit, and we have already considered the excess-weight of that part. Consider now the two dotted circles, their crowns touch, and if completed, their greatest distance apart is at their lowest points, and equals the difference of their diameters. At the points left of the centre, their distance apart is half as much, being nearly the difference of the radii, slightly more or less, according as we measure along a horizontal line through the one or other centre. It will be seen then, that the distance apart of the two dotted circles on fig. 266, measured along the radius through O, at any angle i out from the crown is given closely by $z = 2t_0(1 - \cos i)$, as at the three cardinal points for $i = 0$, 90°, and 180° we have $z = 0$, $2t_0$, and $4t_0$, the proportions indicated, while with $i = 60°$ we have $z = t_0$ as required.

An element of the area between these dotted circles is $z \, . \, ds$ or $zrdi$, and the vertical-load-area from the crown out to i is the definite integral of this between the limits 0 and i. Hence

$$\begin{aligned} P &= r \int zdi = 2rt_0 \int (1 - \cos i)di \\ &= 2rt_0 \left[i - \sin i\right] = 2t_0 \left[ri - r \sin i\right] \\ &= 2t_0 \left[s - x\right]. \qquad (4) \end{aligned}$$

Interpreting the two terms separately :—The first is a uniform load along the arc equivalent to the area between two concentric circles $2t_0$ apart;—The second term is a negative load uniform along the span. That is a parallelogram of height $2t_0$ to be *taken off* the span. But since the excess-weight is only $\frac{1}{2}w$, each of the uniform loads is to be an area of uniform breadth t_0, and of the normal density w, the one added along the rib and the other taken off the span. The corresponding conjugate-loads we already know.

On fig. 266, suppose the load only to reach up to the horizontal through the crown A, that is, the pair of parallelograms AE and ao simultaneously removed. The conjugate horizontal-load-area is drawn inside the arch in dotted lines. One boundary is the dotted 45° line AgB, and the other the dotted treble-batter boundary $bglk$. This last is to recede $\frac{3}{15}$ths of the area bO, by pricking back $bb' = \frac{3}{15}bA$ and $kk' = \frac{3}{15}kO$, and similarly at g and l. One-fifteenth part is due to the uniform part of the ring, and the other two-fifteenths to the first term of the spread-out part. At the same time the boundary AB is to recede $\frac{1}{15}$th of a radius into the position $a'n'$, due to the second term. Lastly, $a'o'$ is to be added on when we restore AE. The conjugate horizontal-load-area as modified for the thickening-out ring is shown shaded, Q_1 is the portion of it pushing in below p'', while Q_2 is the part pulling out above p''.

Note that at the point g, where the two dotted boundaries cross, the effect of the weight of the thickening-out part of the arch-ring makes both boundaries move to the right the same distance. For the one moves a $\frac{1}{15}$th part of r, and the other $\frac{2}{15}$ths of the breadth of the figure bO, which at g is almost half of r. Hence p' and p'' are practically at one level whether the ring be uniform or thicken outwards.

Note, however, that the assumption that the area between the dotted circles acts along the soffit makes the approximation much coarser than in the case of the uniform ring.

RANKINE'S POINT AND JOINT OF RUPTURE.

We are now in a position to define Rankine's point of rupture, and to exhibit it to the eye by our graphical construction on fig. 266. The point p' or p'' projected horizontally on to the circle gives P the point of rupture, that is, the point on the rib where the conjugate horizontal-load changes sign. Below P there is required a thrust Q_1 on the back of the arch, but above P an outward pull Q_2. The angle of rupture is the slope which the tangent to the rib at P makes with the horizon, which in the circle is the same as *AOP*. For the shaded load, fig. 266, between the circle itself and an extrados $\frac{1}{10}r$ over the crown, together with a slight additional load due to the excess density of the voussoirs, the shaded conjugate horizontal-load-area may be readily drawn on a large scale on square-ruled paper, taking *nine sides* of the square to represent the radius. *P* is to be projected from p'', and the angle of rupture measured with a protractor, when it will be found to be $AOP = 43° \ 15'$.

The joint at *P*, between two voussoirs there, is the joint of rupture, and the point where this joint meets the back of the masonry-ring gives Rankine's level of heavy backing.

In the simpler case, with no excess load along the linear rib *p* determines the point of rupture: see fig. 266. This is where the 45° line *uo* meets the treble-batter *bglk*, or more strictly the gentle curve to which that treble-batter closely approximates.

It is instructive to observe how *p* behaves as the extrados moves lower or higher over the crown. First let the extrados touch the crown *A*, then the 45° line *an* meets the treble-batter exactly at the joint *g*, the positive and negative parts of the conjugate horizontal-load-area are *glkn* and *gab*, respectively. They each contain exactly *six squares* or

$$gnk = gab = 6\left(\frac{r}{9}\right)^2 = \cdot\dot{0}7\dot{4}r^2.$$

Their algebraic sum is zero, and H_0 the thrust at the crown is zero. As the extrados cannot come lower down, *g* projected over to *G* gives the lowest possible position of the joint of rupture, and putting i_0 for the corresponding biggest value of the angle of rupture we have the cosine of i_0 given by the ratio of the height of *g* above *dk* to the radius, but the height of *g* is five sides of the square, that is, $\frac{5}{9}r$, so that $\cos i_0 = \cdot\dot{5}$ and $i_0 = 56° \ 15'$.

As the extrados *DEu* is placed higher, *p* moves up and *P* approaches the crown and i_0 is decreasing. It will be seen, by inspection of fig. 266, that the positive part of the conjugate-load-area is increasing and the negative part decreasing. Their algebraic sum expresses H_0 the thrust at crown, the unit being the weight of a cubic foot of the superstructure. That sum must be equal to the product of the height of the load over the crown, and the radius there. When the extrados reaches a height one-third of *r* above the crown *A*, the conjugate-load-area *bglks* is wholly positive, the point of rupture has reached the crown, i_0 is zero, and counting the squares

$$bks = as = 27\left(\frac{r}{9}\right)^2 = \frac{r^2}{3}\cdot$$

The area remains wholly positive for all higher positions of the extrados.

Since the triangle *peg* has one side sloping at 1 in 1, and the other at 1 in 4, it follows that the height of the triangle is $\frac{4}{3}$rds of its base *eg*, that is $\frac{4}{3}$rds of *AD*, the height of the load over the crown of the rib. It is convenient to express the height of the load over the crown as a fraction of the radius; if s is this important ratio, then $eg = AD = sr$.

For the height of p the point of rupture, we have the sum of the heights of g above OB, and of p above g. It is

$$r \cos i = \tfrac{5}{9}r + \tfrac{4}{3}sr,$$

and

$$\cos i = \tfrac{5}{9} + \tfrac{4}{3}s \tag{5}$$

is a close approximation to the angle of rupture.

For the positive part of the conjugate horizontal-load-area, we have

$$pok = gnk + go + peg$$

$$= 6\left(\frac{r}{9}\right)^2 + 5\left(\frac{r}{9}\right)\overline{eg} + \tfrac{2}{3}\overline{eg}^2$$

$$= r^2\left(\tfrac{2}{27} + \tfrac{5}{9}s + \tfrac{2}{3}s^2\right). \tag{6}$$

The negative portion equals in area

$$gab - ea + peg \quad \text{or} \quad r^2\left(\tfrac{2}{27} - \tfrac{4}{9}s + \tfrac{2}{3}s^2\right), \tag{7}$$

giving as their algebraic sum sr^2, which is the product of the height of the load over the crown and the radius there.

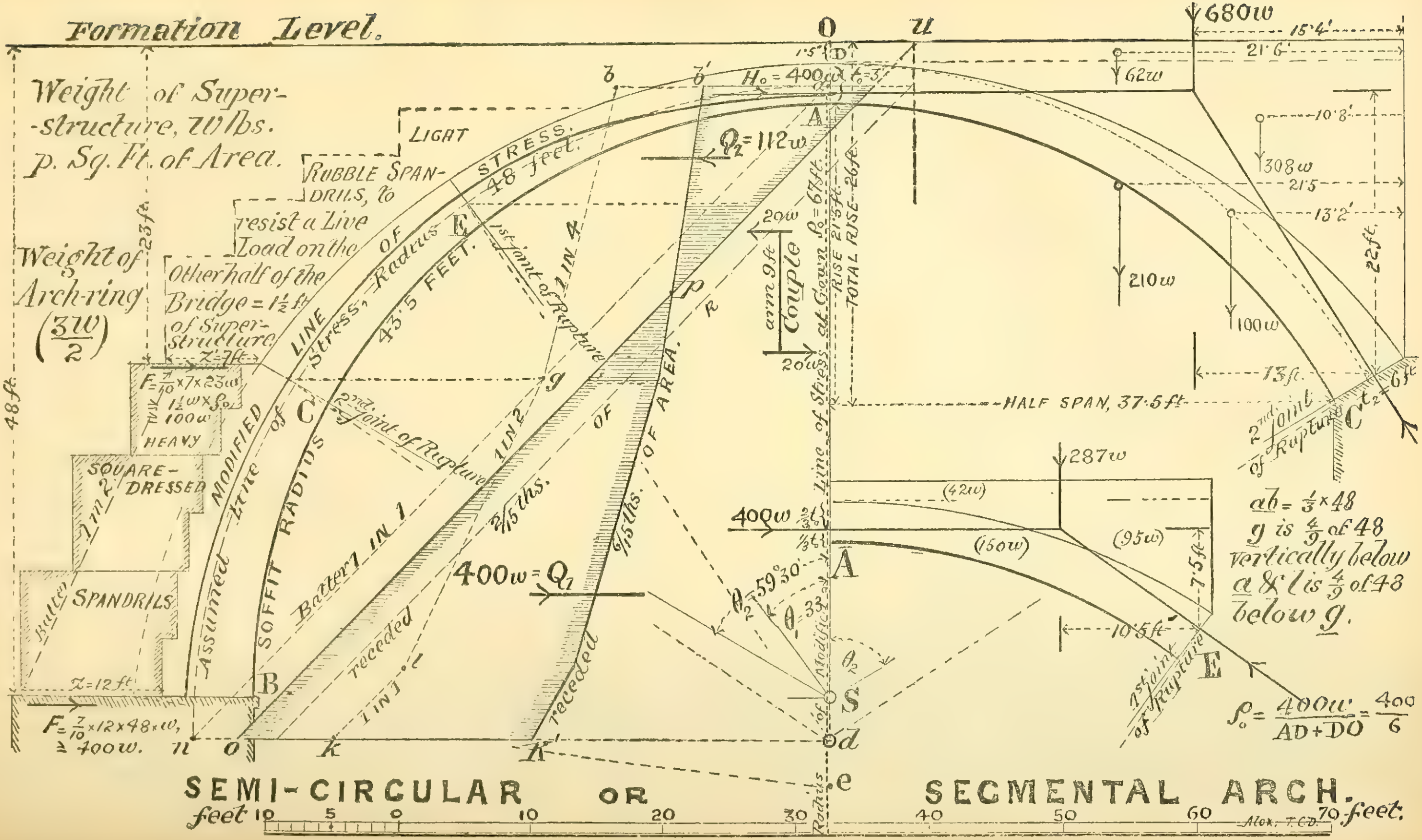
Formation Level.
Weight of Super-structure, w lbs. p. Sq. Ft. of Area.
Weight of Arch-ring (3w/2)
LIGHT
RUBBLE SPANDRILS, to resist a Live Load on the other half of the Bridge = 1½ ft of Super-structure
HEAVY
SQUARE-DRESSED
SPANDRILS
Batter
1 in 2
F = 7/10 × 7 × 23w; 1½w × ρ₀; ≥ 100w
7·7ft
23ft
48ft
z = 12ft
F = 7/10 × 12 × 48 × w, ≥ 400w.
n
o
k
K
B
C
E
g
p
R
b
b′
a
A
D
O
u
S
d
e
Assumed Modified Line of Stress
LINE OF STRESS, 48 feet.
SOFFIT RADIUS
Stress, Radius 43·5 FEET.
1st Joint of Rupture
2nd Joint of Rupture
1 IN 4
1 IN 2
Batter 1 IN 1
1 IN 1·2
receded
2/15 ths.
6/15 ths. OF AREA.
Q₂ = 112w
H₀ = 400w
t₀ = 3′
1·5′
400w = Q₁
20w
arm 9ft
Couple
Line of Stress at Crown ρ₀ = 67ft.
RISE 21·5ft.
TOTAL RISE 26ft.
HALF SPAN, 37·5ft.
680w
15·4′
21·6′
62w
10·8′
308w
21·5
13·2′
22ft.
210w
100w
13ft.
6ft
t₂
2nd Joint of Rupture
C′
287w
(42w)
(150w)
(95w)
7·5ft
10·5ft
400w
⅔t
⅓t
θ₂ = 59°30′
θ₁ 33°
θ₂
Modified
Radius of
1st Joint of Rupture
E
ab = ⅓ × 48
g is 4/9 of 48 vertically below a & l is 4/9 of 48 below g.
ρ₀ = 400w/(AD + DO) = 400/6
SEMI-CIRCULAR OR SEGMENTAL ARCH.
feet 10 5 0 10 20 30 40 50 60 70 feet.
Alex. T. Cd.

Fig. 267.

SEMICIRCULAR MASONRY ARCH.

In the application to the semicircular arch, the left quadrant of which is shown on fig. 267, the assumed line of stress is the quadrant *an* with the centre at *d*, and having a radius of 48 feet. It is taken up the middle of the masonry ring, which has a thickness $AD = t_0 = 3$ feet at the keystone, and a thickness twice as great, $t_2 = 6$ feet at the joint near C which is 60° out from the crown *a*. So far the ring is mapped out by two eccentric circles as already explained, one is the soffit $AECB$ drawn from the centre S with radius $SA = 43{\cdot}5$ feet, and the other from the centre *e* with radius $eD = 52{\cdot}5$ feet. Beyond the joint C the back of the ring is stepped to receive the heavy backing, and the ring ceases to widen out any further.

From *a* the middle of the keystone *ab* is laid off horizontally and equal to one-third of 48 the mean radius; then *bg* is drawn at the batter 1 in 4 till *g* is at a level $\frac{4}{9}$ths of 48 lower than *b*; next *gl* is drawn at the batter 1 in 2 till *l* is $\frac{4}{9}$ths of 48 lower in level than *g*, when *lk* is drawn at the batter 1 in 1. Also *agn* is drawn though *a* the crown of the assumed line of stress at the batter 1 in 1. These dotted lines are readily drawn on squared-paper, making *da* nine sides, they include the conjugate horizontal-load-area, neglecting altogether the weight of the half of the ring below the assumed line of stress, and the excess weight of the other half over the average weight of the superstructure which is w lbs. per cubic foot. The load between a horizontal through the crown *a* and the formation is in the meantime also neglected. Now the arch-ring is to be built of masonry half again as heavy as the superstructure so that the *excess* density of the lower half is $\frac{3}{2}w$, and of the upper half it is $\frac{1}{2}w$, or conjointly the excess weight of the arch-ring is w, and as this is twice as great as it was in fig. 266, the boundaries must now recede twice as much. So that, on fig. 267, the treble-batter-boundary *bglk* recedes $\frac{6}{15}$ths of the area into the position $b'k'$, while the 45° boundary *an* recedes $\frac{2}{15}$ths of 48 when there is added to it a parallelogram, by drawing *uo*, equal to the parallelogram between the horizontal through the crown *a* and the formation through O, which is at a height $aO = 3$ feet above it.

The area of $pk'o$, the positive part, is readily found by measurement or by counting the squares, or it may be calculated from the manner in which it was constructed. It is almost exactly 400 square feet, so that $Q_1 = 400w$ is the inward thrust with which the solid backing must resist the tendency of the arch to spread at the haunches. For this purpose the square-dressed heavy backing must stretch out at the springing joint B, a distance $z = 12$ feet. For the weight pressing the backing down on its base is then $12 \times 48w$; and taking the coefficient of friction of masonry on masonry at ·7, the frictional stability of the backing at the joint B is $F = \frac{7}{10} \times 12 \times 48w = 400w$ nearly. The level of this heavy backing might be up to the point p. Above this level the conjugate horizontal-load required is $Q_2 = 112w$ outward, and if it were practicable to apply this in any way, for instance, by the centring which might still be inside the arch, then the line of stress would be the circle of radius 48 feet as assumed, and the thrust at the crown $H_0 = Q_1 - Q_2 = 288w$. This is exactly the value of H_0, got by the rule for the thrust at the crown of the circular linear rib of 48 feet radius with the normal load at a of AO made up of the two parts $DO = 1{\cdot}5w$ and $AD = 3 \times \frac{3}{2}w$, in all equivalent to $6w$, so that $H_0 = 48 \times 6w = 288w$.

In practice the negative part Q_2 of the horizontal-load-area has to be left out, and for a masonry ring of uniform thickness the heavy backing may come up to the level of p, as recommended by Rankine. He gives the joint at 45°, as being always on the safe side. With the arch-ring as in fig. 267, we need only bring the backing up to the level of g, as the very thickening of the ring outwards is sufficient backing down to that point. It will be seen that we are leaving out a part of the positive area from p down to g, as well as the negative part Q_2. This positive part, shown shaded across from p down to g, is not completely left out; it is only applied lower down, so that the whole inward thrust $Q_1 = 400w$ of the square-dressed heavy backing is applied to the back of the arch between the joints B and C. A portion of the negative area from p upwards, $20w$ is also shaded across; equal in area to the shaded part below p. In this way we determine *two joints of rupture* E and C, one on each side of Rankine's joint. And just as the average position of his joint is 45°, so the average positions of our pair are 30° and 60° out from the crown.

The leaving out of the horizontal load above C requires the line of stress from that joint to the same joint on the other side of the crown to be *modified*. For want of the outward pull Q_2, the arch tends to shrink horizontally and the crown to go up. This brings the centre of stress at the crown joint down from the bisecting point to the lower trisecting point. As the thrust

H_0 at the crown is now greatly increased, we must divide $400w$ by $AD + DO$ or $6w$, when we have the radius of curvature of the modified line of stress at its crown, $\rho_0 = 67$ feet. A circle swept out with this radius, having its centre on the vertical through the crown, and beginning at the lower trisecting point of the crown joint A, crosses the assumed line of stress, and approaches the upper boundary of the middle third of the arch-ring. Consider now the thrust at E, the upper end of the portion of the arch-ring CE. At E, its upper end, a thrust a little greater than $400w$ has shifted from the bisecting point to nearly the upper trisecting point of the joint, a distance nearly half a foot. This is the same as applying at E, the upper end of the block CE, a left-handed couple $400w + \frac{1}{2} = 200w$, and takes the place of the couple $20w \times 9$ feet, constituted by the shaded-across parts of the horizontal-load-area, above and below p, which ought to act on the back of the block CE, but which have been left out as far as that block is concerned.

Three points on the left half of the *modified line of stress*, fig. 267, are the lower trisecting point of the *crown joint* A, the upper trisecting point of the *first joint of rupture* E, and a point slightly above the middle of the *second joint of rupture* C. This is to be verified as follows. Consider the structure rigid from A to C, and examine whether the three forces acting on it meet at a point. Namely, the weight of the structure from C to O $680w$, acting vertically downwards through the centre of gravity, the horizontal force $H_0 = 400w$ acting horizontally through the centre of the stress at the crown joint assumed to be its lower trisecting point and the oblique thrust at the centre of stress of the joint C assumed to be sensibly at its middle point. Taking moments about this last point, we have $680w \times 13$ feet and $400w \times 22$ feet sensibly equal. It is only necessary then to show that the whole weight from A down to C is $680w$, and the centre of gravity 15·4 feet measured horizontally from the right vertical boundary. The weights and leverages of four parts into which the area OAC can be divided are printed on the right half of the fig. 269. A simpler way is to take half the depth of the area at each end and five intermediate equidistant depths amplifying by adding, with the dividers, to the depth at any point one-half of that part which crosses the ring, its density being 50 per cent. in excess. These are given in the first column below. The breadth of the area is 43 feet, so that the common interval between the depths is $7\frac{1}{6}$ feet. In the second column the levers 0, 1, 2, 3, 4, 5, 6 are in terms of the common interval. The third column gives the products of each depth by its lever measured from the right-hand vertical boundary:—

12	0	00
29	1	29
20	2	40
15	3	45
9	4	36
7	5	35
3	6	18
95		95)203
$7\frac{1}{6}$		2·137
		$7\frac{1}{6}$
680		
		15·4 nearly.

Deduct 2·4, and 13 feet is the horizontal distance from *C* to the centre of gravity.

The sum of the first column when multiplied by $7\frac{1}{6}$ feet, the common horizontal interval, gives the area; and the quotient obtained by dividing the sum of the third column by the sum of the first column, when multiplied by $7\frac{1}{6}$, gives the position of the centre of gravity.

Again, consider the structure rigid from *A* to *E* when the three forces must again meet at a point: namely, $287w$ vertical through the centre of gravity, $400w$ horizontal through the lower trisecting point of the joint *A*, and an oblique force parallel to the tangent of the line of stress acting through the upper trisecting point of *E*. Taking moments about this last point, we have $400w \times 7·5$ ft. and $287w \times 10·5$ ft. sensibly equal.

The areas and levers of the various regular figures into which the area under consideration can be divided are printed on the lower part of the right half of fig. 267. Or taking the amplified depths as above:—

6	0	00
15	1	15
9	2	18
7	3	21
3	4	12
40		40)66
$7\frac{1}{6}$		$1·65 \times 7\frac{1}{6} = 11·8$ nearly
		off 1·3
287		
		10·5

In this way we have indirectly designed a Segmental Arch, the half span of soffit being 37·5 feet, the rise of the soffit 21·5 feet, the formation level being 4·5 feet above the crown of soffit, or otherwise the total rise of the formation above the springing level is 26 feet. (See the right half of fig. 267.) We know *five points* on the line of stress: the lower trisecting point of the crown joint A, the upper trisecting points of the first pair of joints of rupture E and E at about 30° out from the crown one on each side, and a pair of points slightly above the central points of the two springing joints C and C. These five points being just comfortably located in the "Kernel," consisting of the middle third of the arch-ring, we will prove that the line of stress lies wholly inside that "Kernel."

It is *not sufficient*, as Rankine does in his *C. E.*, p. 421, to insure the centres of stress to be in the middle third of crown-joint and his (45°) joint of rupture, as there is a point of *maximum curvature* on line of stress near 30°, of which he was unaware.

Light elastic rubble spandrils are shown riding out to a stretch of $z = 7$ feet on the heavy backing or abutments, but they only exert a horizontal reaction when the live load comes on the opposite half of the arch. They can offer a horizontal resistance $7 \times 23w \times \cdot 7 = 113w$. Now the height of superstructure representing the live load is $1\cdot 5w$; this when multiplied by ρ_0 or 67 feet, gives $100w$, the excess of the horizontal thrust of the loaded half over the unloaded half.

Equilibrium Rib or Transformed Catenary.

It will be seen on fig. 267 that the modified line of stress, in the segmental arch-ring CC, bears only a vertical load, the bulk of which is given by the area included between itself and the formation level. The remainder of the load is spread in a very *similar manner* because of the thickening of the arch-ring from the crown outwards.

The curve which is a balanced rib for the vertical-load-area alone included between itself and a horizontal straight extrados is known as the Equilibrium Curve. Let AB, fig. 268, be a finite arc of the curve balanced under the vertical-load-area $NOAB$, so that H and T meet at the same point g on W, drawn vertically through the centre of gravity of that area.

$H = T \cos \theta$, $W = T \sin \theta$, $W = H \tan \theta$. Expressing the area by an integral, and the tangent of the slope at B by a differential coefficient

$$w \int y dx = H \frac{dy}{dx},$$

and differentiating each side

$$wy = H \frac{d^2y}{dx^2}, \quad \text{or} \quad y = \frac{H}{w} \frac{d^2y}{dx^2},$$

or with m^2 as a short name for $\frac{H}{w}$, the differential equation to the equilibrium curve is

$$y = m^2 \frac{d^2y}{dx^2}. \tag{8}$$

There are only two functions of x which differ from their original values by a constant m^2, when differentiated twice successively, so that their sum is the general solution of this equation. It is

$$y = A\varepsilon^{\frac{x}{m}} + B\varepsilon^{-\frac{x}{m}}. \tag{9}$$

When $x = 0,\ y = y_0 = OA,$ giving $y_0 = A + B.$

By differentiating (9), we have

$$\tan\theta = \frac{dy}{dx} = \frac{1}{m}\left(A\varepsilon^{\frac{x}{m}} - B\varepsilon^{-\frac{x}{m}}\right). \tag{10}$$

Now when $x = 0$, $\tan\theta = 0$, giving $0 = A - B$, so that $A = B = \frac{1}{2}y_0$, and

$$\left.\begin{aligned} y &= \frac{y_0}{2}\left(\varepsilon^{\frac{x}{m}} + \varepsilon^{-\frac{x}{m}}\right), \\ \text{or} \qquad y &= r\frac{m}{2}\left(\varepsilon^{\frac{x}{m}} + \varepsilon^{-\frac{x}{m}}\right), \end{aligned}\right\} \tag{11}$$

where $y_0 = OA$ is put equal to rm.

If $r = 1$, the curve is the common catenary. It is shown inverted by a thick line on fig. 268. Any other of the curves can be described by dividing the ordinates of the catenary in a constant ratio r. The catenary might be drawn on a sheet of indiarubber fixed along its edge OX, and previously stretched in the direction OY only. If the sheet were now to contract in that direction only, the catenary would become in succession the *Transformed Catenaries AB*, *ac*, *bd*, &c. In this way a whole family of transformed catenaries is derived from the common catenary by assigning a succession of graduated values to r, the ratio of transformation. There is only *one such family*, for the catenary itself like the circle admits of no variety, you may make m larger or smaller, but you only draw the catenary to a coarser or finer scale. It is therefore convenient in tabulating numerical values of the different dimensions of the family of transformed catenaries to assign to m the value unity, and to r a series of fractional values graduated at close intervals.

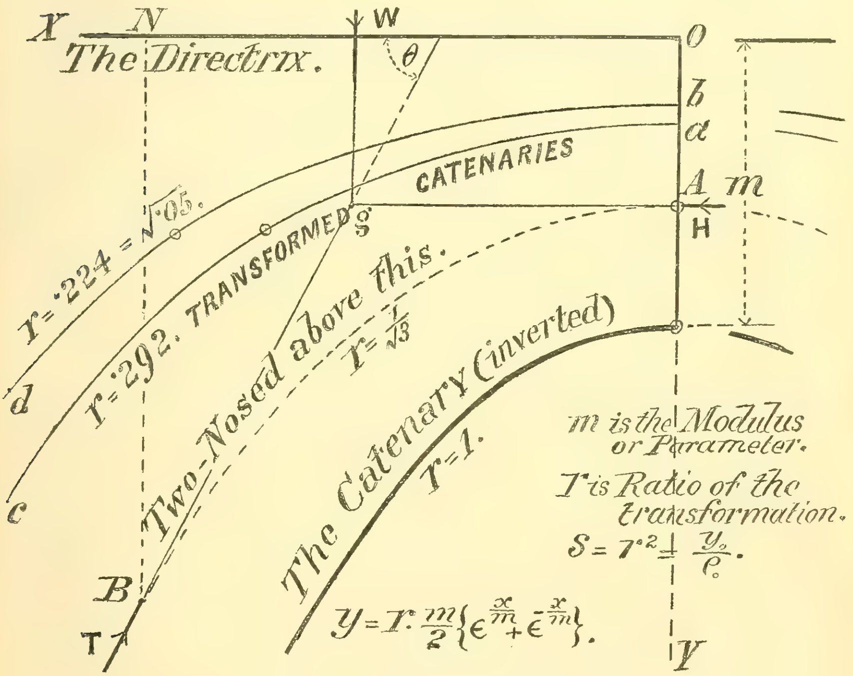
X
N
W
O
The Directrix.
θ
b
a
A
m
CATENARIES
g
H
$r = \cdot 224 = \sqrt{\cdot 05}$.
$r = \cdot 292$. TRANSFORMED
Two-Nosed above this.
$r = \frac{1}{\sqrt{3}}$
The Catenary (inverted)
$r = 1$.
d
c
m is the Modulus or Parameter.
r is Ratio of the transformation.
$S = r^2 \pm \frac{y_0}{\rho_0}$.
B
T
$y = r \cdot \frac{m}{2}\left\{\epsilon^{\frac{x}{m}} + \epsilon^{-\frac{x}{m}}\right\}$.
Y

Fig. 268.

If the curve *AB* with its tangents *Ag* and *Bg* be transformed towards *ON* by the shrinking of the stretched sheet of indiarubber, it is evident that g travels on a vertical locus. It follows that the centres of gravity of the areas from *ON* down to *bd*, to *ac*, to *AB*, &c., all lie on one vertical, and the same is true of the areas included between any pair of the curves. Hence any one of the curves is a balanced rib for the vertical-load-area alone between any other pair.

As a matter of fact, *bd* is the modified line of stress, in the segmental arch, fig. 267. Its soffit is a circle coincident nearly with *ac*, while *ONX* is the formation level. The portion of extra density lies between another pair of which *ac* is one; the other being a curve a little higher than *bd*, but not shown on fig. 268. Of course for this comparison between the figs. 267 and 268, the scale of fig. 268 must be such that *Oa* shall measure absolutely 4·5 feet. We then have $Ob = 3·5$ feet $= y_0$, while ρ_0 is 67 feet, so that for the curve *bd* we have the ratio $y_0 : \rho_0 = ·05$ nearly.

Now, we only used m^2 as a convenient abbreviation of $H_0 \div w$, that is the thrust at the crown of the equilibrium curve expressed in square units of the vertical-load-area so that $H_0 = wm^2$, but from the general rule for the thrust at the crown of a rib we have $H_0 = wy_0\rho_0$, so that $\rho_0 = m^2 \div y_0 = m \div r$, but $y_0 = rm$, so that

$$\frac{y_0}{\rho_0} = r^2 = s. \qquad (12)$$

That is—The important ratio *s*, namely, that of the depth of the load at the crown to the radius of curvature there, is, in the transformed catenary, exactly the square of the ratio of transformation. On fig. 268, *AB* is the transformed catenary for which $s = \frac{1}{3}$. All the members of the family of curves below this are like the catenary or parabola in this, that their curvature is sharpest at the vertex. They are of little importance in their application to arches.

On the other hand, all the curves for which *s* is less than one-third such as *ac*, *bd*, differ in form completely from that of the catenary or parabola. They have a flat curvature at the vertex *a* or *b*. The curvature becomes sharper and sharper as you travel out from the crown on both sides, till it is sharpest of all at a pair of points on each curve, one on each side of the vertex, one of the pair being shown with a little ring on fig. 268. From this, as you go outward, the curvature begins to flatten till at *d* or *c* it is again exactly the same as it was at the crown. This *finite part* of the curve, having the same curvature at its middle point and at its two ends, must now engage our attention. Beyond *d* or *c* the curves flatten indefinitely outwards.

Observe that the value $s = \frac{1}{3}$, which divides the transformed catenaries into the two sets, those sharpest at the crown and those flattest at the crown, corresponds exactly to the same value of s in fig. 266. On that figure $s \geq \frac{1}{3}$ makes the conjugate horizontal-load-area wholly positive. So that in an example like fig. 267, but with the formation level above the crown of the assumed line of stress by a third or more of its radius, the horizontal-load-area would be wholly positive or inwards. If then an upper part were left out, that would *decrease* the thrust at the crown, so that the modified line of stress would sharpen at the crown instead of flattening. In bridge practice the formation would never be at such great heights.

THE TWO-NOSED CATENARIES.

We have ventured to use this name for the important part of the family of transformed catenaries having their ratio of transformation less than the square root of one-third. We shall only give the part of the treatment necessary for the construction of the tables. That their importance is appreciated is shown by their adoption by Professor Howe in the chapters

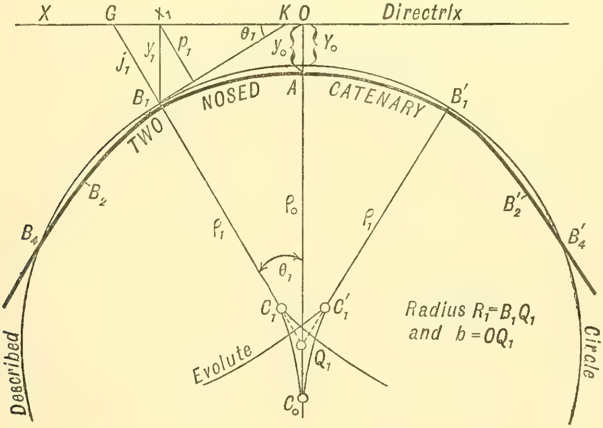

Fig. 269.

of his elegant treatise which deal with the masonry arch.* We are indebted to the courtesy of the Royal Irish Academy for the use of their blocks, and especially for a photo-block from the design of an arch by this method engraved on a grand scale by the splendid liberality of the Academy.†

For simplicity, put m equal to unity, and

$$y = \frac{r}{2}(\varepsilon^{x} + \varepsilon^{-x}) = \frac{d^2y}{dx^2}.$$

$$\frac{dy}{dx} = \frac{r}{2}(\varepsilon^{x} - \varepsilon^{-x}) = \tan\theta = \sqrt{(y^2 - r^2)}.$$

$$\rho = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{\sec^3\theta}{y} = \frac{(y^2 + 1 - r^2)^{\frac{3}{2}}}{y},$$

or

$$\rho = \frac{(y^2 + a^2)^{\frac{3}{2}}}{y} \quad \text{if} \quad a^2 = 1 - r^2 = 1 - s.$$

$$\frac{d\rho}{dy} = \frac{\sqrt{(y^2 + a^2)}\,(2y^2 - a^2)}{y^2}.$$

$$\frac{d^2\rho}{dy^2} = \frac{\sqrt{(y^2 + a^2)}}{y^2} \times 4y + (2y^2 - a^2)\,\frac{d}{dy}\,.\,\frac{\sqrt{(y^2 + a^2)}}{y^2}.$$

Now $(2y^2 - a^2) = 0$ makes $\frac{d\rho}{dy} = 0$, and as a first step makes

$$\frac{d^2\rho}{dy^2} = 4\,\frac{\sqrt{(y^2 + a^2)}}{y} = 4\sqrt{3},$$

a positive quantity upon a further substitution from $2y^2 - a^2 = 0$.

Hence $2y^2 - a^2 = 0$ gives the values of y, which make ρ a minimum, namely, a pair of points symmetrical about the crown or vertex. They are the noses, and are shown at B_1 and B'_1 on fig. 269. The abscissa, ordinate, slope, and radius of curvature of the "nose" B_1 are designated by x_1, y_1, θ_1, and ρ_1. If ρ_1 be produced to meet the vertical through O at Q_1, and if now a circle be drawn from Q_1 with a radius $Q_1B_1 = R_1$, it will touch the curve externally at its two "noses." This is the *described circle*, and is outside of the finite arc $B_2AB'_2$ with which we are concerned.

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* "A Treatise on Arches," by Malverd A. Howe, C.E., Professor of Civil Engineering, Rose Polytechnic Institute, Terre Haute, Indiana. New York: John Wiley & Son. London: Chapman & Hall, Ltd., 1897.

† "On Two-Nosed Catenaries and their Application to the Design of Segmental Arches," by T. Alexander, M.A.I., Professor of Engineering, Trinity College, Dublin, and A. W. Thomson, B.Sc., Lecturer in the Glasgow and West of Scotland Technical College, *Transactions of the Royal Irish Academy*, Vol. xxix., Part iii., 1888.

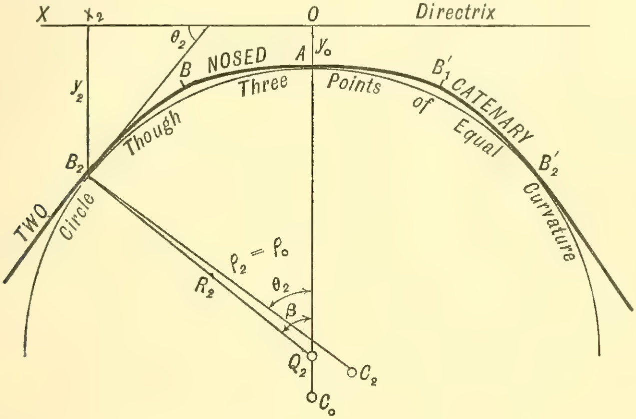


Fig. 270.

On the other hand, fig. 270 shows a circle drawn from the centre Q_2 with a radius $Q_2B_2 = R_2$ passing through the three points B_2, A, and B'_2 of common curvature. It is *the three-point* circle. It touches the curve internally at A, and sensibly touches it at B'_2 and B_2, as the slope of the curve θ_2 and that of the circle β at B_2 never differ by more than two and a half degrees. For all practical purposes this circle is inscribed in the curve. There is evidently a mathematically inscribed circle, but its equations are very troublesome.

For the described-circle, we have

$$y_1 = \frac{a}{\sqrt{2}} = \sqrt{\frac{1 - r^2}{2}} = \sqrt{\frac{1 - s}{2}}. \tag{13}$$

$$\rho_1 = \frac{(y_1^2 + a^2)^{\frac{3}{2}}}{y_1} = 3\sqrt{3}\, y_1^2 = \frac{3\sqrt{3}}{2}(1 - s), \tag{14}$$

$$\tan\theta_1 = \sqrt{(y_1^2 - s)} = \sqrt{\frac{1 - 3s}{2}}. \tag{15}$$

So that when $s = \frac{1}{3}$ then $\theta_1 = 0$, and the crown is the sharpest point.

Again, by adding the expressions for y and for its first differential coefficient, we get

$$r\varepsilon^x = y + \tan\theta,$$

$$x = \log\frac{y + \tan\theta}{r}, \qquad x_1 = \log\sqrt{(1-s)} + \sqrt{(1-3s)},$$

$$R_1 = x_1 \operatorname{cosec}\theta_1, \qquad Y_0 = y_1 + R_1\cos\theta_1 - R_1.$$

For the three-point circle, we have,

$$\sec^3\theta = \rho y = \rho\sqrt{(r^2 + \tan^2\theta)}.$$

$$\sec^6\theta = \rho^2(r^2 + \tan^2\theta),$$

$$\text{but} \quad \rho_2 = \rho_0 = \frac{1}{r} = \frac{1}{\sqrt{s}} \quad \text{and} \quad y_0 = r = \sqrt{s}\,;$$

$$\therefore \quad r^2\sec^6\theta_2 = r^2 + \tan^2\theta_2,$$

or

$$\frac{1}{r^2} = \frac{\sec^6\theta_2 - 1}{\sec^2\theta_2 - 1},$$

or

$$\frac{1}{s} = \sec^4\theta_2 + \sec^2\theta_2 + 1 = \frac{\rho_0}{y_0}. \tag{16}$$

Solving this like a quadratic equation

$$\left.\begin{aligned} \sec^2\theta_2 &= \sqrt{\left(\frac{1}{s} - \tfrac{3}{4}\right)} - \tfrac{1}{2}, \\ \tan^2\theta_2 &= \sqrt{\left(\frac{1}{s} - \tfrac{3}{4}\right)} - \tfrac{3}{2}. \end{aligned}\right\} \tag{17}$$

So that when $s = \frac{1}{3}$, then $\theta_2 = 0$, and B_2 comes to the crown just as B_1 did. Also

$$y_2 = \sqrt{s}\,\sec^3\theta_2,$$

$$x_2 = \log\left(\frac{y^2}{r} + \frac{\tan\theta_2}{r}\right) = \log\left(\sec^3\theta_2 + \frac{\tan\theta_2}{\sqrt{s}}\right).$$

As $\theta_2 = \beta$ nearly

$$\frac{1}{s'} = \sec^3\theta_2 + \sec^2\theta_2 + \sec\theta_2 = \frac{R_2}{y_0}. \tag{18}$$

By Euc. iii. 35 $$x^2_2 = (y_2 - y_0)\{2R_2 - (y_2 - y_0)\},$$

$$R_2 = \frac{x^2_2}{2(y_2 - \sqrt{s})} + \frac{y^2 - \sqrt{s}}{2},$$

$$\delta_0 = y_0 - Y_0,$$

$$\delta_2 = R_1 - R_2 + \Delta \cos \theta_2 \text{ sensibly},$$

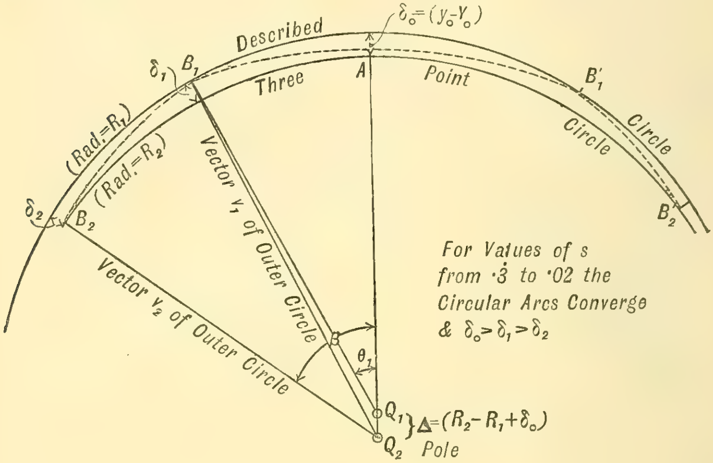
where $$\Delta = R_2 - R_1 + \delta_0. \quad \text{(See fig. 271.)}$$

Table A, for each assigned value of s, the following ten quantities are calculated by the above equations in the order y_0, ρ_1, θ_1, R_1, Y_0, ρ_2, or ρ_0, θ_2, R_2, δ_0, δ_2. The values assigned to s are at equi-distances apart.

TABLE A.

m	$r = \sqrt{s}$	$s = \frac{y_0}{\rho_0}$	θ_1	θ_2	R_1	R_2	ρ_1	$\rho_0 = \rho_2$	$y_0 = r$	Y_0	δ_0	δ_2
1	$\sqrt{\frac{1}{3}}$	·333	0°	0°	1·73	1·73	1·73	1·73	·577	·577	·000	·000
1	$\frac{1}{2}$	·250	19	28	1·97	1·98	1·94	2·00	·500	·499	·000	·000
1	—	·180	25	39	2·24	2·25	2·13	2·35	·424	·419	·004	·002
1	—	·170	26	41	2·29	2·30	2·15	2·42	·412	·406	·005	·003
1	—	·160	27	42	2·34	2·35	2·18	2·50	·400	·392	·007	·004
1	—	·150	27	44	2·39	2·40	2·20	2·58	·387	·378	·009	·005
1	—	·140	28	45	2·44	2·45	2·23	2·67	·374	·363	·011	·006
1	—	·130	28	46	2·50	2·51	2·26	2·77	·360	·347	·013	·008
1	—	·120	29	48	2·57	2·58	2·28	2·88	·346	·330	·016	·010
1	$\frac{1}{3}$	·111	30	49	2·63	2·63	2·30	3·00	·333	·313	·019	·012
1	—	·100	30	51	2·72	2·72	2·33	3·16	·316	·292	·024	·016
1	—	·090	31	52	2·80	2·80	2·36	3·33	·300	·270	·029	·019
1	—	·080	31	54	2·89	2·88	2·39	3·53	·282	·247	·035	·025
1	—	·070	32	55	3·00	2·99	2·41	3·77	·264	·220	·043	·032
1	$\frac{1}{4}$	·063	32	57	3·09	3·07	2·43	4·00	·250	·199	·051	·039
1	—	·050	33	59	3·28	3·24	2·46	4·47	·223	·156	·066	·054
1	$\frac{1}{5}$	·040	33	61	3·46	3·40	2·49	5·00	·200	·115	·084	·073
1	—	·000	35	90	∞	∞	2·60	∞	·000	—	—	—

Fig. 271 shows the finite arc $B_2B_1AB'_1B'_2$ sandwiched between the arcs of these two circles, and $\delta_0 = y_0 - Y_0$ is their distance apart at the crown, while δ_2 is their distance apart at each end. These distances are so small, that as far as being the *boundary* of a load-area the arc of the three-point-circle is practically the same as the finite arc of the two-nosed catenary itself. The dotted curve should pass through A.



$\delta_0=(y_0-Y_0)$
Described
δ_1
B_1
Three
A
Point
B'_1
(Rad.=R_1)
(Rad.=R_2)
Circle
Circle
δ_2
B_2
B'_2
Vector v_1 of Outer Circle
Vector v_2 of Outer Circle
For Values of s from ·3 to ·02 the Circular Arcs Converge & $\delta_0>\delta_1>\delta_2$
B
θ_1
Q_1
$\Delta=(R_2-R_1+\delta_0)$
Q_2
Pole

Fig. 271.

TABLES FOR THE IMMEDIATE DESIGN OF SEGMENTAL ARCHES.

Table B (p. 518, *et seq.*) is derived from Table A (p. 513) line by line by dividing the linear quantities in Table A by the value of R_1. Ratios r, s, θ_1, and θ_2 are unaltered. The description of the table is fully given on the face of it.

In the Supplementary Table B_1, the thickness of the keystone t_0 is made equal to $3\delta_0$. (See *DA* on fig. 272.) This confines the line of stress to a "kernel," which is the middle third of the arch-ring. The upper limit of this "kernel" is the described-circle of the line of stress. It is still the described-circle of the new line of stress due to a uniform live load all over the span, which is equivalent to raising the directrix. The depth of the crown of the *soffit* from the formation-level or directrix is $d = y_0 + \delta_0$. (See *OA* on figs. 267 and 272, and *oa* on fig. 268.) The *soffit* is the three-point-circle of a lower-down member of the same family as the line of stress itself. This is accomplished as follows. In any line of Table B_1, take the value of d, which divide by the value of m in that line produced into Table B. Look for the quotient in Table A, under the heading y_0; take the corresponding value of R_2 and place it in Table B$_1$, under the heading R, but, of course, first multiplying by m. The rise and span of the soffit are $k = R$ vers θ_2, and $2c = 2R \sin \theta_2$.

The quantities engraved on fig. 272 correspond to those set down on the fourth line from the bottom of Tables B$_1$ and B, only multiplied each by 50·07. This line will serve to explain the construction of the Table B$_1$, while inspection of fig. 272 may help to make it clear. The position of the line fourth from the bottom is designated as the line on Table B$_1$ with $s = \cdot 05$. In the third column, 104 is the maximum multiplier which may be used on the linear quantities in this line that there may be a factor of safety 10 against crushing a sandstone-key. First, the thrust at the crown $wd \times \rho_0 = 140 \times \cdot 089 \times 1\cdot 3634 = 16\cdot 987$, and dividing this by $t_0 = \cdot 061$, the number of square feet exposed to it, we get the average thrust on the crown-joint to be 278 lbs. per square foot. Multiplying by 2, as the maximum is double the average (see fig. 263), the maximum stress on the cheek of the sandstone-key is 556 lbs. per square foot. Now this increases directly with the multiplier used; hence dividing 556 into 57,600 lbs. per square foot, the resistance to crushing of sandstone allowing a factor of safety of 10, we get 104. Any smaller multiplier gives a factor of safety proportionately greater. Thus on fig. 272, the multiplier being 50·07, gives the factor of safety 20. In the fourth column 155 feet is the maximum span between the points of rupture for a sandstone bridge with its proportions as on that line; it is $2c = 1\cdot 498$ multiplied by 104. The first two columns are the same numbers divided by three for strong brick, while the fifth and sixth columns are the same numbers doubled for granite. Refer to p. 477 under "strength." In column seven $(d - t_0) = \cdot 028$ is the *surcharge*; it is DO on fig. 272, and should be only sufficient to make a soft bed for the railway that the sleepers may not hammer the top of the keystone.

It only remains to explain how $t_2 = \cdot 123$, the thickness of the *voussoir* at the joint of rupture sloping at θ_2, that is at the skewback or springer of the segmental arch, is obtained. The radius of the *described-circle* (fig. 272), as given on the fourth lowest line of Table B is 1. Also $QA = R = \cdot 869$ is the radius of the *soffit*. If now a circle were drawn from the centre Q, concentric with the soffit and touching the described circle at the crown, its radius would be $(R + 2\delta_0) = \cdot 869 + 2 \times \cdot 0204 = \cdot 9098$.

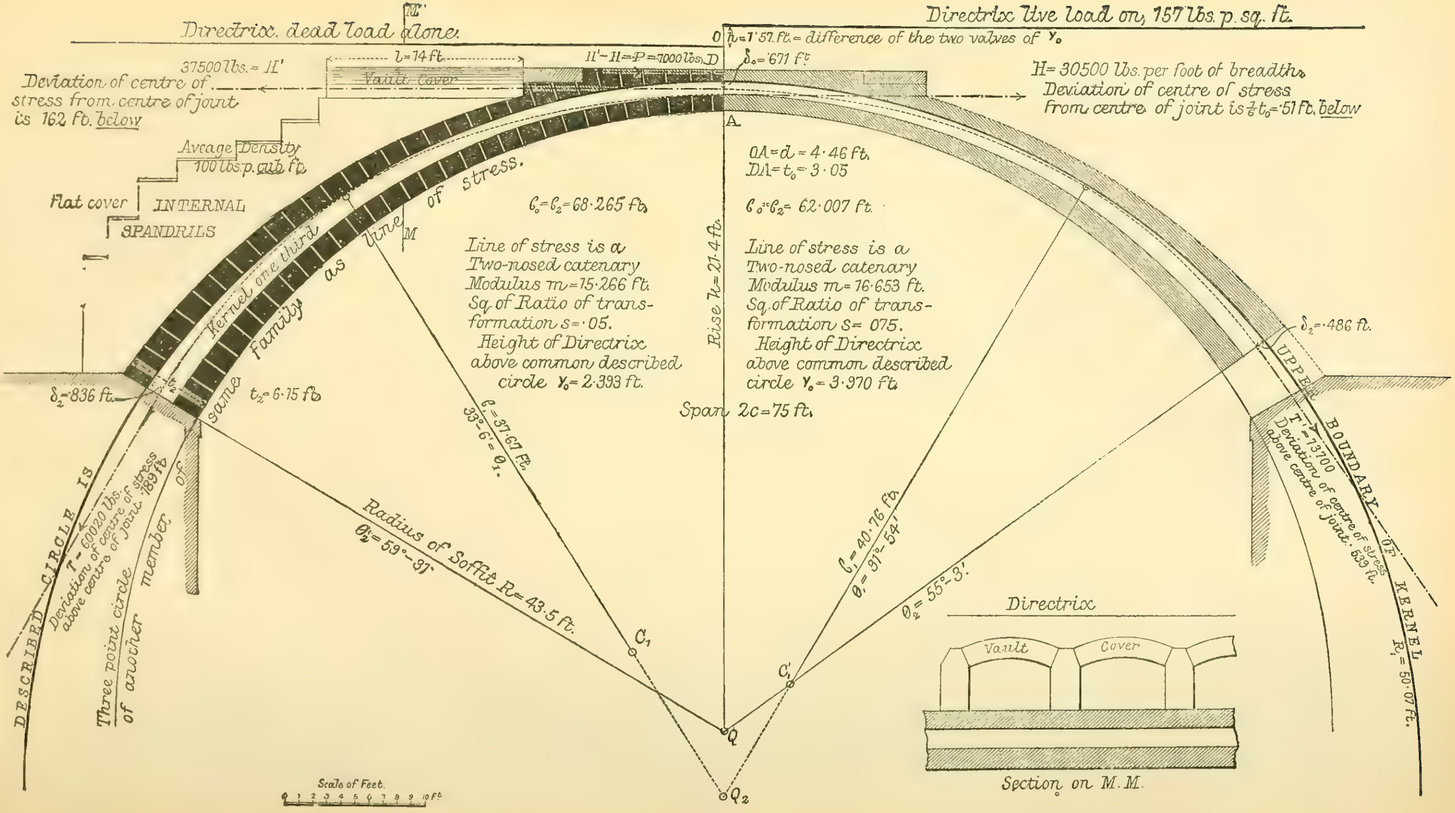

Directrix. dead load alone.
M'
Directrix live load on, 157 lbs. p. sq. ft.
O
p=1'57 ft. = difference of the two values of Y_0
l=14 ft.
H'−H=P=7000 lbs. D
δ_0='671 ft.
37500 lbs. = H'
Vault Cover
Deviation of centre of stress from centre of joint is 162 ft. below
H=30500 lbs. per foot of breadth.
Deviation of centre of stress from centre of joint is $\frac{1}{6}t_0$='51 ft. below
A
Avcage Density 100 lbs. p. cub. ft.
Flat cover
INTERNAL
SPANDRILS
OA=d=4·46 ft.
DA=t_0=3·05
C_0=C_2=68·265 ft.
C_0=C_2=62·007 ft.
Line of stress is a Two-nosed catenary Modulus m=15·266 ft. Sq. of Ratio of transformation s=·05. Height of Directrix above common described circle Y_0=2·393 ft.
Line of stress is a Two-nosed catenary Modulus m=16·653 ft. Sq. of Ratio of transformation s=·075. Height of Directrix above common described circle Y_0=3·970 ft.
Rise h=21·4 ft.
Span 2c=75 ft.
Kernel one third
same family as line of stress.
M
δ_2='836 ft.
t_2
t_2=6·15 ft.
δ_2='486 ft.
UPPER BOUNDARY OF KERNEL
R_1=50·07 ft.
T'=73700
Deviation of centre of stress above centre of joint ·639 ft.
C_1=37·67 ft.
33°-6'=θ_1.
C_1=40·76 ft.
θ_1=31°-54'
θ_2=55°-3'
Radius of Soffit R=43·5 ft.
θ_2'=59°-37'
DESCRIBED CIRCLE IS
T=60020 lbs.
Deviation of centre of stress above centre of joint ·189 ft.
Three point circle of another member of
C_1
C_1'
Q
Q_2
Directrix
Vault
Cover
Section on M.M.
Scale of Feet.
0 1 2 3 4 5 6 7 8 9 10 Ft.

Fig. 272.

Here, then, we have two circles touching at the crown, the described-circle itself of radius 1, and the circle concentric with the soffit of radius ·9098, just like the two dotted circles on fig. 266. By the approximation, page 496, the distance apart of these two eccentric circles at the springing (fig. 272) is the difference of their radii multiplied by $(1 - \cos \theta_2)$ where $\theta_2 = 59° 31'$. Putting Z for this, we have

$$Z = (1 - \cdot 9098)(1 - \cdot 5073) = \cdot 044.$$

Add to this the distance between the soffit and the circle we drew concentric to it, namely $2\delta_0 = 2 \times \cdot 0204$, we have ·084 the distance between the soffit and the *described-circle* at the springing joint on fig. 274. Increasing this by 50 per cent., we get $t_2 = \cdot 126$; and by a closer approximation we make $t_2 = \cdot 123$.

The thickening of the arch-ring from t_0 at the crown outwards to t_2 at the springing (lower joint of rupture) by this tabular method insures that the line of stress, beginning at the *lower* trisecting-point of the crown-joint A (fig. 272), just reaches the *upper* trisecting-point of the joint M (higher joint of rupture at 33° 6′); then it comes back into the white-kernel.

This thickening makes the *extra load* due to the excess density of the ring suited to the line of stress, since the two circles embracing the voussoirs are, approximately, for boundary purposes, two members of the same family of transformed catenaries as the line of stress itself. (See top of page 508.)

The Tables B_2 and B_3 are only wanted for small arches in which the surcharge is relatively larger and the economy of material in the arch-ring has to give place to the consideration of strength.

Apparent Factor of Safety.—In looking at fig. 267, it will appear that the design there made by assuming sizes and drawing the conjugate load-area is the same as that made immediately from Table B_1 on fig. 272.

In fig. 267, the apparent thrust at the crown increased from $288w$ to the actual thrust $400w$ from the leaving out of the negative part of the conjugate load, and the radius at crown of line of stress changed from 48 to $\rho_0 = 67$ feet. Dividing equation (16) by (18), we have

$$\rho_0 = R_2(\sec \theta_2 - 1 + \cos \theta_2), \qquad (19)$$

which may serve instead of drawing the conjugate load-area or using the tables. Thus taking $R_2 = 43 \cdot 5$, the radius of *soffit* roughly, and $\theta_2 = 60°$; then $\rho = 1 \cdot 5 R_2 = 65$ feet. As θ_2 is never greater than 60°, it follows that the real factor of safety against crushing the keystone will not be less than *two-thirds* the apparent factor.

SUPPLEMENTARY TABLE B_1.

Strong Brick. $w = 112$ $f = 154,000$		Sandstone. $w = 140$ $f = 576,000$		Granite. $w = 164$ $f = 1,350,000$									
Max. mult.	Max. span.	Max. mult. to give factor of safety 10.	Max. value of span $2c$ with that mult.	Max. mult.	Max. span.	Surcharge over top of keystone.	Depth of load from extrados (directrix) to crown of soffit.	Thickness of arch-ring normal to soffit at crown.	Thickness of arch-ring normal to soffit at θ_2 joint of rupture.	Radius of the soffit.	Rise of segment of soffit subtending $2\theta_2$.	Span of segment of soffit subtending $2\theta_2$.	Sq. of ratio of transformation $= y_0 \div \rho_0$.
						$d-t_0$.	d.	t_0.	t_2.	R.	k.	$2c$.	s.
w is in lbs. per cub. ft. f is in lbs. per sq. ft. Intensity of thrust on Sandstone Key $\frac{\rho_0 d}{t_0}$ must not exceed $\frac{576000}{10 \times 140} \cdot \frac{1^*}{2}$ its average being $\frac{1}{2}$ its maximum.						"KERNEL" MIDDLE THIRD OF ARCH-RING. The Soffit is the Three-point Circle of a lower curve transformed from same Catenary as Line of Stress, the directrix being surface of Rails.							Equal distances.
					Feet.								
				50	72	·122	·141	·019	·026	·973	·325	1·451	·120
				55	81	·116	·137	·021	·029	·970	·333	1·462	·115
				61	90	·110	·133	·023	·033	·966	·341	1·473	·110
			Feet.										
				68	101	·104	·129	·025	·037	·962	·349	1·482	·105
		38	56	76	112	·098	·125	·027	·041	·957	·357	1·491	·100
		42	63	84	126	·092	·121	·029	·045	·952	·364	1·497	·095
		46	70	92	140	·085	·117	·032	·049	·946	·372	1·504	·090
		51	78	102	156	·079	·113	·034	·055	·940	·380	1·509	·085
	Feet.												
19	29	57	86	114	172	·073	·110	·037	·062	·933	·388	1·514	·080
21	32	63	95	126	190	·066	·106	·040	·071	·925	·395	1·516	·075
23	35	69	105	138	210	·059	·103	·044	·081	·917	·403	1·518	·070
25	39	77	116	154	232	·051	·099	·048	·089	·907	·409	1·516	·065
28	43	85	128	170	256	·043	·095	·052	·097	·896	·416	1·514	·060
31	47	94	141	–	–	·036	·092	·056	·110	·883	·421	1·506	·055
35	52	104	155	–	–	·028	·089	·061	·123	·869	·427	1·498	·050
38	57	–	–	–	–	·018	·085	·067	·141	·850	·431	1·480	·045
42	62	–	–	–	–	·009	·082	·073	·159	·831	·436	1·462	·040
46	67	–	–	–	–	·000	·078	·078	·177	·812	·441	1·444	·035
One-third.	–	*205 ÷ $\frac{\rho_0 d}{t_0}$.	–	Double.	–	Subject to a mult., less than given max., to give factor of safety greater than 10 in inverse ratio.							–

TABLE B.

A Series of "Two-nosed Catenaries" *inscribed in the Circle of Radius* (R_1) *Unity, and having Parallel Directrices at Graduated Distances* ($R_1 + Y_0$) *from its Centre from* (1 + ·187) *to* (1 + ·026).

This Table forms a continuation of the Supplementary Tables B_1, B_2, and B_3, and has then for its purpose the designing of arch-rings, so as to secure the condition of the line of stress lying within a *kernel* forming the middle third, fifth, or ninth of the arch-ring, respectively.

s.	θ_1.	θ_2.	m.	R_1.	R_2.	ρ_1.	$\rho_0 = \rho_2$.	y_0.	Y_0.	δ_0.	δ_2.
Equal distances.	Slope of the circle where line of stress touches with its nose.	Angle of rupture. Line of stress has same curvature as at crown.	Param. of catenary from which the line of stress is transformed.	The Radius of Described Circle is taken as unity. This circle forms the upper boundary of the "Kernel" of arch-ring, *i.e.* of its middle third, fifth, or ninth.	Radius of three-point circle which passes through lower limit of *kernel* at crown.	Radius of curvature at nose of line of stress.	Common radius of curvature at vertex and point of rupture of line of stress.	Depth of vertex of line of stress below directrix.	Depth of crown of circle below directrix.	$y_0 - Y_0$ = Distance of centre of stress below upper boundary of *kernel* at crown.	Do. do. at joint of rupture (springing).
·180	25°37	39°46	·4455		1·0024	·9491	1·0501	·1890	·1869	·0021	·0010
·175	25°58	40°28	·4410		1·0024	·9452	1·0543	·1845	·1821	·0023	·0011
·170	26°20	41°11	·4365		1·0024	·9413	1·0586	·1800	·1774	·0026	·0013
·165	26°40	41°53	·4319		1·0024	·9370	1·0635	·1754	·1726	·0028	·0015
·160	27°01	42°36	·4273		1·0024	·9327	1·0684	·1709	·1678	·0031	·0017
·155	27°20	43°18	·4226		1·0023	·9279	1·0738	·1664	·1629	·0034	·0019
·150	27°40	44°00	·4180		1·0022	·9232	1·0793	·1619	·1581	·0038	·0021
·145	27°59	44°42	·4132		1·0021	·9180	1·0856	·1574	·1532	·0042	·0023
·140	28°18	45°24	·4085		1·0020	·9129	1·0919	·1529	·1484	·0045	·0026
·135	28°36	46°06	·4036		1·0019	·9071	1·0990	·1483	·1434	·0049	·0029
·130	28°55	46°49	·3988		1·0017	·9014	1·1061	·1438	·1384	·0054	·0032
·125	29°12	47°31	·3938		1·0014	·8952	1·1142	·1392	·1333	·0059	·0035
·120	29°30	48°14	·3888		1·0011	·8890	1·1224	·1347	·1283	·0064	·0039
·115	29°47	48°57	·3836		1·0007	·8821	1·1318	·1301	·1231	·0070	·0043
·110	30°04	49°41	·3785		1·0004	·8752	1·1412	·1255	·1180	·0076	·0048
·105	30°20	50°25	·3731		0·9998	·8676	1·1521	·1209	·1127	·0082	·0053
·100	30°37	51°09	·3678		0·9993	·8600	1·1631	·1163	·1074	·0089	·0058
·095	30°52	51°54	·3622		0·9987	·8517	1·1760	·1116	·1019	·0097	·0064
·090	31°08	52°40	·3567		0·9981	·8433	1·1889	·1070	·0965	·0105	·0071
·085	31°23	53°27	·3508		0·9972	·8340	1·2043	·1023	·0908	·0114	·0079
·080	31°39	54°14	·3450	1	0·9963	·8246	1·2197	·0976	·0852	·0124	·0087
·075	31°54	55°03	·3388	1	0·9951	·8141	1·2384	·0928	·0793	·0134	·0097
·070	32°09	55°53	·3326	1	0·9940	·8036	1·2571	·0880	·0735	·0145	·0108
·065	32°23	56°45	·3259	1	0·9925	·7917	1·2803	·0831	·0673	·0158	·0121
·060	32°38	57°38	·3193	1	0·9910	·7798	1·3036	·0782	·0611	·0172	·0134
·055	32°52	58°34	·3121	1	0·9890	·7661	1·3335	·0732	·0544	·0188	·0150
·050	33°06	59°31	·3049	1	0·9870	·7524	1·3634	·0682	·0478	·0204	·0167
·045	33°20	60°34	·2968	1	0·9842	·7363	1·4037	·0630	·0406	·0223	·0190
·040	33°33	61°37	·2888	1	0·9815	·7203	1·4440	·0578	·0334	·0243	·0213
·035	33°46	62°49	·2808	1	0·9788	·7040	1·4843	·0526	·0260	·0263	·0236
Independent of R_1.			Directly proportioned to R_1, and subject to any multiplier.								

SUPPLEMENTARY TABLE B_2.

Strong Brick. $w=112$ $f=154,000$		**Sandstone.** $w=140$ $f=576,000$		**Granite.** $w=164$ $f=1,350,000$		Surcharge over top of keystone.	Depth of load from extrados (directrix) to crown of soffit.	Thickness of arch-ring normal to soffit at crown.	Thickness of arch-ring normal to soffit at θ_2 joint of rupture.	Radius of the soffit.	Rise of segment of soffit subtending $2\theta_2$.	Span of segment of soffit subtending $2\theta_2$.	Sq. of ratio of transformation $= y_0 \div \rho_0$.
Max. mult.	Max. span.	Max. mult. to give factor of safety 10.	Max. value of span $2c$ with that mult.	Max. mult.	Max. span.								
						$d-t_0$.	d.	t_0.	t_2.	R.	k.	$2c$.	s.
w is in lbs. per cub. ft. f is in lbs. per sq. ft. Intensity of thrust on Sandstone Key $\frac{\rho_0 d}{t_0}$ must not exceed $\frac{576000}{10\times 140}\cdot\frac{5^*}{8}$ its average being $\frac{5}{8}$ its maximum.						"KERNEL" MIDDLE FIFTH OF ARCH-RING. The Soffit is the Three-point Circle of a lower curve transformed from same Catenary as Line of Stress, the directrix being surface of Rails.							Equal distances.
					Feet.								
				52	70	·150	·169	·019	·026	·972	·273	1·351	·150
				59	80	·145	·166	·021	·030	·968	·280	1·362	·145
				66	90	·139	·162	·023	·034	·964	·287	1·373	·140
			Feet.	73	101	·134	·159	·025	·038	·960	·294	1·383	·135
	Feet.	40	56	80	112	·128	·155	·027	·042	·956	·302	1·394	·130
		45	63	90	126	·122	·151	·029	·046	·951	·309	1·402	·125
17	24	50	71	100	142	·115	·147	·032	·050	·945	·316	1·410	·120
18	26	55	79	110	158	·109	·144	·035	·056	·939	·322	1·415	·115
20	29	61	87	122	174	·103	·141	·038	·063	·932	·329	1·420	·110
22	32	67	95	134	190	·096	·137	·041	·070	·925	·335	1·424	·105
24	35	73	104	146	208	·089	·134	·045	·077	·917	·342	1·429	·100
27	39	81	116	162	232	·082	·131	·049	·086	·908	·348	1·429	·095
30	43	89	128	178	256	·075	·128	·053	·095	·899	·354	1·429	·090
32	46	97	139	–	–	·068	·125	·057	·105	·888	·359	1·426	·085
35	50	106	151	–	–	·060	·122	·062	·115	·877	·365	1·423	·080
38	54	–	–	–	–	·053	·120	·067	·128	·862	·368	1·414	·075
42	59	–	–	–	–	·044	·117	·073	·142	·848	·372	1·404	·070
46	64	–	–	–	–	·036	·115	·079	·159	·831	·375	1·390	·065
50	69	–	–	–	–	·027	·113	·086	·176	·814	·378	1·376	·060
One-third.	–	·257 ÷ $\frac{\rho_0 d}{t_0}$.	–	Double.	–	Subject to a mult., less than given max., to give factor of safety greater than 10 in inverse ratio.							–

TABLE B.

A Series of "TWO-NOSED CATENARIES" *inscribed in the Circle of Radius* (R_1) *Unity, and having Parallel Directrices at Graduated Distances* ($R_1 + Y_0$) *from its Centre from* (1 + ·187) *to* (1 + ·026).

This Table forms a continuation of the Supplementary Tables B_1, B_2, and B_3, and has then for its purpose the designing of arch-rings, so as to secure the condition of the line of stress lying within a *kernel* forming the middle third, fifth, or ninth of the arch-ring, respectively.

s.	θ_1.	θ_2.	m.	R_1.	R_2.	ρ_1.	$\rho_0 = \rho_2$.	y_0.	Y_0.	δ_0.	δ_2.
Equal distances.	Slope of the circle where line of stress touches with its nose.	Angle of rupture. Line of stress has same curvature as at crown.	Param. of catenary from which the line of stress is transformed.		Radius of three-point circle which passes through lower limit of *kernel* at crown.	Radius of curvature at nose of line of stress.	Common radius of curvature at vertex and point of rupture of line of stress.	Depth of vertex of line of stress below directrix.	Depth of crown of circle below directrix.	$y_0 - Y_0$ = Distance of centre of stress below upper boundary of *kernel* at crown.	Do. do. at joint of rupture (springing).
·180	25°37	39°46	·4455	The Radius of Described Circle is taken as unity. This circle forms the upper boundary of the "Kernel" of arch-ring, *i.e.* of its middle third, fifth, or ninth.	1·0024	·9491	1·0501	·1890	·1869	·0021	·0010
·175	25°58	40°28	·4410		1·0024	·9452	1·0543	·1845	·1821	·0023	·0011
·170	26°20	41°11	·4365		1·0024	·9413	1·0586	·1800	·1774	·0026	·0013
·165	26°40	41°53	·4319		1·0024	·9370	1·0635	·1754	·1726	·0028	·0015
·160	27°01	42°36	·4273		1·0024	·9327	1·0684	·1709	·1678	·0031	·0017
·155	27°20	43°18	·4226		1·0023	·9279	1·0738	·1664	·1629	·0034	·0019
·150	27°40	44°00	·4180		1·0022	·9232	1·0793	·1619	·1581	·0038	·0021
·145	27°59	44°42	·4132		1·0021	·9180	1·0856	·1574	·1532	·0042	·0023
·140	28°18	45°24	·4085		1·0020	·9129	1·0919	·1529	·1484	·0045	·0026
·135	28°36	46°06	·4036		1·0019	·9071	1·0990	·1483	·1434	·0049	·0029
·130	28°55	46°49	·3988		1·0017	·9014	1·1061	·1438	·1384	·0054	·0032
·125	29°12	47°31	·3938		1·0014	·8952	1·1142	·1392	·1333	·0059	·0035
·120	29°30	48°14	·3888		1·0011	·8890	1·1224	·1347	·1283	·0064	·0039
·115	29°47	48°57	·3836		1·0007	·8821	1·1318	·1301	·1231	·0070	·0043
·110	30°04	49°41	·3785		1·0004	·8752	1·1412	·1255	·1180	·0076	·0048
·105	30°20	50°25	·3731		0·9998	·8676	1·1521	·1209	·1127	·0082	·0053
·100	30°37	51°09	·3678		0·9993	·8600	1·1631	·1163	·1074	·0089	·0058
·095	30°52	51°54	·3622		0·9987	·8517	1·1760	·1116	·1019	·0097	·0064
·090	31°08	52°40	·3567		0·9981	·8433	1·1889	·1070	·0965	·0105	·0071
·085	31°23	53°27	·3508		0·9972	·8340	1·2043	·1023	·0908	·0114	·0079
·080	31°39	54°14	·3450	1	0·9963	·8246	1·2197	·0976	·0852	·0124	·0087
·075	31°54	55°03	·3388	1	0·9951	·8141	1·2384	·0928	·0793	·0134	·0097
·070	32°09	55°53	·3326	1	0·9940	·8036	1·2571	·0880	·0735	·0145	·0108
·065	32°23	56°45	·3259	1	0·9925	·7917	1·2803	·0831	·0673	·0158	·0121
·060	32°38	57°38	·3193	1	0·9910	·7798	1·3036	·0782	·0611	·0172	·0134
·055	32°52	58°34	·3121	1	0·9890	·7661	1·3335	·0732	·0544	·0188	·0150
·050	33°06	59°31	·3049	1	0·9870	·7524	1·3634	·0682	·0478	·0204	·0167
·045	33°20	60°34	·2968	1	0·9842	·7363	1·4037	·0630	·0406	·0223	·0190
·040	33°33	61°37	·2888	1	0·9815	·7203	1·4440	·0578	·0334	·0243	·0213
·035	33°46	62°49	·2808	1	0·9788	·7040	1·4843	·0526	·0260	·0263	·0236
Independent of R_1.			Directly proportioned to R_1, and subject to any multiplier.								

SUPPLEMENTARY TABLE B_3.

w is in lbs. per cub. ft.
f is in lbs. per sq. ft.

Intensity of thrust on Sandstone Key $\frac{\rho_0 d}{t_0}$ must not exceed $\frac{576000}{10 \times 140} \cdot \frac{3^*}{4}$ its average being $\frac{3}{4}$ its maximum.

"KERNEL" MIDDLE NINTH OF ARCH-RING.

The Soffit is the Three-point Circle of a lower curve transformed from same Catenary as Line of Stress, the directrix being surface of Rails.

Strong Brick. $w = 112$ $f = 154{,}000$		Sandstone. $w = 140$ $f = 576{,}000$		Granite. $w = 164$ $f = 1{,}350{,}000$		Surcharge over top of keystone.	Depth of load from extrados (directrix) to crown of soffit.	Thickness of arch-ring normal to soffit at crown.	Thickness of arch-ring normal to soffit at θ_2 joint of rupture.	Radius of the soffit.	Rise of segment of soffit subtending $2\theta_2$.	Span of segment of soffit subtending $2\theta_2$.	Sq. of ratio of transformation $= \gamma_0 \div \rho_0$.
Max. mult.	Max. span.	Max. mult. to give factor of safety 10.	Max. value of span $2c$ with that mult.	Max. mult.	Max. span.	$d - t_0$.	d.	t_0.	t_2.	R.	k.	$2c$.	s.
	Feet.		Feet.		Feet.								Equal distances.
				58	72	·178	·197	·019	·028	·964	·223	1·234	·180
				64	80	·173	·194	·021	·031	·963	·230	1·250	·175
				70	88	·167	·190	·023	·034	·962	·238	1·266	·170
		40	50	80	101	·162	·187	·025	·037	·958	·245	1·278	·165
		44	57	88	114	·155	·183	·028	·041	·953	·252	1·290	·160
		49	64	98	129	·149	·180	·031	·047	·948	·258	1·299	·155
		55	72	110	144	·143	·177	·034	·054	·942	·265	1·309	·150
20	27	61	80	122	161	·137	·174	·037	·060	·936	·269	1·316	·145
22	30	67	89	134	178	·130	·171	·041	·067	·929	·274	1·323	·140
25	33	75	99	150	199	·123	·168	·045	·073	·921	·281	1·327	·135
27	37	83	110	166	220	·116	·165	·049	·080	·913	·288	1·331	·130
30	40	91	121	182	242	·109	·162	·053	·089	·903	·293	1·331	·125
33	44	99	132	198	264	·102	·160	·058	·098	·893	·298	1·332	·120
36	48	108	145	–	–	·095	·158	·063	·108	·882	·302	1·334	·115
39	52	118	157	–	–	·088	·156	·068	·119	·870	·307	1·326	·110
42	56	–	–	–	–	·080	·154	·074	·131	·857	·311	1·320	·105
46	61	–	–	–	–	·072	·152	·080	·144	·844	·315	1·315	·100
50	66	–	–	–	–	·064	·151	·087	·158	·828	·317	1·303	·095
55	71	–	–	–	–	·054	·149	·095	·173	·812	·320	1·291	·090
One-third	–	$308 \div \frac{\rho_0 d}{t_0}$.	–	Double.	–	Subject to a mult., less than given max., to give factor of safety greater than 10 in inverse ratio.							–

TABLE B.

A Series of "Two-nosed Catenaries" *inscribed in the Circle of Radius* (R_1) *Unity, and having Parallel Directrices at Graduated Distances* ($R_1 + Y_0$) *from its Centre from* ($1 + {\cdot}187$) *to* ($1 + {\cdot}026$).

This Table forms a continuation of the Supplementary Tables B_1, B_2, and B_3, and has then for its purpose the designing of arch-rings, so as to secure the condition of the line of stress lying within a *kernel* forming the middle third, fifth, or ninth of the arch-ring, respectively.

s.	θ_1.	θ_2.	m.	R_1.	R_2.	ρ_1.	$\rho_0 = \rho_2$.	y_0.	Y_0.	δ_0.	δ_2.
Equal distances.	Slope of the circle where line of stress touches with its nose.	Angle of rupture. Line of stress has same curvature as at crown.	Param. of catenary from which the line of stress is transformed.	This circle forms the upper boundary	Radius of three-point circle which passes through lower limit of *kernel* at crown.	Radius of curvature at nose of line of stress.	Common radius of curvature at vertex and point of rupture of line of stress.	Depth of vertex of line of stress below directrix.	Depth of crown of circle below directrix.	$y_o - Y_o$ = Distance of centre of stress below upper boundary of *kernel* at crown.	Do. do. at joint of rupture (springing).
·180	25°37	39°46	·4455	The Radius of Described Circle is taken as unity. This circle forms the upper boundary of the "Kernel" of arch-ring, *i.e.* of its middle third, fifth, or ninth.	1·0024	·9491	1·0501	·1890	·1869	·0021	·0010
·175	25°58	40°28	·4410		1·0024	·9452	1·0543	·1845	·1821	·0023	·0011
·170	26°20	41°11	·4365		1·0024	·9413	1·0586	·1800	·1774	·0026	·0013
·165	26°40	41°53	·4319		1·0024	·9370	1·0635	·1754	·1726	·0028	·0015
·160	27°01	42°36	·4273		1·0024	·9327	1·0684	·1709	·1678	·0031	·0017
·155	27°20	43°18	·4226		1·0023	·9279	1·0738	·1664	·1629	·0034	·0019
·150	27°40	44°00	·4180		1·0022	·9232	1·0793	·1619	·1581	·0038	·0021
·145	27°59	44°42	·4132		1·0021	·9180	1·0856	·1574	·1532	·0042	·0023
·140	28°18	45°24	·4085		1·0020	·9129	1·0919	·1529	·1484	·0045	·0026
·135	28°36	46°06	·4036		1·0019	·9071	1·0990	·1483	·1434	·0049	·0029
·130	28°55	46°49	·3988		1·0017	·9014	1·1061	·1438	·1384	·0054	·0032
·125	29°12	47°31	·3938		1·0014	·8952	1·1142	·1392	·1333	·0059	·0035
·120	29°30	48°14	·3888		1·0011	·8890	1·1224	·1347	·1283	·0064	·0039
·115	29°47	48°57	·3836		1·0007	·8821	1·1318	·1301	·1231	·0070	·0043
·110	30°04	49°41	·3785		1·0004	·8752	1·1412	·1255	·1180	·0076	·0048
·105	30°20	50°25	·3731		0·9998	·8676	1·1521	·1209	·1127	·0082	·0053
·100	30°37	51°09	·3678		0·9993	·8600	1·1631	·1163	·1074	·0089	·0058
·095	30°52	51°54	·3622		0·9987	·8517	1·1760	·1116	·1019	·0097	·0064
·090	31°08	52°40	·3567		0·9981	·8433	1·1889	·1070	·0965	·0105	·0071
·085	31°23	53°27	·3508		0·9972	·8340	1·2043	·1023	·0908	·0114	·0079
·080	31°39	54°14	·3450	1	0·9963	·8246	1·2197	·0976	·0852	·0124	·0087
·075	31°54	55°03	·3388	1	0·9951	·8141	1·2384	·0928	·0793	·0134	·0097
·070	32°09	55°53	·3326	1	0·9940	·8036	1·2571	·0880	·0735	·0145	·0108
·065	32°23	56°45	·3259	1	0·9925	·7917	1·2803	·0831	·0673	·0158	·0121
·060	32°38	57°38	·3193	1	0·9910	·7798	1·3036	·0782	·0611	·0172	·0134
·055	32°52	58°34	·3121	1	0·9890	·7661	1·3335	·0732	·0544	·0188	·0150
·050	33°06	59°31	·3049	1	0·9870	·7524	1·3634	·0682	·0478	·0204	·0167
·045	33°20	60°34	·2968	1	0·9842	·7363	1·4037	·0630	·0406	·0223	·0190
·040	33°33	61°37	·2888	1	0·9815	·7203	1·4440	·0578	·0334	·0243	·0213
·035	33°46	62°49	·2808	1	0·9788	·7040	1·4843	·0526	·0260	·0263	·0236
Independent of R_1.			Directly proportioned to R_1, and subject to any multiplier.								

EXAMPLES.

298. Design of a sandstone segmental arch with vertical load, directly from the tables, span, 75 feet, and depth of surcharge at crown about 1′ 4″: the springing to be the joint of rupture. (See fig. 272, left half.)

Here $2c = 75$, and $d - t_0 = 1{\cdot}\dot{3}$; their ratio is 56·25. We find, by trials on Table B_1, that $2c \div (d - t_0) = 53$ occurs on the line where $s = {\cdot}05$, and the multiplier required on that line to make $2c$ into 75, is 50·07, about half of the max. mult. given under "sandstone"; so we shall have a factor of safety of about twice *ten*, and need not consult the other supplementary Tables, B_2, B_3.

That line gives the following relative and absolute values:—

s	Mult.	$d - t_0$	d	t_0	t_2	R	k	$2c$
·050	50·07	·028	·089	·061	·123	·869	·427	1·498
		1·41	4·46	3·05	6·15	43·5	21·4	75 ft.

The radius and rise of soffit are 43·5 and 21·4 feet; the thickness of arch-ring at crown and springing 3′ 0″ and 6′ 2″; the surcharge being 1·41 feet, or about 1′ 4″, as required.

On the same line, continued on Table B, we have—

s	θ_2	*Mult.*	R_1	ρ_0	Y_0	δ^0	δ_2
·050	59° 31′	50·07	1	1·3634	·0478	·0204	·0167
			50·07	68·265	2·393	1·02	·836 ft.

At the crown the thrust on the arch-ring per foot of the breadth is $H = w\rho_0 d = 140 \times 68{\cdot}265 \times 4{\cdot}46 = 42624$ lbs.; the average intensity of the stress is $42624 \div 3{\cdot}05 = 13975$; and double of this 27950 lbs. per sq. ft., is the maximum intensity, giving a factor of safety of $576000 \div 27950 = 20$. Otherwise, the multiplier being half the maximum in the Table, the factor of safety is double ten.

At the springing, $T = H \sec \theta_2 = 84023$; the average stress $84023 \div 6{\cdot}15 = 13662$; and since the deviation of the centre of stress is $\frac{1}{6}t_2 - \delta_2 = 1{\cdot}025 - {\cdot}836 = {\cdot}189$ *above* the centre of the joint; we can substitute this for δ, the deviation of the centre of stress from the middle of the joint, equation (1), at page 476,

$$\frac{\text{Max. stress}}{\text{Aver. stress}} = 1 + \frac{6\delta}{t_2} = 1{\cdot}184.$$

Hence the maximum intensity of the stress on the springing joint is 16180 lbs. per sq. ft. Dividing this out of 576000, the crushing strength of sandstone, we get the factor of safety 35.

A simpler way to proceed to get the factor of safety against crushing, sufficiently close for all purposes, is to divide the crushing strength 576000 by 13662, the average, and the quotient 42 is the *apparent factor of safety* against crushing the sandstone-skew-back. For the three deviations of the centres of stress shown on fig. 263, expressed as fractions of t the thickness of the joint, the average bears to the maximum the ratios below.

Deviations of c. of stress, $\frac{1}{18}$th, $\frac{1}{10}$th, $\frac{1}{6}$th.

Ratios of aver. to max., $\frac{6}{8}$, $\frac{5}{8}$, $\frac{4}{8}$.

Now the deviation of the centre of stress on the skewback (fig. 272) is ·189 in 6·15; as this is less than $\frac{1}{18}$, then the factor of safety is at least $\frac{3}{4}$ths of 42, that is, there is a real factor of safety greater than 31.

299. If a live load of 220 lbs. per sq. ft. of the platform be all over the span of the bridge (Ex. 298), find the new line of stress in the arch-ring and the intensities of the stresses at crown and springing. (Fig. 272, right half.)

The height of superstructure equivalent to this live load is

$$h = 220 \div 140 = 1{\cdot}571 \text{ ft.}$$

of sandstone. Here we have to find a new two-nosed catenary still inscribed in the same circle, $R_1 = 50{\cdot}07$, forming the upper boundary of "kernel" (middle third) of the arch-ring, as already designed (Ex. 298), but to a directrix, h higher than before. Adding h to the old value of Y_0, we get $2{\cdot}393 + 1{\cdot}571$ or $3{\cdot}964$, which, divided by the multiplier 50·07, gives us ·0793 as a *new* relative value of Y_0, which is found in the Table B, at the line—

s	θ_2	*Mult.*	R_1	ρ_0	Y_0	δ_0	δ_2
·075	55° 3′	50·07	1	1·2384	·0793	·0134	·0097
			50·07	62·007	3·970	·671	·486 ft.

This is a new two-nosed catenary, of a different modulus and of a different family, so that the soffit already designed will not be mathematically the three-point circle of another member of the family of this line of stress, but it will sensibly be so. The joints of rupture have gone up to 55° 3′, but this is immaterial, as the line of stress is now *closer* to the upper "kernel," and will therefore be wholly in the "kernel," down to 59° 31′, the springing joint. In designing the abutment, the tangent at the joint 53° 3′ should lie in its middle third.

At the crown, now we have the thrust, $H' = w\rho_0\,(d + h) = 140 \times 62{\cdot}007\,(4{\cdot}46 + 1{\cdot}57) = 52346$ lbs.; average intensity $52346 \div 3{\cdot}05 = 17162$; the deviation of centre of stress is $\frac{1}{6}t_0 - \delta_0 = {\cdot}509 - {\cdot}671 = {\cdot}162$ ft. *below* centre of the joint. The apparent factor of safety is $576000 \div 17162 = 33$. The fractional deviation of the centre of stress from the middle of joint being ·162 in 3·05 or 1 in 18 nearly, the factor of safety is of $\frac{3}{4}$ths its apparent value, that is, 27 is the factor of safety at crown when the live load is on the bridge.

At the springing joint, $T' = H' \sec 59° 31' = 103188$ lbs., nearly; the average intensity is $103188 \div 6{\cdot}15 = 16778$; the deviation of centre of stress is $\frac{1}{6}t_2 - \delta_2 = 1{\cdot}025 - {\cdot}486 = {\cdot}539$ ft. *above* the centre of joint. The fractional deviation of centre of stress is ·539 in 6·15 or less than 1 in 10. The apparent factor of safety is $576000 \div 16778 = 34$, and the factor of safety not less than $\frac{5}{8}$ths of this or 21.

300. Let the live load (Ex. 299) cover only one-half of the span: find the horizontal thrust to be balanced by the backing of the voussoirs.

The horizontal thrust due to dead load is (Ex. 298) 42624 lbs.; and due to dead and live loads, combined, it is (Ex. 299) 52346 lbs.; the difference is 9722 lbs. per ft. of breadth.

301. Suppose the arch-ring, spandrils, &c., of Ex. 298, have, by means of voids in the superstructure, an average density of 100 lbs. per cub. ft. Find results corresponding to those of Ex. 298.

For stability, and to give the required surcharge, the dimensions in Ex. 298 are required, just as before, but the stresses will be altered in the ratio 140 : 100. $H = 30500$ lbs., nearly; $T = 60320$ lbs., giving factors of safety, 28 at crown and 49 at springing. These values are engraved on fig. 272.

The voids in the superstructure should be so arranged that their boundary may be roughly a member of the same family as line of stress, by making the ordinates of their boundary a constant fraction of those of the soffit.

302. Let a live load of 157 lbs. per sq. ft. of platform be over the whole span of bridge (Ex. 301): find the line of stress and the intensities of stress at crown and springing.

The equivalent height of structure is 1·57 ft., taking the new density into account; so that the solution is the same as Ex. 298, only we must alter the quantities in the ratio 140 : 100.

$H' = 37500$ lbs., nearly; $T' = 73700$, nearly, and the factors of safety are increased to 35 at crown and 31 at springing. These values are engraved on fig. 272.

303. Let the live load in Ex. 302 be only over one-half of the span: find the amount of horizontal thrust to be balanced by the frictional stability of vault covers butting against the higher voussoirs. Find also the distance back to which the vault covers must extend to balance it. (See fig. 272.)

The thrust is $P = H' - H = 37500 - 30500 = 7000$ lbs. per foot of breadth. If the underside of the vault covers come up to the level of the crown of the soffit, then the weight per foot of breadth of bridge on the spandrils due to the vault covers, and dead load over them alone, is $wdl = 140 \times 4{\cdot}46l = 624l$. Taking the coefficient of friction at ·7; then $\cdot 7 \times 624l = 7000$ or $l = 16$ ft. to the nearest foot. The voussoirs near the key-stone should have square-dressed side-joints until the sum of their vertical projection is t_0, the depth of key-stone, so as to receive the horizontal longitudinal resistant thrust of the vault covers truly; and the covers are to be built with square-dressed side joints set closely so as to yield as little as possible. Light spandril walls may be built up to level of crown of soffit, and extend back so as to give 16 feet longitudinally of vault covers; then the spandrils may step rapidly down to lower levels.

304. Design of a semicircular arch-ring of common sandstone, the span to be 100 ft., and a surcharge of at least $1\frac{1}{2}$ ft. being required for the formation of the roadway, laying of gas-pipes, &c. The data are $R = 50$, and $R \div (d - t_0)$ not to be greater than 33. On Table B_1, the lines above that with $s = \cdot 08$ (in order to make R into 50) require a multiplier greater than the maximum given for sandstone; these lines are therefore excluded on the question of strength, while the lines below that with $s = \cdot 05$ give $R \div (d - t_0)$ greater than 33, and are excluded by requirements of the roadway. Those two limiting lines give—

s	*Mult.*	R	d	t_0	θ_2	Factor of Safety.
·08	53·6	50	5·9	2	54° 14′	$\frac{57 \times 10}{53{\cdot}6} = 10{\cdot}6.$
·05	57·5	50	5·1	3·5	59° 31′	$\frac{104 \times 10}{57{\cdot}5} = 18.$

The upper gives *greatest economy* of material in *arch-ring*, which is only 2 ft. at crown, but *less economy* of material in superstructure, as d is larger, and also *less economy* of solid backing, which has to be built to a joint 5° higher. Hence the line midway between them would be most suitable all round. For a single arch a line a little nearer the upper may be adopted; and for a series of arches a line nearer the lower, that is, in favour of a heavier arch-ring to withstand the shocks transmitted from arch to arch. The best lines, then, are for a Single Arch, or a Series, respectively—

s	*Mult.*	R	d	t_0	t_2	θ_2	Factor of Safety.
·07	54·526	50	5·6	2·4	4·4	55° 53′	$\frac{69 \times 10}{54{\cdot}5} = 12{\cdot}7.$
·06	55·804	50	5·3	2·9	5·4	57° 38′	$\frac{85 \times 10}{55{\cdot}8} = 15{\cdot}2.$

Compare Rankine's empirical rule, *Civ. Eng.*, Art. 290, giving

$$t_0 = \sqrt{\cdot 12 \times 50} \quad \text{and} \quad \sqrt{\cdot 17 \times 50}$$

$$= \quad 2\cdot 45 \quad ,, \quad 2\cdot 92 \text{ respectively.}$$

The solid backing must be brought up to the point where the joint at θ_2 meets the back of the arch-ring, and below that joint the arch-ring may be of the uniform thickness t_2. The superstructure may readily be reduced by voids and the employment of material of less density than sandstone, till the average density of the whole is a fifth less than that of sandstone, which would raise the factors of safety at crown to 16 and 19. The factors of safety at joint of rupture are even greater as the centre of stress is nearer the centre of the joint, and $t_2 \div t_1 > \sec \theta_2$. By means of the values obtained for w, ρ_0, d, δ_2 the thrust at crown and joint of rupture, and the centre of stress at joint of rupture are calculated, as in preceding example. A tangent from this last point enables a suitable abutment to be designed.

305. Design of a segmental brick arch, having the joints of rupture at the springing, a span of 30 ft., and a surcharge of $2\frac{1}{2}$ ft., slightly more or less.

Here the ratio $2c \div (d - t_0)$ is 12. But this ratio (nearly) occurs in the Tables B_1, B_2, and B_3, as noted under—

	$\frac{2c}{d-t_0}$	s	Max. Span Brick.	Factor of Safety.
Table B_1	11·9	·120	12ft.	10
,, B_2	11·5	·125	21ft.	10
,, B_3	11·5	·130	37ft.	10

It is clear, then, that we cannot adopt the proportions from Tables B_1 or B_2; for although the surcharge and span would be in the required proportion, the multiplier to make $2c$ equal to 30 ft. would only give in Table B_1 a factor of safety $\frac{12}{30} \times 10 = 4$, and in Table B_2 a factor $\frac{21}{30} \times 10 = 7$ which are not sufficient. From Table B_3, then,

s	θ_2	*Mult.*	$d-t_0$	d	t_0	t_2	R
·130	46° 49′	22·54	·116 2·61	·165 3·72	·05 1·2	·08 1·8	·913 20·6

ρ_0	k	$2c$	Factor of Safety.
1·106 24·92	·29 4·5	1·331 30 ft.	$\frac{37 \times 10}{30} = 12\cdot 5$

The arch is to spring at about 45°, have a rise of $4\frac{1}{2}$ feet, the radius of soffit being 20′ 8″, while the thickness of the crown is to be 14 inches, and at the springing 50 per cent. greater. These dimensions would suffice for the arch-ring built of the strongest red bricks moulded into the proper wedge-shaped forms. The line of stress may be exactly located, as in previous examples; but it would be sufficiently near, in this case, to take it as being up the very centre of the arch-ring; and a line drawn normal to the springing joint from its middle point is sensibly the line of thrust on the abutment. The superstructure could easily be arranged with voids to make the average mass of the whole $w = 90$ lbs. per cubic ft., or $\frac{3}{4}$ths that of brickwork, which would increase the factor of safety to about 20. The thrust then would be—

$$\text{At crown,} \quad H = w\rho_0 d \quad = 9262 \text{ lbs. per ft. of breadth.}$$

$$\text{At springing,} \quad T = H \sec \theta_2 = 13532 \text{ lbs.} \quad ,, \qquad ,,$$

For ordinary-shaped bricks the arch-ring ought to be built of a uniform thickness, the line of stress and thrust remaining sensibly unchanged, as the extra part of the arch-ring at crown will really form part of the superstructure. For a single arch, an average of t_0 and t_2 could be taken; but for a series of arches the greater had better be chosen, giving, respectively,

$$t = 1{\cdot}5 \text{ and } 1{\cdot}8 \text{ feet.}$$

Compare Rankine's empirical formula, *Civ. Eng.*, Art. 290—

$$t = \sqrt{20{\cdot}6 \times {\cdot}12} \quad \text{and} \quad \sqrt{20{\cdot}6 \times {\cdot}17}$$

$$= \quad 1{\cdot}57 \qquad ,, \qquad 1{\cdot}87 \quad \text{respectively.}$$

306. If the bridge, left-half of fig. 272, is to carry a road, approaching with a slope 1 in 100, and passing over with a curved profile sensibly circular; to find position of profile.

To make the profile sensibly a member of the same family as line of stress, it is sufficient to derive it from the soffit, which is sensibly a member of the family. Let u_0 and u_2 be depth to profile at crown and springing. The tangent to profile, being like that to a parabola, gives slope of half chord of profile 1 in 200; hence $u_2 - u_0 : 37{\cdot}5 : : 1 : 200$, and $u_2 : u_0 : : 4{\cdot}6 + 21{\cdot}4 : 4{\cdot}46$. Hence $u_0 = {\cdot}04$ ft., and $u_2 = {\cdot}23$ ft. The profile is to be below the directrix about $\frac{1}{2}$ an inch at crown, and about 3 inches at springing. The load left out between directrix and profile increases the factors of safety as $4{\cdot}46$; $(4{\cdot}46 - 0{\cdot}04)$.

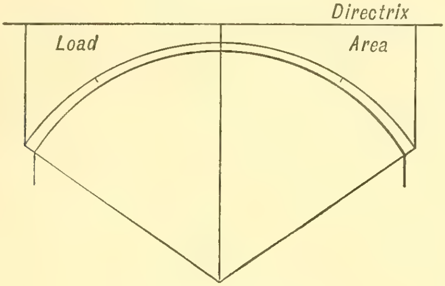
307. Compare the segmental masonry arch of 75 feet span as already designed in Example 298, for the moderate surcharge of 1′ 4″ with others with the surcharge *heavy* and *light*, respectively.

Deriving the dimensions from the three lines of Table B₁ where $s = {\cdot}08$, $s = {\cdot}05$, and $s = {\cdot}035$, the multipliers are 49·6, 50·07, and 51·94, that in each case the span may be 75 feet. The corresponding maximum spans for sandstone in the fourth column are 86, 155, and 201 feet; and 10 increased in the ratios which these bear to 75 gives the factors of safety against crushing the keystone if the ring be sandstone. The values for strong brick are one-third part, while those for granite are double that for sandstone. The three designs are sketched on fig. 273, and we have the following dimensions and factors of safety against the crushing of the keystone:—

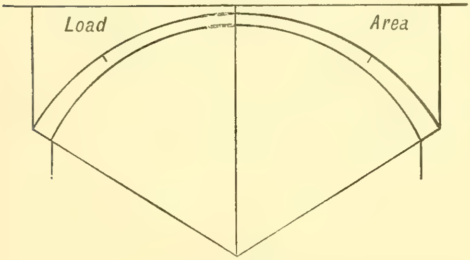
Surcharge, .	3′ 8″	1′ 4″	0′ 0″
Span, .	75′ 0″	75′ 0″	75′ 0″
Keystone, .	1′ 10″	3′ 0″	4′ 0″
Skewback, .	3′ 1″	6′ 2″	9′ 0″
Rise, .	19′ 3″	21′ 5″	22′ 10″
Total rise, .	24′ 8″	25′ 10″	26′ 10″
Strong brick, .	4	7	9
Sandstone, .	12	20	27
Granite, .	24	40	54

It is to be observed of Table B₁ that it gives designs of the utmost economy of the material in the arch-ring itself consistent with equilibrium under the vertical-load only. But as the surcharge gets heavier, as for instance on the first part of fig. 273, there is not sufficient strength with brick or sandstone, though granite would suffice. With still heavier surcharges (see Example 305), we are forced to sacrifice economy of material and use the other Tables B_2 and B_3. The problem is passing from one of *stability* to one of mere *strength*. See page 478, under "strength." As the surcharge further increases relative to the span, the line of stress becomes practically up to the very centre of the ring, and we have *buried arches* as in Example 293, and in tunnel roofs for the deep lengths.

Heavy Surcharge.



Moderate Surcharge.



Zero Surcharge.

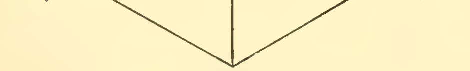


Fig. 273.

Inspection of the Tables B_1, B_2, and B_3 shows clearly that for balanced segmental masonry arches strong brick is economical for spans from 20 to 50 feet, sandstone from 50 to 120 feet, and granite from 60 to 250 feet.

LINEAR TRANSFORMATION OF BALANCED RIB.

We had an example of this in fig. 268, where we supposed one of the curves *AB* to be painted on a sheet of indiarubber pinned down along its edge *OX*, and already stretched in the vertical direction. The sheet is, of course, made of a purely imaginary kind of indiarubber which can be stretched in one direction without changing in the direction at right angles to it. At the ends of the arc *A* and *B* are to be painted the forces which balance it, *H* and *T* to scale; while the vertical-load-area, *the only load on the arc AB*, is mapped out by the four painted lines, the arc itself and the three straight boundaries *OA*, *NB*, and *ON*. Then, by further stretching the sheet vertically or by allowing it to contract, we obtain a new balanced rib with new forces at its ends and a new load-area. Otherwise the lines may be painted or traced through upon a sheet of glass lying flat on the paper and *hinged* to the paper along its upper edge *OX*. The glass is then to be tilted up on its edge *OX*, and the work projected normally on the paper. The sheet of glass, although hinged to the plane of the paper along *OX*, may extend beyond the hinge. It may also have been originally tilted up when the work was projected normally from the paper on to it, then flattened down before projecting the work back on the paper. This is the simplest case of the orthogonal or linear transformation to obtain, from a rib balanced under a vertical-load-area alone, another rib balanced under the transformed vertical-load-area. In the work which follows, it is more convenient to transform horizontally.

In dealing with a rib balanced by a vertical-load-area and horizontal-load-area conjointly, both of these areas themselves may not be painted on the stretching-sheet or tilting-plate. For convenience, the vertical-load-area may be supposed to be painted on the tilting-plate, but after the transformation the density or weight per square foot of area is to be altered. That is, the superstructure, if transformed linearly, must then be reckoned to be of a new material of different density or weight per cubic foot.

All forces, thrusts along the rib or their components, cross the *hinge* at the same points after the transformation as before. Forces or component forces in the direction of the transformation alter in the *ratio of the transformation*, those at right angles remain unaltered.

Resultant stresses or arrows representing the extern-load, such as r, r, &c., on the south-west quadrantal-rib, fig. 264, also cross the *hinge* at the same points after the transformation and before. The new arrows obtained this way although correct in direction, are not of the proper length. Consider one of the arrows r (fig. 264): it represents to scale the extern load normal on a *unit* tangent-plane to the back of the rib with a certain southern *exposure* to stress and a certain western *exposure*. After the transformation the tangent plane is no longer a *unit* plane, and one exposure to stress has altered, while the other has remained unaltered. But the length of the new arrow must give the extern-load on a *unit* plane, allowing for every alteration.

Let the north-west quadrant, fig. 264, in equilibrium under the *fluid load* represented by the two equal rectangles, be pulled out horizontally till it is twice as broad as before. It is now shown on fig. 274. The quadrant of the circle $AB'S'$ has become ABS, the quadrant of an ellipse twice as long east and west as it is north and south. Since T_0 has doubled in value, so must also the area of the east load, and as this load stands on the same base or platform as before, so the height of the rectangle must have doubled. On the other hand, T_1 is unaltered by the transformation, so that the south-load is unaltered in area; but as it stands now on a platform or base doubled in length, the height of the rectangle is *half* what it was at first. The ratio of q to p is now 4, which is the duplicate of the ratio of transformation. The shaded part of the east-load-area before and after the elongation is the horizontal component of T' and of T, respectively. The shaded part has therefore doubled in area. By starting with B at S and moving it up to A, the shaded area has added to it strip after strip, each strip being stretched to double what it was at first. We see that the new east-load-area is of the same *form* as at first, only with the height at each point of its base *increased in the ratio of transformation.* In the same way it will be seen that the south-load-area is also of the same original form, but with the height at each corresponding point of its base *decreased in the ratio of transformation.*

The thrusts T and T' along the ribs at B and B' meet at the same point j on the *hinge* Oj. The extern resultant stresses on the backs of the ribs r and r'_1 also meet at the common point O on the *hinge*.

The whole stress at B' for a unit plane, its edge towards you and its face turned in all directions, is given by a circle of which r' is a radius normal to the circular rib. The whole stress at B for a unit plane, its edge towards you and its face turned in all directions, is given by an ellipse of which r is the radius-vector parallel to BO, and whose major and minor semiaxes are q and p, the heights of the east- and south-load-areas, respectively, for the point B. For clearness, a quadrant of this ellipse is shown with its centre at D, the point on the base of the east-load-area in a line with B. It is called the *Ellipse of Stress.* It completely determines r in direction as well as in length. Thus through D draw CC, the unit plane, parallel to Bj the tangent to the rib at B; draw DN normal to CC laying off DM equal to the half sum of q and p, make

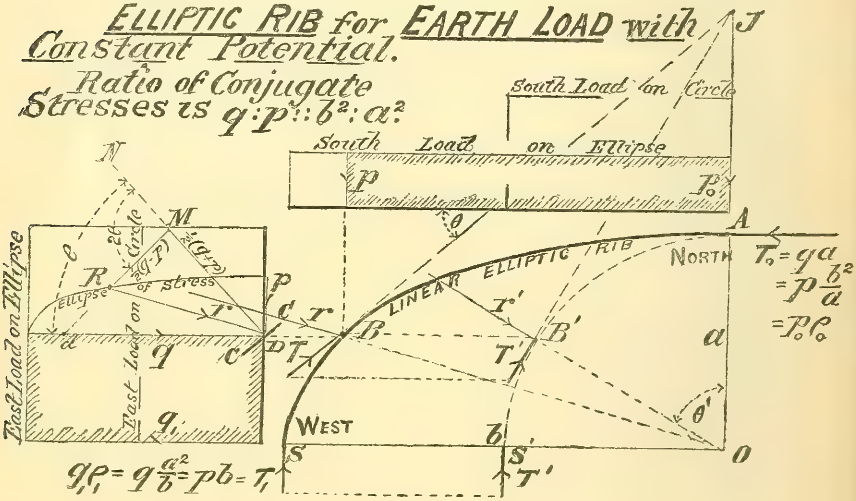


Fig. 274.

Md isosceles with MD, and lay off MR equal to the half difference of q and p. This gives RD or r, which is to be shifted parallel to itself to act at B. It is not necessary to know the point O on the *hinge* through which r must pass; it is only necessary to know the shape of the rib itself so as to be able to draw a tangent to it at B, and further to know q and p the semi-axes of the ellipse of stress at that level.

In the example, fig. 274, both of the rectangles, giving the south- and east-loads on the elliptic-quadrant, may be doubled in height, and their density per square foot taken as half that of the rectangles giving the loads on the circular quadrant. The new south-load-area is then obtained simply by projection, while the new east-load-area has been obtained by projection in the duplicate ratio.

Generally then a balanced quadrantal rib and its vertical-load-area, that is its superstructure, can be projected by a pull-out horizontally, provided *that the conjugate horizontal-load-area be pulled out in the duplicate ratio.* After which the density of the new load-areas, that is, *the weight per cubic foot of the new superstructure is to be reckoned as less than that of the old superstructure in the ratio of the transformation.* (See figs. 274 and 275.)

GEOSTATIC LOAD UNIFORM OR VARYING POTENTIAL.

The pair of conjugate-loads shown on fig. 274 can be produced simultaneously by placing the elliptic ring horizontally in a frictional granular mass with a free level surface, the mass being excluded from the inside. The east- and west-load q on the pair of short barricades being uniform, and four times as intense as p the north- and south-load on the pair of long barricades. Compared to the greater, the lesser load is passive. If the shaft or well were a thin elastic cylinder erected first and stayed inside by an elliptic centre, which could be gradually shrunk upon itself, and if now the granular mass were spread loosely around it in horizontal layers, the genesis of the horizontal stresses would be as follows. The potential is the weight of the column of grains from the surface down to the ring of the cylinder under consideration; it is the active force. Every cube of grains around the elliptic ring would try to spread equally in all horizontal directions. The horizontal load on all the four barricades around the elliptic ring would be p, just as on the circular ring, fig. 264. But p would only be a *fraction* of the potential, for the friction among the grains assists in supporting the overhead-load on a cube of grains. Let us suppose that p is about a fourth part of the potential. This would be the proportion if the granular mass were crushed stone or, in a model, small-shot. Any cube of the small-shot would cease spreading from the load overhead, when the horizontal load applied to its four vertical faces was a fourth of the overhead-column.

Here we have the elliptic ring, fig. 274, with a *fluid-load* p one-fourth of the potential outside as in fig. 264, and supported inside by an elliptic *centre*. If now the *centre* were gradually shrunk, the elliptic ring would gradually collapse or flatten, due to the advantage which the fluid-load has on the long barricades. In elongating, the elliptic ring would *consolidate* the granular mass opposite the short barricades, till it pressed back with a stress $q = 4p =$ the potential, when the ring would cease elongating and the centre could be removed. The ellipse would now be more than twice as long as it was broad. To remedy this, the granular mass is to be spread around the shaft in *thin* layers and punned or rammed hard, layer after layer opposite the ends of the ellipse, but filled loose opposite the sides. The granular mass will be earth or shivers in the case of masonry cylinders or wells, and must be filled as described, as the distortion would be fatal to an elliptic brick cylinder long before the earth at its ends would be sufficiently consolidated. The load on the elliptic rib, fig. 274, is called shortly an earth or *geostatic-load of uniform potential.* One ellipse of stress, a quadrant being shown with its centre at D, serves to give completely the stress on the vertical faces of every unit-cube-of-grains all around the rib. If the unit-cube lie north and south, p and $(q = 4p)$ are the loads on its vertical pairs of faces, while the potential also q is the load on the horizontal faces. The whole stress at D is given by *an oblate spheroid.* If the unit-cube be turned till a pair of faces is parallel to CC, then the vector r gives the load on that pair of faces, and is therefore the mutual pressure between the face of the unit-cube and the back of the rib at B which are in contact.

In the same way the ribs shown on fig. 275 are subjected to the one a fluid-load, and the other a *geostatic-load of uniformly varying potential.* The vector r' gives the load on the face of a unit-cube parallel to jj at the depth B' below the surface of the earth, that is the stress on the back of the rib at B'. One ellipse of stress serves for all cubes at depth B', xF being the semi-major axis which is itself the potential. The load on the face of the unit-cube parallel to the paper is equal to the minor semi-diameter, and the whole stress at the depth B' is given by *a prolate spheroid.* As the depth of B' increases, this spheroid's axes increase at the same rate; but the ratio of any pair of its axes remain the same.

EQUILIBRIUM OF A FRICTIONAL GRANULAR MASS.

If the loads on the elliptic rib, fig. 274, were actually due to a granular mass of crushed stone, filled around the cylinder in the manner described, then, in order that the equilibrium may be permanent it is necessary that the unit-granular-cube at D

should itself be in equilibrium with the load q on its east and west faces 4 times as great as the load p on its north and south faces. These loads are called the *principal stresses* at D because they are normal to the faces of the cube lying north and south. If we map out a unit-cube at D with a pair of faces parallel to CC, the load on these faces is r which is no longer normal, but makes an angle RDM. The manner of constructing r is to lay off DM equal to the half-sum of the principal stresses along the normal DN, and then to lay off MR equal to their half-difference along Md isosceles to MD. Hence r is *most oblique* when the unit-cube-of-grains is so mapped out that MRD is a right angle. It is only necessary then to see that grains, in the cube mapped out in this most critical position, shall not slide on each other due to the extreme obliquity RDM of the load on its faces. But when R is a right angle, then sine RDM equals RM divided by MD. If we put $RDM = \phi$, the angle of friction between the grains, then for *the equilibrium of the granular mass at D it is necessary that sine ϕ shall not be greater than the ratio of the difference of the principal stresses to their sum.* Or what is the same thing, if the friction, among the particles of the unit-cube-of-grains mapped out by the principal planes (north-and-south and east-and-west on fig. 274) is to be sufficient to enable the minor principal stress to resist the tendency of the major to flatten out the cube, then *the ratio of the minor to major stress must not be less than* $(1 - \sin\phi) : (1 + \sin\phi)$.

The value of the angle of friction ϕ for a granular mass may be found by shooting the mass out on a level plane, and when the surface of the conical mass *ceases to run*, its slope to the horizon is to be measured. For crushed stone $\phi = 37°$, and $\sin\phi = \cdot 6$, so that

$$(1 - \sin\phi) : (1 + \sin\phi) = 1 : 4$$

which would exactly suit the conditions required by the rib, fig. 274.

For dry granular earth spread in layers horizontally or gently sloping $\phi = 30°$, so that $(1 - \sin\phi) : (1 + \sin\phi) = 1 : 3$.

Compare the preceding chapter on Rankine's Method of the Ellipse of Stress, Chapter IV., pp. 45, 52, 65, 71.

APPROXIMATE ELLIPTIC RIB.

Consider the circular quadrantal rib ACB, fig. 266, loaded with the vertical-load-area between itself and the straight extrados $\frac{1}{3}r$ above its crown A. The conjugate horizontal-load-area $bglks$ is wholly positive or inwards, and its area is

$\frac{1}{3}r^2$. On fig. 275 the same quadrant of a circle $AB'S'$ is shown, having the same directrix OE at a height $AO=\frac{1}{3}DA=\frac{1}{3}r$ as before. The conjugate-load-area is $bglke$, only it has been turned round so that $bglk$ the treble-batter-boundary is outside, while the other boundary be is now vertical instead of sloping at 45°. The area is still $\frac{1}{3}r^2$, and the breadth at each point

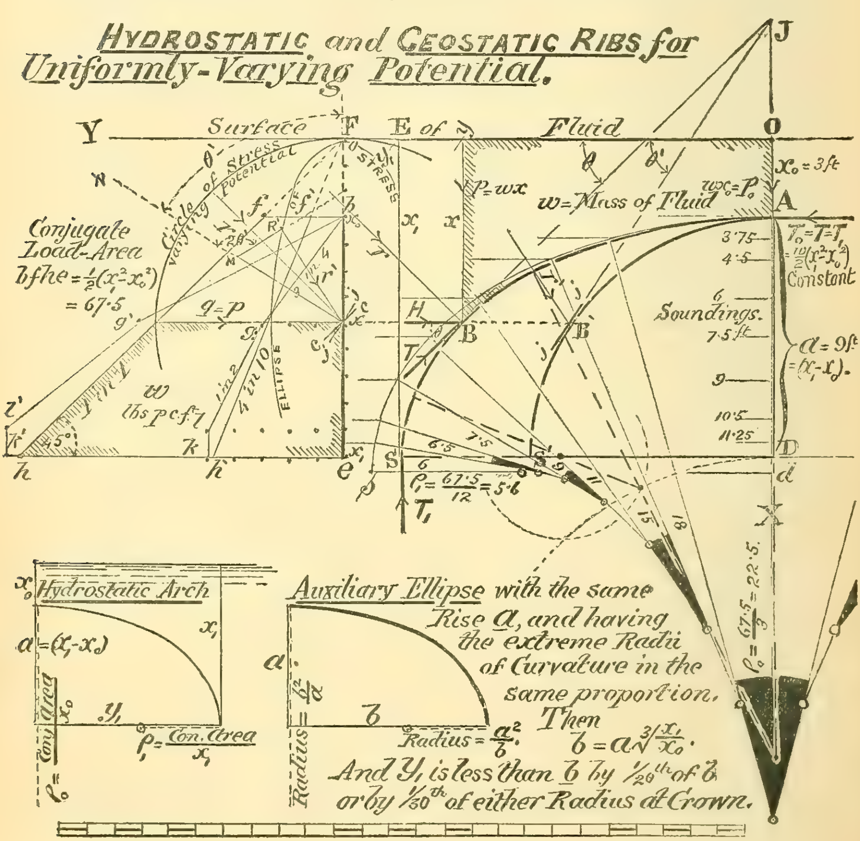

HYDROSTATIC and GEOSTATIC RIBS for Uniformly-Varying Potential.
J
Y
Surface
F
E of
Fluid
O
Circle of Varying Stress Potential
STRESS
P=wx
w=Mass of Fluid
$wx_0=P_0$
$x_0=3$ft
A
$T_0=T=T_1$
$=\frac{w}{2}(x_1^2-x_0^2)$
Constant
Conjugate Load-Area
$bfhe=\frac{1}{2}(x_1^2-x_0^2)$
$=67{\cdot}5$
3·75
4·5
6
Soundings.
7·5ft
9
10·5
11·25
$a=9$ft
$=(x_1-x_0)$
g=p
H
B
B'
T
D
d
w lbs p.c.f.
in 2
4 in 10
ELLIPSE
45°
h
h'
k
k'
l'
e
S
S'
$\rho_1=\frac{67\cdot5}{12}=5{\cdot}6$
T_1
$\rho_0=\frac{67\cdot5}{3}=22{\cdot}5$
x_0 Hydrostatic Arch
$a=(x_1-x_0)$
x_1
Con. Area
y_1
$\rho_1=\frac{\text{Con. Area}}{x_1}$
ρ_0
Auxiliary Ellipse with the same Rise a, and having the extreme Radii of Curvature in the same proportion. Then
$b=a\sqrt[3]{\frac{x_1}{x_0}}$.
And y_1 is less than b by $\frac{1}{20}$th of b or by $\frac{1}{30}$th of either Radius at Crown.
Radius $=\frac{b^2}{a}$
Radius $=\frac{a^2}{b}$

Fig. 275.

exactly the same as before. The slopes of the treble-batter are now, bg battering at 3 in 4, gl at 1 in 2, while lk is vertical. Of course the exact outer boundary is a gentle curve to which that treble-batter is a close approximation. Let now a *single boundary* $f'h'$ be drawn so that when produced it will pass through F, and such that the enclosed area $bf'h'e$ shall equal *bglke*, that is $\frac{1}{3}r^2$. It is only necessary to draw $Ff'h'$ at the batter 4 in 10. We have then $bf'h'e$ as a rough approximation to the conjugate horizontal-load-area for the circular quadrantal rib $AB'S'$ loaded up to a straight extrados at a height OA, one-third of the rise DA above the crown. Suppose now that the quadrant be slightly modified in form so as to exactly balance under these two load-areas. We have $AB'S'$ the quadrant of some curve differing but slightly from a circle, and balancing under the vertical-load-area between itself and a horizontal extrados at a height over the crown OA which is one-third of the rise DA, together with the horizontal-load-area standing on the platform *be* equal to DA the rise, and mapped out by the straight slope $Ff'h'$ battering at 4 to 10.

The modified quadrant $AB'S'$ is called *a geostatic rib*, because both loads can be simultaneously imposed on the rib by immersing it in a frictional granular mass of which OEY is the free horizontal surface, and the friction among the particles such that the unit-cubes mapped out horizontally and vertically are themselves in equilibrium with the horizontal load on each *four-tenths* of the vertical. Dry granular earth spread outside the arch in horizontal layers can, as already explained, produce such a load which is shortly called a geostatic load, the potential uniformly increasing with the depth below the free surface. At x, a point at the same depth as B', the ellipse of stress is shown on fig. 275, FR' is an arc, and the semi-diameters are in the ratio of 4 to 10.

By transforming horizontally the rib $AB'S'$, in any ratio, we obtain a new rib ABS. If $AB'S'$ be the quadrant of a circle, then ABS is the quadrant of an ellipse, the new horizontal load area being $bg'l'k'e$ derived from *bglke* by a horizontal transformation in the *duplicate ratio.* But if $AB'S'$ be a true geostatic quadrantal rib, differing little from a circular quadrant, then the new rib ABS is also the quadrant of a true geostatic rib differing but little from the quadrant of an ellipse, and the horizontal-load-area is *bfhe* derived from $bf'h'e$ by a horizontal transformation in the *duplicate ratio.*

By choosing a suitable ratio for the horizontal transformation, namely, one for which the duplicate ratio is 10 to 4, we have Efh sloping at the batter 1 to 1, the load is then a *fluid load*, and the rib ABS is called a *hydrostatic rib.* Now the ratio whose duplicate is 10 to 4 is approximately 3 to 2; hence the hydrostatic rib, of which ABS is the left half, has its span *three times the rise*, and the depth of the load over the crown *one-third part of the rise.* The ellipse of stress has become a circle of stress.

In this way we see that a semi-elliptic rib, or one which in courtesy (see fig. 262) is called semi-elliptic, may with like propriety be called a complete hydrostatic rib. That if the span be *treble the rise* the surface of the fluid must be over the crown at a height *one-third the rise.*

It is better now to suppose that, on fig. 275, we start with this hydrostatic rib *ABS* loaded outside up to the fluid surface *OEF*, so that the 45°, or 1 to 1 line *Ffh*, defines the horizontal-load-area. Then to suppose the geostatic rib *AB′S′* to be derived from *ABS*, by a crush-in in the ratio 2 to 3, the horizontal-load-area crushing-in in the ratio 4 to 9, or 4 to 10 nearly, and defined by the 4 to 10 battering line *Ff′h′*: the circle of stress at x becoming the ellipse of stress with diameters in this duplicate ratio. The particular geostatic rib *AB′S′* derived by this crushing-in of the particular hydrostatic rib *ABS* is sensibly the quadrant of a circle. One might readily fall into the error of supposing that all hydrostatic ribs might be crushed-in so as to be sensibly quadrants of circles by choosing a fractional ratio of transformation to make the semi-span equal to the rise. But by considering the genesis we have employed, it will appear that only those hydrostatic ribs, with the surface of the fluid at a height over the crown at least *one-third of the rise*, can be crushed-in to give derived geostatic-ribs with half span equal to rise, so that such derived ribs may be, in any other respect, like the quadrant of a circle. Because starting as we do on fig. 275 with a horizontal-load-area *bglke* wholly positive or inwards, the rougher approximation *bf′h′e* is reasonable, but with a lower directrix to start with, the horizontal-load-area would be partly positive and partly negative, to which one sloping boundary *Ff′h′* could, in no sense, give an approximation.

We have this important practical distinction, that all sensibly semi-elliptic-ribs may be treated as sensibly geostatic whether derived from a hydrostatic rib by a *pull-out* or a *push-in*, provided the height of the load over the crown be at least *one-third of the rise.* For loads over the crown less than one-third of the rise the hydrostatic rib itself, and geostatic ribs got by a pull-out, are sensibly elliptic, but not those got by a push-in.

HYDROSTATIC AND GEOSTATIC RIBS.

Let *ABS* be the quadrant of a hydrostatic rib, fig. 275. The origin is the point over the crown, while the axis *OY* is the surface of the fluid, the vertical is the axis *OX*. The depths of *A* and *S* the crown and springing are x_0 and x_1. The rise of the arch is $a = (x_1 - x_0)$, and the half-span is y_1. The vertical-load-area is *OESA*, while the horizontal-load-area, mapped out by the 45° or 1 to 1 line, is *bfhe*; it is the difference of two half squares, namely $\frac{1}{2}(x_1^2 - x_0^2)$, and this multiplied by w, the weight of a cubic foot of the fluid, gives us the thrust at the crown T_0 equal to $\frac{1}{2}w(x_1^2 - x_0^2)$. Since the extern load is *normal* at each point on the back of the rib the thrust along the rib is constant. So that we have, at *S* the springing point, at *B* any intermediate point, and at the crown *A*,

$$T_1 = T = T_0 = \frac{w}{2}(x_1^2 - x_0^2). \qquad (20)$$

The rule, p. 486, for the thrust at the crown of a rib applies alike to every point of the hydrostatic rib. To get the radius of curvature at any point of the rib it is only necessary to divide the constant thrust T by the intensity of the extern *normal load*, which itself is the same as the mass of the fluid w multiplied by the depth of the point on the rib below the surface. As w is common to both expressions, the same quotient is obtained by dividing the horizontal-load-area by the depth of the point. The radii of curvature of the points *S*, *B*, and *A*, are—

$$\rho_1 = \frac{x_1^2 - x_0^2}{2x_1}, \quad \rho = \frac{x_1^2 - x_0^2}{2x}, \quad \rho_0 = \frac{x_1^2 - x_0^2}{2x_0}. \qquad (21)$$

The central of these expressions is an equation to the hydrostatic rib, and is to be expressed simply in words thus—*at any point of the rib the radius of curvature and the depth of the point below the surface have a constant product*;—the value of the constant being the horizontal conjugate-load-area for the quadrant. The constant involves two linear quantities called parameters; they are x_0 and x_1, or, what is the same, $a = (x_1 - x_0)$ the rise of the rib, and x_0 the depth of the load over the crown. Any two values assigned to those determine a *particular* hydrostatic rib, when y_1 the half-span can be approximated to. In having two parameters this rib is like the ellipse. Further, as the depth to the rib increases continuously from the crown *A* down to the springing *S*, it follows that the radius of curvature decreases continuously between those points just as in the quadrant of an ellipse.

Rankine approximates to the half-span y_1 as follows: see the lower part of fig. 277, which shows the quadrant of the hydrostatic rib, and the quadrant of an auxiliary ellipse having the same absolute rise $a = (x_1 - x_0)$, and having its extreme radii of curvature in the same ratio to each other that they have on the other quadrant. If b is the half-span of this ellipse it will be a fair approximation to y_1. Now, the extreme radii of curvature of the elliptic quadrant are $b^2 \div a$, and $a^2 \div b$, so that their ratio is $b^3 : a^3$. For the quadrant of the hydrostatic rib their ratio is $\rho_0 : \rho_1$, or $x_1 : x_0$. Equating and arranging we have

$$b = a \sqrt[3]{\frac{x_1}{x_0}} \cdot \qquad (22)$$

That is, an approximation to the half span of the hydrostatic rib is the rise multiplied by the cube root of the ratio of the depth of the springing point to that of the crown.

This is always about 5 per cent. too great, or, more definitely, from b is to be deducted *one-thirtieth* of the radius of curvature at the crown of either the auxiliary ellipse or of the hydrostatic rib as it may be convenient. The closest approximation to the half span of the hydrostatic rib is

$$\left.\begin{aligned} y_1 &= b - \tfrac{1}{30}\frac{b^2}{a} \\ &= b - \tfrac{1}{30}\rho^0 \end{aligned}\right\} \cdot \qquad (23)$$

In this way we shall find the span of the hydrostatic rib having the depth of the springing points four times as great as that of the crown, or, what is the same thing, having the rise of the rib treble the depth of load over crown. Putting unity for the depth of the load at the crown A, fig. 275, then $x_0 = OA = 1$, and the rise is $a = DA = 3$, while $x_1 = ES = 4$. By equation (22)

$$b = 3\sqrt[3]{4} = 4{\cdot}76.$$

Squaring this and dividing by a we get 7·55, the radius of curvature of crown of auxiliary ellipse. Or for the hydrostatic rib, the horizontal-load-area being $\frac{1}{2}(4^2 - 1^2)$, we divide it by x_0, and get $\rho_0 = 7{\cdot}5$. Also a thirtieth part of these radii is ·252, which is the same as 5 per cent. of b. Reducing b by this amount, we get $y_1 = 4{\cdot}51$, and doubling, the span of the rib is 9·02.

Hence for the hydrostatic rib with the load over the crown *a third part* of the rise the span is *three times* the rise, a result which we arrived at in the last section by a quite independent approximation of our own.

For this particular hydrostatic rib the approximation given by (Rankine's) equations (22) and (23) are verified graphically on fig. 277, the curve being drawn in small arcs beginning at the crown *A*. It is now more convenient to have $OA = 3$ feet, and $OD = 12$ feet. The scale was double before reduction by photography. Horizontal lines were ruled at different *soundings* below *OE* the surface of the fluid. The conjugate horizontal-load-area $\frac{1}{2}(144 - 9)$, or 67·5 square feet, being divided by 3, gives $\rho_0 = 22{\cdot}5$. With this radius the first arc is drawn beginning at *A*, and ending when it cuts the horizontal at the *sounding* 3·75. Along the old radius at this point a new shorter radius is laid off, with which a second arc is drawn till it cuts the horizontal through the next sounding 4·5. In this way arc after arc is drawn till the last one has a vertical tangent at (script) *s*, which completes the approximate quadrant. As each arc has been drawn of constant radius, while the radius ought to have decreased, it follows that the semi-diameters of this approximate quadrant are too large. By measurement *Ad* is found to be 5 per cent. greater than the given rise *AD*, and we conclude that *ds* is 5 per cent. too big also; hence *DS* is laid off at that reduced value when *ABS*, the false quadrant, is struck from two main centres, and an intermediate one as already explained at fig. 262, the whole rib being struck from five centres. By measurement the half-spans *ds* and *DS* verify the values already calculated by the equations (22) and (23).

The soundings and radii used in the graphical construction of the curve are given below in two rows. Each radius is found in turn by dividing the sounding out of the area 67·5. These radii begin with the largest $\rho_0 = 22{\cdot}5$, and end with $\rho_1 = 5{\cdot}6$.

3·00,	3·75,	4·5,	6·0,	7·5,	9·0,	10·5,	11·25,	12.
22·5,	18·0,	15,	11,	9·0,	7·5,	6·5,	6·00,	5·6.

Observe that in this particular rib, 5 per cent. of the half-span and $\frac{1}{30}$th of the crown radius are the same, but in other ribs the fraction of the radius is to be preferred.

In the same way the spans of two other particular hydrostatic ribs may be calculated from the equations (22) and (23). The depth of load over crown (x_0) being taken as unity, then for the rise ($a = x_1 - x_0$) three times the load, the span ($2y_1$) was found to be 3·01 times the rise. With the rise four times the load in the same way the span is 3·22 times the rise, and with the rise five times the load the span is 3·41 times the rise.

As already pointed out, only the first of these hydrostatic ribs may be transformed by pushing-in and the resultant geostatic rib reckoned as sensibly elliptical or circular. All three may be transformed by pulling them out horizontally, and the resulting geostatic ribs are sensibly semi-elliptic. Let the pull-out, in the first place, be such that the 1 to 1 sloping boundary of the horizontal-load-area becomes a 1½ to 1 sloping boundary. That is, the duplicate of the ratio of transformation is 1·5, so that the ratio of transformation is 1·225, which is the multiplier to give the three new spans. In the second place the 1 to 1 sloping boundary is to become a 2 to 1 sloping boundary, so that the ratio of transformation is now 1·414, which again is the multiplier to give the three new spans. The hydrostatic ribs themselves may be called geostatic ribs with the ratio of transformation *unity*, for, as already pointed out, a fluid load is one of the varieties of the earth-load.

We have then the following nine geostatic ribs, that is, ribs sensibly semi-elliptic, able to be struck out from five centres as on fig. 262, and such that, when loaded between the ribs themselves and a straight extrados at the given height above the crown, require for equilibrium a horizontal-load-area mapped out by a sloping line at the batter given in the table. In the following table the depth of the load over the crown is given as a fraction of the rise of the rib, and the span as a multiple of the rise:—

Spans in Terms of the Rise of Semi-Elliptic Ribs which are sensibly Geostatic.

Boundary of horizontal-load-area.	Load at Crown ⅓rd of Rise.	Load at Crown ¼th of Rise.	Load at Crown ⅕th of Rise.
1 to 1	3·01	3·22	3·41
1½ to 1	3·68	3·94	4·18
2 to 1	4·25	4·55	4·82

With the semi-elliptic rib, it being desirable to have the span between four and five times the rise, it appears from the above table, that the depth of the load over the crown should be between ¼th and ⅕th of the rise.

SEMI-ELLIPTIC MASONRY ARCH.

In this arch the soffit is a false semi-ellipse struck from five centres as on fig. 264. The span being determined upon, the rise of the soffit may be any fraction of the span from a fourth to a fifth. The radius at the crown is now determined, being the square of the half-span divided by the rise. Then a mean proportional between this radius and ·12 gives Rankine's empirical thickness of the keystone; to this a suitable surcharge over the keystone being added, we have the depth of the load over the crown, which will be from a third to a fifth of the rise. For example, take the lower part of fig. 276, showing a quadrant of an elliptic masonry arch, half-span of soffit 38 feet and rise 19 feet, the radius of crown will be $38^2 \div 19 = 76$ feet. By Rankine's formulæ the thickness of the keystone is $\sqrt{76 \times \cdot 12} = 3 \cdot 02$ for a single arch, or for one of a series it is $\sqrt{76 \times \cdot 17} = 3 \cdot 6$. Now, it is usual to have the ring of stone of uniform thickness, instead of thickening out as in the segmental arch, so that 4 feet had better be allowed for the uniform thickness of the ring. To this a surcharge of *not less than* a foot and a half is to be added. With this minimum surcharge the depth of load at crown of soffit is sensibly a fourth of the rise. But we have on fig. 276 adopted a surcharge of 3 feet, so that the depth of the load at crown of soffit is 7 feet. This we have chosen, that the *total rise* may be 26 feet, so that this semi-elliptic arch may have the same span and total rise or head-room as the segmental arch, figs. 272 and 267, the two patterns being rival designs for the same railway viaduct of many arches.

The line of stress is to be supposed to be along the soffit in the first place. The soffit is now, instead of elliptical, to be considered as sensibly a geostatic rib. The quadrant is shown on the upper part of fig. 276. The three data are $OA = x_0 = 7$, $OD = x_1 = 26$ feet, and the half span $c = sy_1 = 38$ feet, where x_0 and x_1 are the extreme depths of the hydrostatic rib from which the geostatic rib is derived by a horizontal transformation in the ratio s. The thrust T_0 at A the crown of the geostatic rib is the mass of the superstructure w multiplied by the horizontal-load-area $bf'h'e$, while T_1 the thrust at the springing is an sth part of T_0, for before the transformation they were equal, and only T_0 has increased. We shall return to the strict numerical solution of the arch on fig. 276 among the examples following.

ELLIPTIC MASONRY ARCH TREATED AS GEOSTATIC

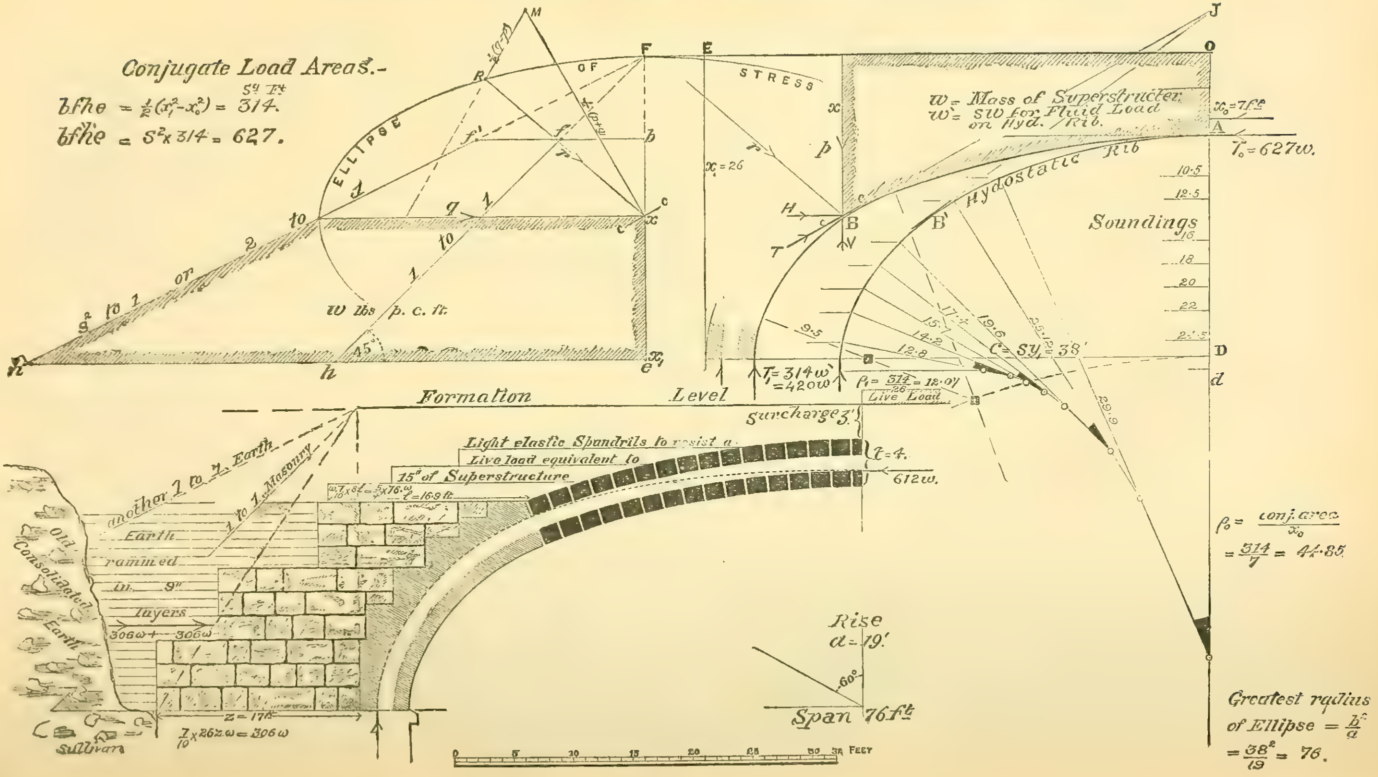


Fig. 276.

In the meantime fig. 276 may serve to illustrate the quite general treatment.

Since $x_0 = 7$ is sensibly $\frac{1}{3}$rd of $a = 19$, we may look in the second column of the table, and since the span 76 is exactly 4 expressed in terms of the rise a, the slope of the boundary of the horizontal-load-area will lie between $1\frac{1}{2}$ to 1 and 2 to 1, just as 4 lies between 3·68 and 4·25. By proportional parts this slope is $1\frac{3}{4}$ to 1. Quite strictly it is 1·853 to 1 : see the numerical examples following. On fig. 276, s^2 to 1 is taken as sensibly 2 to 1. At A the thrust is $T_0 = w \times$ area $bf'h'e = 627w$, and dividing by s, $T_1 = 420w$ nearly.

Consider the quadrant of the masonry ring as rigid, and apply a pair of equal and opposite couples, one at the crown, and the other at the springing. Let the couple at the crown be of a moment such that it shifts T_0 just into the middle third of the keystone as shown on the lower part of the figure; then the other couple will shift T_1 well into the middle third. In no case will T_1 be shifted past the middle third, as T_0 is never twice as great as T_1. The modified line of stress begins and ends in the middle third, and as its curvature is continuously varying, the whole curve is confined to the middle third of the masonry ring, shown white on the figure.

For the middle elastic part of the ring, shown dense black, the heavy backing is left out. It stretches between the pair of joints where the tangents to the soffit are inclined at 30° to the horizon. For one thing T_0 is now slightly reduced. The soffit for this middle part of the ring is actually all struck from the main centre, and is really a segmental circular masonry ring springing at 30° to the horizon. The modified line of stress of this central part, for the most perfect economy of masonry, would start at the lower limit of the middle third at the keystone and would reach the upper limit of the middle third at the joint of rupture, the springing joint for this black part of the ring, that is at the joint which, when produced, goes through the main centre, and makes 60° with the vertical. This will now determine whether the thickness adopted for the arch-ring is sufficient, for the horizontal through the lower limit of the middle third of the crown joint, and the line at 30° to the horizon drawn through the upper limit of the joint of rupture must meet on the vertical through the centre of gravity of the corresponding superstructure. The construction will be similar to that shown at the lower right-hand corner of fig. 267. In this case the thickness of the ring is the least for permanent equilibrium, and at both crown and joint of rupture the maximum stress is double the average. With a more liberal thickness of ring, different pairs of tangents may be drawn to meet on the proper vertical, and the actual line of stress will be determined by the ceasing of the subsidence of the keystone when the centre is slowly removed. It is always safe to assume that the maximum intensity of the stress at the crown joint is double the average, or that the real factor of safety is only half the apparent factor against crushing.

If a uniform live load be added all over the span equivalent to an additional height of the superstructure, $1\frac{1}{4}$ feet on fig. 276, the calculations could be repeated, making x_0 and x_1 greater by this amount. The increase of T_1 is simply w multiplied by the area of the rectangle of base 38 feet, and height $1\frac{1}{4}$ feet, or $(T_1' - T_1) = 48w$. And for the increase of T_0 we have (by the rule for the thrust at the crown of a rib, p. 486) merely to multiply $1{\cdot}25w$ the extra *normal* load at the crown by 76, the radius of curvature, when we have $(T_0' - T_0) = 95w$. Light spandrils to *resist* this load are shown on fig. 276. The depth from the formation to the top of the heavy spandril walls upon which the light spandrils ride measures 8 feet on the figure. If l be the distance they stretch out longitudinally; then $8wl$ is the over-head load pressing the spandrils down on their base. Then with ·7 for the coefficient of friction of stone on stone we have the frictional resistance of the light spandrils $5{\cdot}6\,wl$ equal to $95w$, so that $l = 16{\cdot}9$ feet.

Examples.

308. In a semi-elliptic masonry arch the span of the soffit is 76 feet and the rise is 19 feet, and the level of rails is 7 feet above the crown of the soffit. The soffit which is to be struck from 5 centres is to be taken, in the first place, as the line of stress. Assuming it to be sensibly a *geostatic rib*, find the thrust at the crown and springing.

The extreme depths of a quadrant of the *hydrostatic rib* from which the *geostatic rib* is derived are $x_0 = 7$ and $x_1 = 26$. A rough and a close approximation to the half-span of this *hydrostatic rib* are by equations (22) and (23),

$$b = 19\sqrt[3]{\frac{26}{7}} = 29{\cdot}424,$$

$$y_1 = 29{\cdot}424 - \tfrac{1}{30}\,\frac{b^2}{19}$$

$$= 29{\cdot}424 - 1{\cdot}518 = 27{\cdot}906.$$

The half-span of the *geostatic rib* being given $sy_1 = 38$, we have then the ratio of transformation or pull-out by which the half-span 27·906 has become 38 to be

$$s = \frac{38}{27{\cdot}906} = 1{\cdot}3617$$

and $$s^2 = 1{\cdot}853.$$

Hence the 1 to 1 boundary of the horizontal-load-area for the hydrostatic rib having been pulled-out in the *duplicate ratio* 1·853 to 1, this furnishes the slope of the boundary mapping out the horizontal-load-area of the *geostatic rib*.

Multiplying w the weight of a cubic foot of superstructure into this area the thrust at the crown is

$$T_0 = 1·853 \times \frac{w}{2}(26^2 - 7^2) = 581w,$$

and
$$T_1 = T_0 \div s = \frac{581w}{1·362} = 427w.$$

Note.—These are the more exact values which should have appeared on fig. 276, only that the slope of the boundary of the *geostatic rib* was taken 2 to 1 instead of 1·853 to 1 for clearness of explanation.

309. If a live-load equivalent to an additional height of superstructure of 1·25 feet be added all over the masonry arch in last example, find now the thrusts at crown and springing.

We have now $x_0 = 8·25$ and $x_1 = 27·25$,

$$b = 19\sqrt[3]{\frac{27·25}{8·25}} = 28·297,$$

$$y_1 = 28·297 - \tfrac{1}{30}\frac{b^2}{19} = 26·892,$$

$$s = \frac{38}{26·892} = 1·414 \quad \text{and} \quad s^2 = 2,$$

$$T_0' = 2\frac{w}{2}(27·25^2 - 8·25^2) = 675w.$$

$$T_1' = T_0' \div s = 478w.$$

Note that now the boundary of the conjugate horizontal-load-area slopes at 2 to 1. Also $T_0' - T_0 = 675w - 581w = 94w$, about the product of $1·25w$ and $38^2 \div 19$ already used for an approximation: see fig. 277. Again $T_1' - T_1 = 51w$, a little more than the simple product of $1·25w$ and 38 the half-span.

310. The pier between two arches is 12 feet thick at the top and the faces batter 1 in 20: find the centre of stress at a joint 45 feet below the springing level.

That joint is 16·5 feet thick, the weight of the pier $641w$. The half-arches rest $427w$ and $478w$ on its top; the whole weight above the joint is $1546w$. The excess of the horizontal load on one side over the other is $95w$. If z be the deviation of the centre of stress from the middle of the joint 45 feet down, the equation of moments is

$$1546wz = 45 \times 95w,$$

$$z = 2·76 \text{ feet}.$$

This is *one-sixth* of 16·5, so that the centre of stress is just within the middle third of the joint.

311. The abutment pier is 20 feet thick at top and its faces batter at 1 in 20. At a joint 45 below the springing, find z the deviation of the centre of stress from the middle, when the pier bears a half-finished arch on one side only.

The joint is 24·5 feet thick; and if we take the mass of the pier $2w$ and reckon a complete arch on one side only, then weight of pier is $2002w$, and the equation of moments is

$$2002wz + 427w(z + 8) = 581w \times 45,$$

so that $z = 9·3$ or *three-eighths* of 24·5; the thickness of the joint and the line of stress does not come closer to the face of pier than an eighth of its thickness.

312. If, when the viaduct is finished, an arch falls, it leaves a whole arch on one side only of the abutment pier.

The first term in the equation above must be halved, as we cannot now consider the pier of double density; at the same time we have now a mass of masonry 20 feet by 26 feet over the pier, so we must add $520wz$ and deduct $1001wz$ from the first term and

$$1521wz + 427w(z + 8) = 581w \times 45.$$

And now $z = 11{\cdot}6$ less than half of $24{\cdot}5$, so that the centre of stress is still within the masonry.

MASONRY ABUTMENTS.

In the case of the segmental arch (fig. 272), the abutment is readily designed by building the masonry in benches stepped at the back, the oblique thrust of the arch, shown by a dot-and-dash line, to lie in the middle third of the abutment if it be no deeper than the rise of the arch. The masonry may be entirely in square-dressed courses, but the central part may, with advantage, be in radial courses. The abutment is best honeycombed at the back with thick walls for the thin spandrils to ride out upon. The spaces between the thick wall to be thoroughly drained and packed with heavy ballast and broken shivers of stones. The spaces between the thin spandrils to be coated with waterproof material, arched or bridged over, and preferably left void, drain-pipes and ventilating holes being provided.

In the semi-elliptic arch, lower part of fig. 276, the heavy spandrils, with square-dressed joints, are shown in three benches, from the springing level up to the joint of rupture. The thickness z, at springing level, is calculated in the first place, as if the backing required were like that of a fluid load, or as if the horizontal-load-area were mapped out by a 1 to 1 boundary. In the example $\cdot 7 \times 26lw' = \frac{1}{2}w\,(26^2 - 8^2)$, or $l = 17$ feet, taking w' and w as equal. A right-angled triangle, with the base z, maps out the backing. The re-entrant angles of the benches are to lie on the hypotenuse of this triangle, shown with a dot-and-dash line; then the middle and top benches are slid along inwards till they butt against the square-dressed backs of the *voussoirs*. As w' is likely to be 20 per cent. greater than w, that is, the mass of the backing greater than that of the superstructure, we will have the reduced value $z = 14$ feet.

We have shown that the conjugate horizontal-load-area for elliptic arches has its boundary sloping at 2 to 1 at most, that is, double the horizontal action of a fluid load. But if the abutment, as on the lower part of fig. 276, is built in an excavation in old consolidated earth, then the earth filled between the abutment and the face of the excavation can be made to press horizontally, like a fluid, by punning it hard in thin horizontal layers. This reinforcing the frictional stability of the masonry abutment, as already designed, supplies the necessary resistance to the spreading of the arch. The centre of the arch is to be struck when about half of the superstructure is built, and before the earth is rammed behind the abutment. When the masonry has settled and consolidated, the earth is to be first rammed behind the abutment, and then the superstructure finished. On the other hand, if the abutment is built, and a loose embankment built over it, then even by filling the best stuff behind the abutment, it will only press horizontally with half a fluid load, so that the masonry abutment must now be equal to a $1\frac{1}{2}$ to 1 bounded load area. That is, z must be increased by 50 per cent. from 14 feet to 21 feet. Adding to each of these, 4 feet for the thickness of the springing stone, we have the total thickness of an abutment for the semi-elliptical masonry arch (fig. 276) to be 18 feet and 25 feet in the best and worst circumstances respectively. These are almost exactly $\frac{1}{4}$th and $\frac{1}{3}$rd of 76, the radius of curvature of the crown of the soffit. And so we have verified Rankine's proportion: see his *Civil Eng.*, p. 514: *The thickness of the abutments of a masonry bridge are from a third to a fifth of the radius of curvature at the crown.*

MASONRY PIERS AND ABUTMENT PIERS.

The design for a common pier is shown on the left half of fig. 277. It is for a viaduct of many arches, of which half an arch is shown on fig. 276. Every fourth or fifth pier is to be an abutment pier, the design of which is shown on the right half of fig. 277. Both piers are 45 feet high, and are divided into three blocks of 15 feet each. The common pier is 12 feet thick at the top, and the abutment pier is 20 feet thick. Their faces batter at 1 in 20. The arch to the left of the common pier bears a live load equivalent to $1\frac{1}{4}$ feet of extra height of its superstructure. Hence the excess horizontal thrust of this arch above the one on the right side of the pier is $1{\cdot}25w \times 76$, or $95w$. It is shown on fig. 277, acting on the horizontal line, through the centre of gravity of the horizontal-load-area, as drawn on the figure above. Then $468w$ and $420w$ are the vertical loads on the springing stones at the

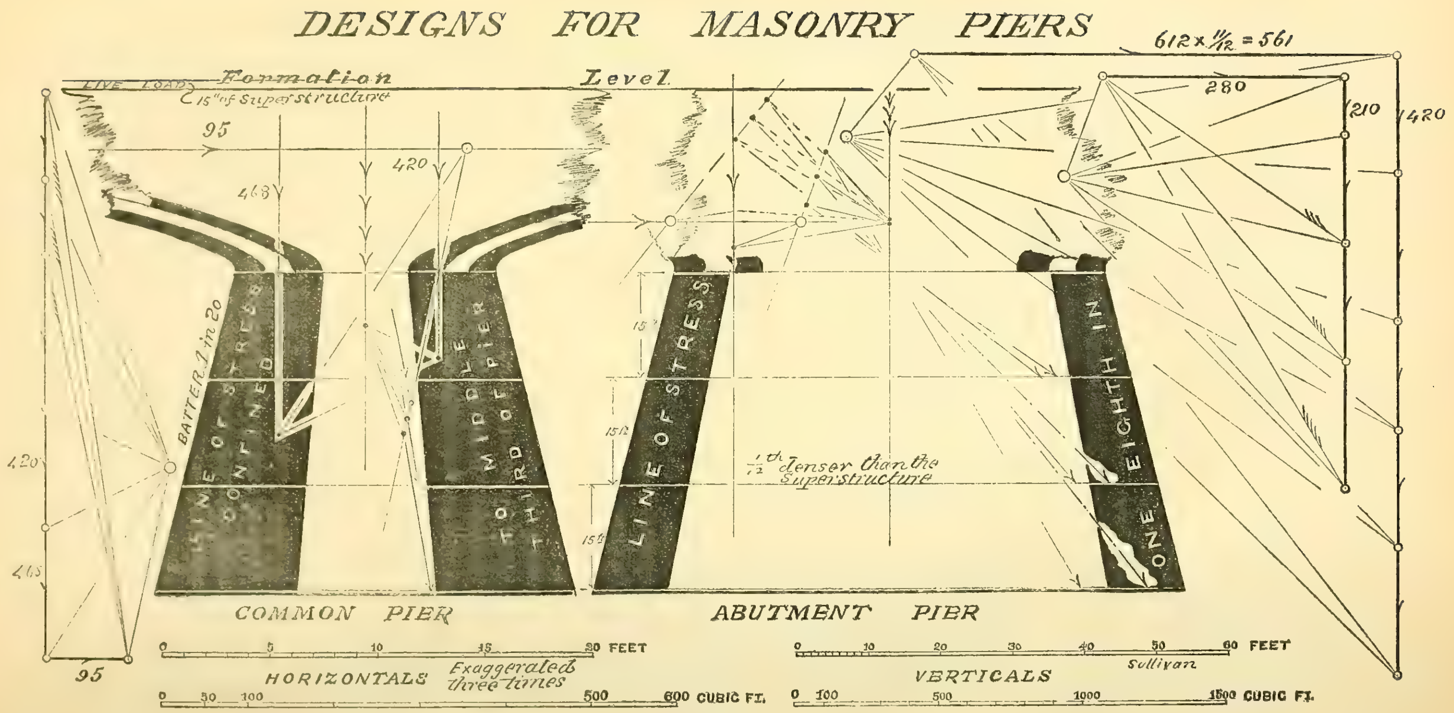

DESIGNS FOR MASONRY PIERS
612 × 11/12 = 561
LIVE LOAD
Formation Level
15th of superstructure
95
420
468
280
210
420
BATTER 1 in 20
LINE OF STRESS CONFINED
TO MIDDLE THIRD OF PIER
LINE OF STRESS
ONE EIGHTH IN
15'
15½
15½
1/12th denser than the Superstructure
420
465
95
COMMON PIER
ABUTMENT PIER
0 5 10 15 20 FEET
HORIZONTALS Exaggerated three times
0 50 100 500 600 CUBIC FT.
0 10 20 30 40 50 60 FEET
VERTICALS
Sullivan
0 100 500 1000 1500 CUBIC FT.

Fig. 277.

corners of the pier, due to the half arch loaded, and the half arch unloaded. They are quoted from fig. 276. There are also shown four barbs on the vertical line down the middle of the pier, which are the weights of the column of masonry between the two half arches, and of the three blocks of masonry into which we divided the pier itself. The mass of the masonry w is taken as unit. The vertical scales, both for dimensions and loads, are three times as fine as the horizontal. By drawing a force, and a link polygon, the centres of stress at each of the three joints below the springing are defined by the barbs of oblique arrows. Each centre of stress is well within the white "kernel," which is the middle third of the masonry of the pier.

It is absolutely necessary, for the permanent stability of the pier, that the line of stress be confined to the middle third, otherwise the joints would open, as the pier was rocked by the live load shifting from one arch to the other, the mortar drop out, and the whole be destroyed.

The everyday work of the abutment pier is to reflect vibrations passing from arch to arch along the viaduct. Once in its history it has to act as an abutment, the arches on one side only being finished, or partly built. Now the arch resting on the abutment pier will have its centre struck when itself *half completed.* On fig. 277 the horizontal and vertical loads, $280w$ and $210w$, at the left corner of the abutment pier, are due to the half finished arch on that side, and are quoted from the fig. 276 being slightly modified because the pier is of dense masonry. The smaller force polygon, and its corresponding link polygon, define the centres of stress to be within the white kernel. That is, the centres of stress do not approach so close to the face as *one-eighth* of the thickness, which is a suitable limit, as the load is steady, and only in one direction, just as in retaining walls.

If by an accident, such as the failure of a foundation, an arch of the viaduct should fall, the damage should only reach to the nearest abutment pier. The larger polygon shows the full load of an arch on one side, and the oblique dot-and-dash arrows show the centres of stress to be still within the masonry, so that the abutment pier could be expected to sustain the complete arch for a short time till it could be shored up. A study of the construction shows that the second last full line oblique arrow produced to the lowest joint defines the same centre as the last oblique dot-and-dash line. This saves the trouble of drawing the larger polygon.

Tunnel Shell.

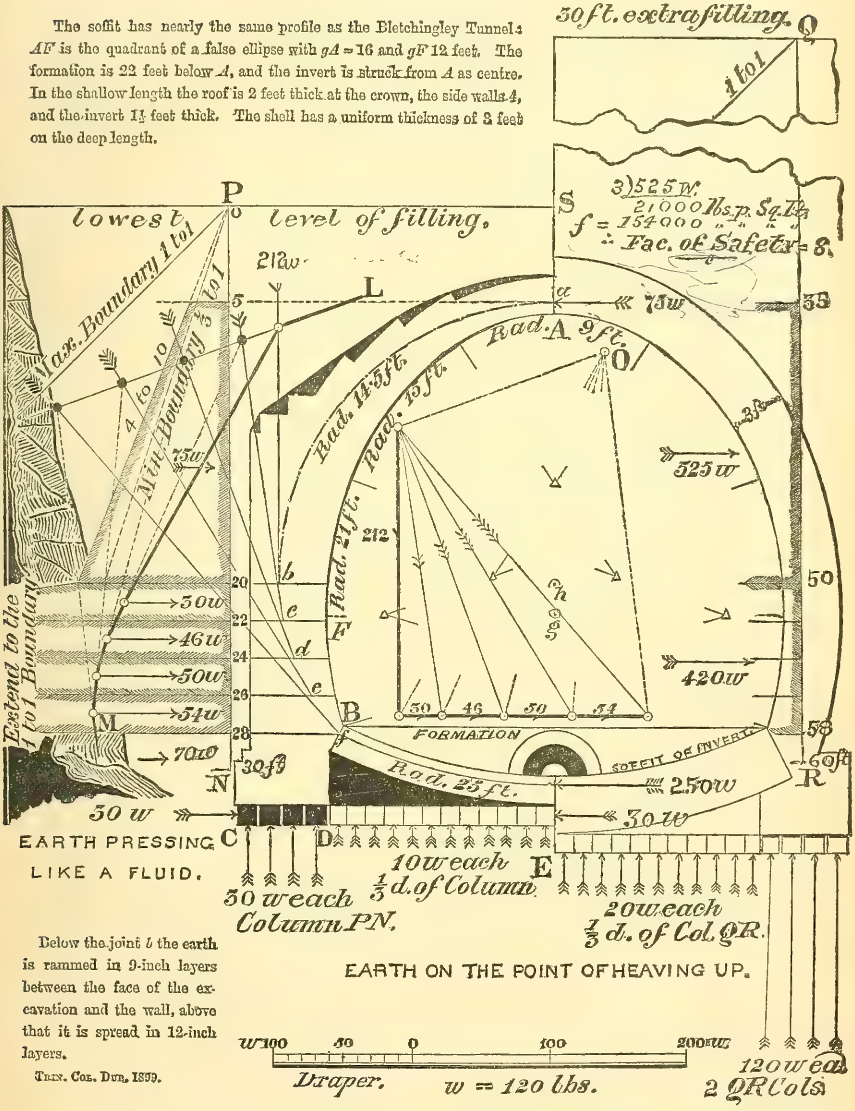
On fig. 278 is shown a tunnel shell; it has nearly the same profile as the Blechingley tunnel (see plate 1 of Simms' *Practical Tunnelling*). In other respects the shell is designed to illustrate the mutual stability established between the shell and the surrounding cubes of earth.

The left part is a half cross-section of the *front length* of the tunnel, where a deep cutting is first made in old consolidated earth, the sides as nearly vertical as may be. The thick side-walls are first built, 4 feet thick, and the roof turned on a centre. It is 2 feet thick at the crown, thickening by a ring of bricks at intervals outwards to the haunches. The notches are covered with a band of asphalt to render the roof waterproof. From *A*, the crown of the soffit, the invert is swept out truly with a suitable versine, to accommodate a covered drain between the rails. The earth is now filled between the back of the walls and the face of the cutting. Up to the level of *b* it is punned in thin layers, that the cubes may press horizontally like a fluid; above that it is filled more loosely in thick layers, so as to be pressing horizontally, with little more than a *third of column* overhead. The lowest level of the filling when completed is to be *SP*, so that *Sa* may be a third part of 14·5, the radius of the circular quadrant *ab* assumed to be the line of stress. This quadrant, then, is a geostatic rib, whose horizontal-load-area is bounded by the 4 to 10 sloping line lying close to the $\frac{1}{3}$rd to 1 slope, indicating the minimum horizontal-load necessary for the equilibrium of the cubes of earth. The depths of *a* and *b*, below *SP*, are 5 and 20 feet, so that the horizontal-load-area measures $\frac{4}{10} \cdot \frac{1}{2}(20^2 - 5^2) = 75$. The horizontal thrust at *a*, the centre of the crown joint, is $75w$. At *b* the vertical thrust is $75w$, multiplied by $\frac{3}{2}$, the ratio of transformation by which the quadrant *ab* is converted into the corresponding hydrostatic quadrant with the same extreme depths (see fig. 275). To this vertical thrust of the rib at *b* there falls to be added a column of earth, 20 feet by 5 feet, making in all $212w$, acting down through *b*. The total upward thrust, acting partly on *CD*, the base of the thick wall, and partly on *DE*, the half invert, must also be $212w$. Now, from the manner of construction, most of this load is thrown on *CD*, the invert only offering the minimum reaction to the swelling of the earth beneath, it being built later and being elastic and yielding.

TUNNEL SHELL

Cut and Cover.

The soffit has nearly the same profile as the Bletchingley Tunnel: AF is the quadrant of a false ellipse with $gA = 16$ and gF 12 feet. The formation is 22 feet below A, and the invert is struck from A as centre. In the shallow length the roof is 2 feet thick at the crown, the side walls 4, and the invert $1\frac{1}{2}$ feet thick. The shell has a uniform thickness of 3 feet on the deep length.



Below the joint b the earth is rammed in 9-inch layers between the face of the excavation and the wall, above that it is spread in 12-inch layers.

Trin. Coll. Dub. 1899.

Fig. 278.

Suppose all the unit-cubes of earth at the level CE, 30 feet below the free surface PS, to be pressing horizontally with $30w$, a load equal to the overhead column. Then the least vertical pressure on the 12 unit-cubes (white) under the invert DE, consistent with the stability of those cubes themselves, is $10w$, or one-third of column. From the ramming of the earth at C, the vertical load, on each of the unit-cubes (black) under CD the base of the wall, may be taken as column, or $30w$, more or less. This furnishes an equation to verify if the thickness of the walls' foundation is sufficient. For we must have

$$30w \times t + 10w \times 12 = 212w.$$

This gives t nearly 4 feet, and taking t exactly 4 feet, and modifying the equation, we have the vertical load on the four (black) unit-cubes below the wall to be $23w$, and

$$23w \times 4 + 10w \times 12 = 212w.$$

The radius of the line of stress in the invert is 25 feet, and it has the upward load of $10w$ uniform along the span, just as in fig. 264, so that the thrust along it is constant, and equal to $10w \times 25 = 250w$.

Next to join together these two lines of stress, in roof and invert, by a line of stress confined to the middle third of the lower part of the thick wall, we must consider the horizontal load applied to the back of the wall by the punned earth behind it. This load is given by an area mapped out by any boundary lying between the 1 to 1 and $\frac{1}{3}$ to 1 boundaries. On fig. 278 these areas are measured up to the 1 to 1 boundary, and at intervals of 2 feet of depth are $30w$, $46w$, $50w$, and $54w$. Beginning with $212w$, at b, and compounding those in order by means of the force polygon, pole O, and the link polygon LM, the centres of stress are defined—c by the arrow with two feathers; d by the arrow with three feathers; e by the arrow with four feathers; and lastly, B, by the arrow with five feathers. In another example the points c, d, e, &c., are to be laid down one after another, in suitable position, each in its own joint, and the horizontal areas necessary to make them so lie determined in succession, and provided the boundary of those load areas lies between the two practical limits, the 1 to 1 and $\frac{1}{3}$ to 1 boundaries; then $bcdeB$ is a possible line of stress, consistent with the equilibrium of the punned earth itself.

Adding $70w$ for the horizontal load below B, we have the sum of $30w$, $46w$, $50w$, $54w$, and $70w$, equal to $250w$, the thrust at crown of invert.

Both the top and bottom quadrant, then, of the left half of fig. 278 are in horizontal and vertical equilibrium, consistent with the equilibrium of the surrounding cubes of earth, and throwing as light a load on the invert as may be. There still remains the *equilibrium of moments* of the shell as a whole. This is satisfied by the fact that the principal axes of stress for the cubes of earth are vertical and horizontal, the free surface being horizontal, and as well the axes of the shell are also vertical and horizontal. With the free surface of the earth, at a gentle side-long slope, it may become necessary to find the cant which the principal axes of the earth-pressure makes with the vertical, which is easily done by the formulæ on the model (fig. 49), and then to prevent the shell of the tunnel from twisting like a stalk of barley-sugar under the unbalanced couples, one of two things has to be done. One is to build the shell of the tunnel at the same cant, an example of which is the Dove Tunnel, a section of which can be seen on the plate xxii. of the fourth edition of Simms' *Practical Tunnelling*. But the more modern way is to load the shell by an extra mass of masonry at one foot, so as to cant the *virtual* axes where it may be required.

As the filling, over the tunnel shell, gets deeper and deeper, the problem becomes less and less one of stability, and more and more one of strength. The right half of fig. 278 shows an extra 30 feet of filling. The total depth is now $QR = 60$ feet, the usual maximum for tunnels built in cut and cover. Even for a tunnel built in a cutting greater than 60 feet in depth, and covered, it is likely that only to this height will the load overhead affect the shell, as the earth itself forms a relieving arch. And, again, in a bored tunnel, the disturbance is only likely to affect the earth to a limited height above the hole. If on the diameter of the hole as base, an isosceles triangle be constructed, with its sides sloping at 75° to the horizon, the vertex gives the probable extent of the disturbance due to the loosening of consolidated earth due to the bore-hole. For cracks will run out on the plane on which the direction of the thrust is most oblique. Thus, in all cases, the height of load shown on the left half of fig. 278 is the probable maximum (see quotations, p. 561, from Simms' treatise).

Consider the right half of fig. 278: the additional vertical load is a column, 30 feet by 16 feet, or $480w$. The upward push of the unit-cubes under the right half of invert is doubled, as the depth is doubled; and taking the upward load on the four unit-cubes under the thick wall at R as $120w$, the total extra upward push on the right half of the shell is $10w \times 12$, together with $90w \times 4$, or, in all, $480w$. The horizontal load among the cubes, to the right of E, must be equal to the column QR, so that no one of these sixteen cubes shall have the rectangular pair of loads on it in a ratio exceeding 3 to 1. This requires that the horizontal thrust of the punned earth, at the depth R, shall equal the column, but at the depth 35 feet below Q, it may be less, say, instead of $35w$, it be only $20w$, while at the depth 50 below Q, it is $50w$. Then the load-area standing on the base, from 35 to 50, has a height half the sum of $20w$ and $50w$, that is $35w$, and multiplying by the base 15, then we have $525w$ as the probable horizontal extern load on the upper right quadrant of the shell. With 15 feet as the radius of the new line of stress at the crown, and the load over it at $35w$, we have, by the rule for the thrust at the crown, $35w \times 15 = 525w$. This establishes the horizontal equilibrium of the upper right quadrant of shell. For the lower right quadrant, the horizontal load due to the punned earth is $\frac{w}{2}(58^2 - 50^2) = 432w$ nearly, and adding $68w$ for the load below 58 at R, we have in all $500w$, the same as the new thrust at crown of invert. Thus the thrust at both crowns is the same, and the shell is now of a uniform thickness of 3 feet.

The thrust at crown is $525w \div 3 = 21000$ lbs. per sq. foot of brick, taking $w = 120$ lbs. per cubic foot. The crushing strength of strong brick is 154000 lbs. per square foot, so that the apparent factor of safety against crushing is 8, and the real factor cannot be less than 4, one-half of it. (See p. 486, and Rankine's *Civil Engineering*, p. 514, on buried arches.)

Allowance for Excess-load along the Elliptic Masonry Ring.—Rankine's assumption of the line of stress along the soffit of the elliptic arch makes allowance for the *excess load of* $\frac{1}{2}w$ *along the ring*, because the soffit, not being so elongated as the line up the middle of the ring, the ratio of transformation is greater.

Compare the two quadrants ABS (fig. 275) and ABS (fig. 265). In the first the half-span is 50 per cent. greater than the rise, and it balances under the fluid load alone. In the second the half-span equals the rise, and it balances under the fluid load together with a uniform load along the rib, equivalent to *voussoirs* of the same mass as the fluid, and of a depth half (the radius) of the rise. By proportional parts the half-span of a quadrant of given rise, to balance under a fluid load, and an excess uniform load along the rib, can be found by reducing the half-span for fluid load above by a proportionate reduction.

Thus, in the numerical example (fig. 276), our second approximation is as follows:—

Assume the line of stress to be up the middle of the masonry ring: then $x_0 = 5$, $x_1 = 26$, and $cy_1 = 40$. For the corresponding hydro-rib: then $b = 36\cdot382$, and $y_1 = 34\cdot28$.

If the ring be 50 per cent. heavier than the superstructure, the excess-mass is w, for the upper half having been *included*, its excess is $\frac{1}{2}w$, and the under half having been excluded, its excess is $1\cdot5w$ where w is the weight of a cubic foot of the superstructure.

Now, with no load along the rib, in addition to the fluid load, $y_1 = 34\cdot28$, with *voussoirs* $10\frac{1}{2}$ feet deep, that is half of the rise 21, $y_1 =$ the rise 21, being a reduction of 13·28 feet. Hence with the actual *voussoirs* of 4 feet in depth, the proportionate reduction is $13\cdot28 \times 4 \div 10\frac{1}{2} = 5\cdot06$. So that we have the reduced value, $y_1 = 29\cdot22$, giving $s = 1\cdot37$, and $s^2 = 1\cdot88$. The horizontal-load-area is, therefore, $1\cdot88 \times \frac{1}{2}(26^2 - 5^2) = 612$. Then $T_0 = 612w$. This is practically the same as the result of Rankine's approximation (see fig. 276).

In an aqueduct, when the masonry ring has a mass greatly denser than water, our second approximation becomes essential to accuracy (see numerical example which follows).

Examples.

313. Design of a semi-elliptic masonry arch of sandstone. Span 80 feet, and with a surcharge of 1·5 feet.

Taking the rise a *fifth* of the span 16 feet, the radius of crown of soffit is $40^2 \div 16 = 100$ feet. For depth of keystone, $t_0 = \surd(\cdot17 \times 100) = 4$ feet, say.

First approximation.—Take line of stress along the soffit and $x_0 = 5\cdot5$, $x_1 = 21\cdot5$, $a = x_1 - x_0 = 16$, and $sy_1 = 40$. The horizontal-load-area is $\frac{1}{2}(x_1{}^2 - x_0{}^2) = 216$ for the corresponding hydro-arch. By equation (22) and (23),

$$b = 16\sqrt[3]{\frac{21\cdot5}{5\cdot5}} = 25\cdot204.$$

$$y_1 = 25\cdot204 - \frac{1}{30}\frac{b^2}{16} = 23\cdot881.$$

$$s = \frac{40}{23\cdot88} = 1\cdot676, \quad s^2 = 2\cdot808.$$

Hence the thrust at the crown is $2\cdot808 \times 216$, or $T_0 = 606w$.

Second approximation.—Take line of stress up the middle of the masonry ring, and $x_0 = 3{\cdot}5$, $x_1 = 21{\cdot}5$, and now $b = 32{\cdot}965$, and $y_1 = 30{\cdot}95$ is the half span of corresponding hydro-arch. The half-rise is 9, and *voussoirs* this depth, besides the fluid load, would require the half-span to equal rise, that is 18 feet. But the *voussoirs* are 4 feet, and have an excess density w, for the ring is 50 per cent. heavier than the superstructure, but top half has been included, and has excess $\frac{1}{2}w$, lower half excluded has excess $\frac{3}{2}w$. Now *voussoirs* 9 feet reduce y_1 from 30·95 to 18, or by 12·95. Hence we must reduce y_1 by four-ninths of this, or 5·76, so that the reduced value to allow for excess weight along the ring is $y_1 = 25{\cdot}20$. Hence $s = 42 \div 25{\cdot}20 = 1{\cdot}667$ and $s^2 = 2{\cdot}778$. But the horizontal-load-area is now $\frac{1}{2}(21{\cdot}5^2 - 3{\cdot}5^2) = 225$, and $T_0 = 1{\cdot}667 \times 225w = 625w$, which differs only by 4 per cent. from the result of the first approximation.

314. The extreme depths of a quadrant of a hydrostatic rib are $x_0 = 7$ and $x_1 = 26$ feet below the surface of the fluid (fig. 276). At a point B', five feet lower than the crown, find the slope of the rib to the horizon.

The horizontal-load-area is $\frac{1}{2}(x_1^2 - x_0^2) = 313{\cdot}5$, and if the weight of the fluid be unity, this is the thrust at crown, and also at B', as it is constant along the rib. Now $x = 12$ is the depth of the point B', and $\frac{1}{2}(x_1^2 - x^2) = 266$ is the part of the horizontal-load-area from the springing up to B', and is therefore the horizontal component of the thrust at B', so that $266 \div 313{\cdot}5$, or ·8484, is the cosine of θ, the slope of the rib at B'.

315. In a complete geostatic rib span 76 feet, rise 19 feet, height of earth load over crown 7 feet (fig. 276), find the slope of the rib at B, a point five feet below the level of the crown. So that $x_0 = 7$, $x_1 = 26$, and $sy_1 = 38$ feet.

As in last example, we must find the slope θ of the corresponding hydro-arch, and also $s = 1{\cdot}362$, as in example 307, and $\cot\theta' = s\cot\theta$ determines the slope at B.

316. Design of an aqueduct with a semi-elliptic masonry ring twice as dense as water. The clear span is to be 40 feet, and the depth of water over the keystone 2·5 feet.

We proceed by trial and error, assuming values for the *rise*, and calculating the span. If there were no excess load along the ring, a first trial value for the rise would be a third part of the span, but a greater value is indicated, because of the large excess load along the rib. Try the rise of the soffit 15 feet. This gives the radius of crown of soffit $20^2 \div 15 = 27$. As an actual fluid is a troublesome load, the greater of Rankine's values is preferable for the thickness of the keystone, or $t_0 = \sqrt{(\cdot17 \times 27)} = 2\frac{1}{4}$ feet nearly. Adopting the value $2\frac{1}{2}$, and assuming the line of stress up the centre of the ring, we have $x_0 = 3{\cdot}75$ and $x_1 = 20$, and for the horizontal-load-area $\frac{1}{2}(20^2 - 3{\cdot}75^2) = 192{\cdot}97$. Also the rise of the line of stress is $a = 16{\cdot}25$, so that

$$b = 16{\cdot}25\sqrt[3]{(x_1 \div x_0)} = 28{\cdot}39, \quad \text{and} \quad y_1 = b - \tfrac{1}{30}b^2 \div a = 26{\cdot}74.$$

Now the excess load of the upper part included is w, and of the lower half excluded is $2w$, or, on an average, $1{\cdot}5w$. This is the same as if the *voussoirs* were of a thickness $1{\cdot}5\,t_0 = 3{\cdot}75$ feet. With no load along rib $y_1 = 26{\cdot}74$, with imaginary *voussoirs* half the rise, that is $8\frac{1}{8}$ feet, $y_1 =$ the rise $= a = 16{\cdot}25$, or is reduced by 10·5 feet. The proportionate reduction for *voussoirs* 3·75 feet is 4·84, so that the reduced value of the half span of line of stress is $y'_1 = 26{\cdot}74 - 4{\cdot}84 = 21{\cdot}9$. The half span of soffit would then be $21{\cdot}9 - 1{\cdot}25 = 20{\cdot}6$. This is near enough to 20 feet, the required half-span of soffit, so that further trial values of the rise of soffit are not required.

For the line of stress, then, the horizontal-load-area is 192·97 square feet, multiplied by $w = 64$ lbs., gives the thrust at crown $T_0 = 12350$ lbs., and the factor of safety can be evaluated.

317. A semicircular rib of radius, 27 feet, is loaded between itself and a horizontal straight line 3 feet over the crown. Find the angle of rupture, and the inward and outward parts of the horizontal-load-area.

By equations (5), (6), and (7), p. 499,

$$\text{Cos } i = \frac{5}{9} + \frac{4}{3} \cdot \frac{1}{9} = \cdot 704.$$

$$Q_1 = \left(\frac{2}{27} + \frac{5}{9} \cdot \frac{1}{9} + \frac{2}{3} \frac{1}{81}\right) (27)^2 w = 105w.$$

$$Q_2 = \left(\frac{2}{27} - \frac{4}{9} \cdot \frac{1}{9} + \frac{2}{3} \frac{1}{81}\right) (27)^2 w = 24w.$$

Their algebraic sum is $81w$, the product of the radius and load over crown.

318. One of the shafts in the Hoosac Tunnel, Western Massachussetts, is elliptical in form, the axes being 27 by 15 feet (Simms' *Tunnelling*, 6th edition, p. 527). Show that 2 feet of brick lining is strong enough for such a shaft, surrounded by earth, even at great depths.

The major diameter of the bore hole is 31 feet, so that 50 feet is about the height of the isosceles triangle, with base 31 feet, and vertical angle 30°. This is the deepest column of earth likely to be disturbed by the bore hole. Consider a ring of brickwork 50 feet below the surface of the earth, the ring itself 2 feet thick and 1 foot deep. Assuming the line of stress, in the first place, to be up the middle of the ring, its diameters are 29 and 17 feet, their ratio is 1·7, almost exactly $\sqrt{3}$. Hence the diameters of the ellipse of stress for the extern cubes of earth may be in the ratio 1 to 3. Now the lesser horizontal stress may be a third of the potential $50w$, so that the greater horizontal stress is equal to $50w$. The horizontal-load-area for a quadrant, in the direction of the major axis, is $50w \times 8{\cdot}5$, or $425w$. This also is the thrust T_0 (see fig. 274), at the end of the minor axis of the ring. But the area of the section of the ring is 2 square feet so that the thrust per square foot of brick is $212{\cdot}5w$, or 25500 lbs. For strong brick the strength is 154000 lbs. per square foot. Hence the apparent factor of safety is *six*.

NOTES TO CHAPTER XXII.

QUOTATIONS FROM RANKINE'S CIVIL ENGINEERING.

Page 425.—To determine, with precision, the depth required for the keystone of an arch by direct deduction from the principles of stability and strength would be an almost impracticable problem from its complexity. That depth is always many times greater than the depth necessary to resist the direct crushing action of the thrust. The proportion in which it is so in some of the best existing examples has been calculated and found to range from 3 to 70 . . . good medium values are those ranging from 20 to 40.*

An empirical rule, founded on dimensions of good existing examples of bridges is—depth of keystone, a mean proportional between the radius of curvature of the intrados at the crown, and a constant ·12 or ·17 for a single arch, or one of a series.

Page 428.—In some of the best examples of bridges, the thickness of the abutments ranges from *one-third* to *one-fifth* of the radius of curvature of the arch at its crown.

Page 429.—The thickness adopted for piers in practice range from *one-tenth* to *one-fourth* of the span of the arches; the latter thickness, and those approaching to it, being suitable for "abutment-piers." The most common thickness for ordinary piers is from one-sixth to one-seventh of the span of the arches.

Page 435.—It appears that, in the brickwork of various existing tunnels, the factor of safety is as low as *four*. This is sufficient, because of the steadiness of the load; but in buried archways, exposed to shocks, like those of culverts under high embankments, the factor of safety should be greater; say, from *eight* to *ten*.

QUOTATIONS FROM SIMMS' PRACTICAL TUNNELLING (4th edition, pp. 202, 203).

It is known, too, that the pressure on tunnels in comparatively shallow ground, say, less than 40 feet below the surface, may be localised and concentrated upon the crown of the arch with peculiar severity. Mr. Simms accounts for the greater pressure upon the work in shallow ground by the supposition that the whole superincumbent mass acts vertically downwards, whilst at greater depths it is more or less sustained as an arch over the tunnel, which is proportionally relieved of the pressure. In the building of the Stapleton Tunnel the arch was first built with only four rings of brickwork in mortar. When the autumnal rains set in, the ground began to press heavily; so much so, that the line of the tunnel could be traced as a hollow on the surface of the ground, which was not more than 40 feet above the tunnel. A portion of the tunnel fell in, and was rebuilt with five and six rings of brick.

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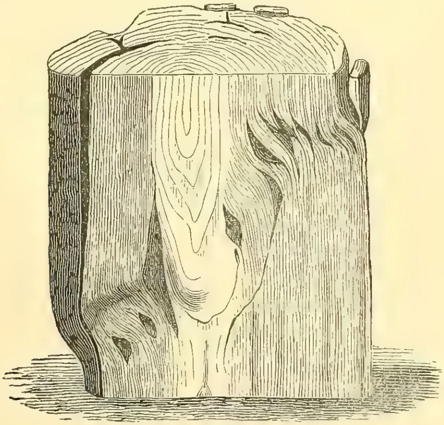
* Factors of safety calculated from existing bridges are likely to be what we have called *apparent* ones, so that the actual factors will range from 10 to 20, the 10 being the lowest assumed in our tables.

The pressure of clay and shale, when disturbed by excavation, is, in some situations, something almost immeasurable. The phenomena of disturbance supply powerful examples of the flow of solids. There is the familiar phenomenon of the bending and snapping of huge poles, and strongly-timbered frames, by the moving pressure of clay. In the construction of the Primrose Hill Tunnel, through the London Clay, the lengths were limited to 9 feet, and strongly timbered, till the arching was completed. In virtue of the mobility, however, the moist clay exerted so great a pressure on the brickwork, as to squeeze the mortar from the joints, to bring the inner edges of the bricks into contact, to grind them to dust by degrees, and to reduce the dimensions of the tunnel slowly but irresistibly. The evil was counteracted by using very hard bricks, laid in Roman cement, which, setting before the back pressure accumulated, hardened and resisted the pressure, and so saved the bricks. The thickness of the brickwork was augmented to 27 inches.

A similar accident occurred to a portion of the invert of the Netherton Tunnel, built on a foundation of "blue-bind," or marl. Some weeks after it was built, the invert was forced up in several places by the swelling of the ground, and at one point the bricks were crushed almost to powder. The invert was rebuilt with a greater versed sine.

The following illustration represents "A column of deal 14″ cube. It supported for some hours 1000 tons, being half the weight of the tube of the Britannia Bridge, when by accident the supports at one end failed. A bed of planks over it was crushed through 7″ by the shock. The compressed column was harder to cut out than brickwork."



APPENDIX.

I.

On figure 279 is shown a skeleton elevation of an iron roof-frame, span 32 feet, rise 12 feet. The rafter is trisected by two isosceles struts *PN* and *PO*, also *PQU* is 60°. The frames or principals are 20 feet apart, so that a fully-loaded joint such as *R* has apportioned to it all the load on an oblong of the roof-cover 20 feet × 7 feet nearly, since *MQ* is 20 feet, being the hypotenuse of a triangle of which the sides are 16 and 12 feet. Taking the weight of the cover at 4 lbs. per square foot, and of one foot deep of snow at 12 lbs. per square foot, the vertical load (5) concentrated at the joint *R* is $140 \times 16 = 2240$ lbs. $= 20$ cwts. The vertical loads, then, are 20 cwts. at the joints *N*, *O*, *Q*, *R*, and *S*, but only 10 cwts. at *M* and *T*. The wind-load is on the left side only, and its direction is at 45° to the rafter *MQ*. We take the normal component wind-loads as 25 per cent. greater than the vertical load. That is, the two loads at *N* and *O* normal to the rafter are each 25 cwts., those at *Q* and *M* being half as much.

The Reactions at the Supports.—The frame is securely anchored to the wall at the storm-end *M*, while the other end *T* has a free horizontal motion on expansion-rollers on the top of the wall there. This is only the first case; later on we will develop the procedure if both ends be anchored.

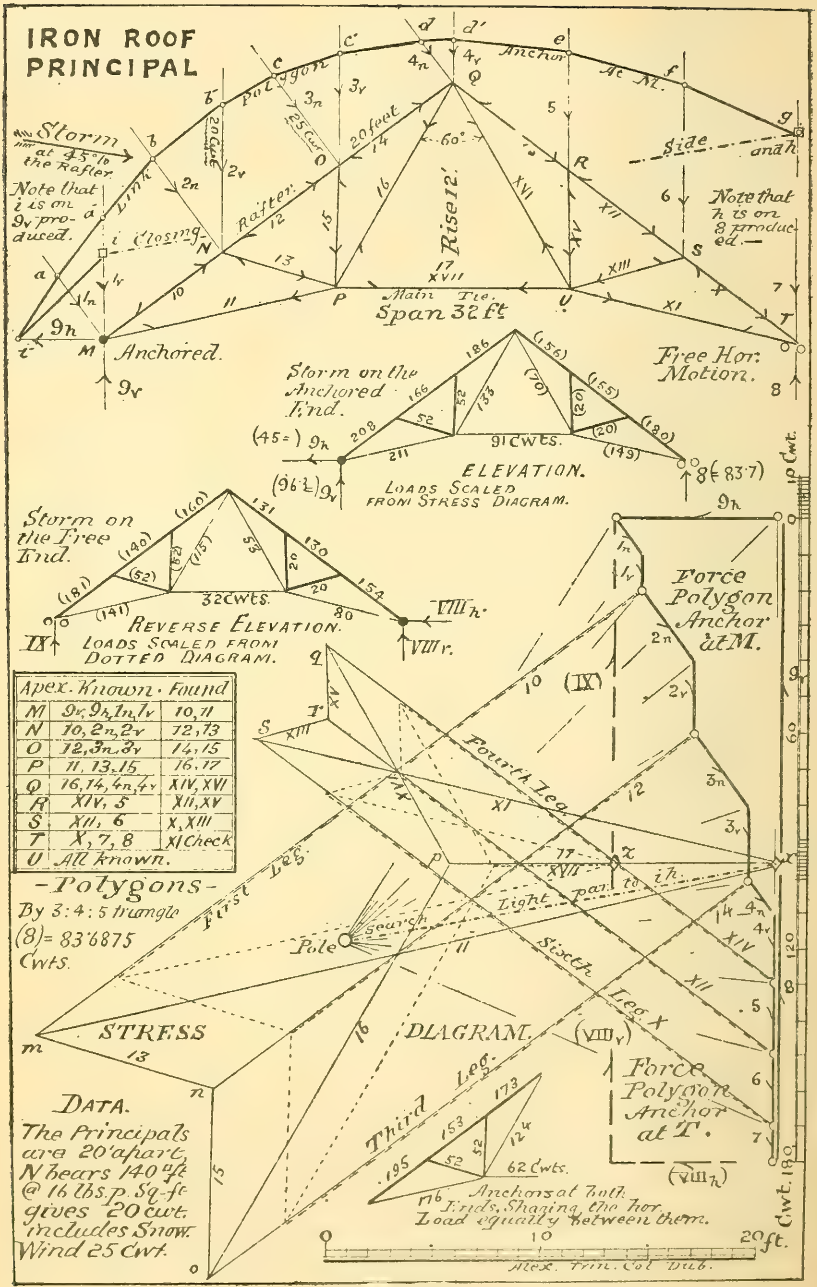
The Force-polygon, or load line, is now drawn to a scale of cwts. Beginning at *M*, and going round the frame in *right-cyclic-order*, the loads at the joints have the numbers 1 to 7 allotted to them. Thus at *M* there is 1_n and 1_v for the normal and vertical component loads there. Because of the trolly, the reaction at *T* is necessarily vertical, and 8 is its number. The reactions at the anchored end *M* are 9_v and 9_h, being vertical and horizontal component reactions, respectively. The sides of the force-polygon are now drawn in the same cyclic order. It will be found to be complete, with one exception, namely, the joint x between the vertical reactions 8 and 9_v is unknown. Because the half-truss is a 3, 4, 5 right-angled triangle, it will be found that all the joints in the force-polygon are horizontally opposite multiples of 10 cwts. on the scale.

The Link-polygon.—By consulting fig. 74, p. 100, this part of the construction will be more readily followed. On fig. 279 a pole O is chosen just opposite the last bend in the force-polygon, and so as, roughly, to be the vertex of an equilateral triangle standing on the force-polygon. Vectors are to be drawn (in pencil only) from the pole O to each joint of the force-polygon except x, which is not yet determined.

Since the joint x is at the junction of 9_v and 8, the vector Ox must be parallel to the link of the link-polygon joining up those two vertical forces. For this reason we should begin the link-polygon at a point on the vertical line of action 9_v, and end it at a point on the line of action of 8. Choose any point i on the line of action of 9_v. On fig. 279 it is chosen on 9_v, produced about an inch above M, and is marked with a little square, as a memorandum that i is the beginning point. The link ii' is drawn between the lines of action of 9_v and 9_h, parallel to the vector from the pole to the junction of those two forces. In the same way the other links, $i'a$, aa', $a'b$, bc', $c'd$, dd', $d'e$, ef, fg, and gh, are drawn consecutively between the lines of action of the forces in pairs, each link parallel to that vector which comes from the pole to the junction of the corresponding pair of forces. The line gh has no finite length, the lines of action of 7 and 8 coinciding, hence g and h coincide too, and have a little circle with a little square round their common position. Joining the two little squares i and h with a dot-and-dash line, we have found the *closing side* of this link-polygon, while a vector (or search-light) drawn from the pole parallel to it determines x the *closing point* of the force-polygon. The judicious choice of the pole, and of the starting point i, ensure that the link-polygon will 'rainbow' over the truss, out of the way of the further construction, and also that the links will go on continuously without looping back on each other.

The Stress Diagram.—Put half-barbs on the sides of the force-polygon, and full-barbs on the lines of action of the forces external to the frame. It will then be seen that at the apex M six forces concur. Four of them are known, that is, they are already drawn to scale on the force-polygon. In naming these four they are to be taken in *right-hand cyclic* order round M, thus, 9_v, 9_h, 1_n, 1_v. It is convenient to call the remaining two forces exerted on the joint M by the rafter and tie 10 and 11, keeping to the same cyclic order round M.

Now, the four-sided open polygon 9_v, 9_h, 1_n, and 1_v is to be completed as a closed six-sided polygon, by drawing a side 10 parallel to the rafter to meet a side 11 drawn backwards from the initial point x. In this way the magnitudes of 10 and 11 are now determined, and by going round the six-sided closed polygon in the direction indicated by the half-barbs, it will be seen that 10 acts along the rafter *towards* M, while 11 acts along the tie *away* from M. On 10 and 11, in the neighbourhood of M, place half-barbs pointing accordingly; but at the other end of 10 put a full-barb in the opposite direction from that of the half-barb, so as to indicate the reaction of the rafter there, and a *full-*


IRON ROOF PRINCIPAL
Storm at 45° to the Rafter.
Note that i is on 9v produced.
Link
Polygon
Anchor
At M.
Closing
Side
and h
Note that h is on 8 produced.
20 cwt
25 cwt
20 feet
Rafter
Rise 12'
60°
Main Tie.
Span 32 ft
Anchored.
Free Hor. Motion.
a b b' c c' d d' e f g
i M N O P Q R S T U
1n 1v 2n 2v 3n 3v 4n 4v 5 6 7 8 9h 9v
10 11 12 13 14 15 16 17 XI XII XIII XIV XV XVI XVII
Storm on the Anchored End.
186 (156) 166 52 133 (70) (155) (20) 208 52 (20) (180) 211 91 cwts. (149)
(45 =) 9h
(96·1) 9v
8 (= 83·7)
ELEVATION.
LOADS SCALED FROM STRESS DIAGRAM.
Storm on the Free End.
(160) 131 (140) (62) (115) 53 130 20 (52) 20 (181) 154 (141) 32 cwts. 80
REVERSE ELEVATION.
LOADS SCALED FROM DOTTED DIAGRAM.
IX
VIII h.
VIII r.
10 cwt.
Force Polygon Anchor at M.
(IX)
Fourth Leg.
First Leg.
Light par. to ih.
Search
Pole
Sixth Leg X
(VIII v)
STRESS DIAGRAM.
Third Leg.
Force Polygon Anchor at T.
(VIII h)
173 153 195 52 52 124 176 62 cwts.
Anchors at both Ends, Sharing the hor. Load equally between them.
m n o q r s t p x z
0 10 20 ft.
0 60 120 180 Cwt.
Alex. Trin. Col. Dub.

Apex.	Known.	Found
M	9v, 9h, 1n, 1v	10, 11
N	10, 2n, 2v	12, 13
O	12, 3n, 3v	14, 15
P	11, 13, 15	16, 17
Q	16, 14, 4n, 4v	XIV, XVI
R	XIV, 5	XII, XV
S	XII, 6	X, XIII
T	X, 7, 8	XI Check
U	All known.	

-Polygons-
By 3:4:5 triangle
(8) = 83·6875 Cwts.

DATA.
The Principals are 20' apart, N bears 140 sq ft @ 16 lbs. p. Sq-ft gives 20 cwt. includes Snow. Wind 25 Cwt.

Fig. 279.

barb to indicate that it is *already drawn to scale* on the force-polygons. In the same way put a *full*-barb reversed at the other end of the tie 11.

It will now be seen that of the five forces that concur at *N*, all are furnished with barbs except 12 and 13, which reminds us that all the sides of the five-sided closed polygon for the apex *N* are *already* drawn on the force-polygon, and the lines drawn parallel to 12 and 13 close it, when the *sense* of 12 and 13 are then found to be both towards *N*, so that they are to be furnished with half-barbs in the neighbourhood of *N*, and with full-barbs reversed at their other ends. The next apex that can be solved is *O*, for here the group have all barbs but two. In this way the apexes are solved in the order *Q*, *R*, and *S*, the known and found parts being duly registered in the table for that purpose on fig. 279, while the letters *m*, *n*, &c., give the corresponding closing points. The next apex *T* is abnormal: there all the forces are known but *one*, namely, XI., which cannot be compelled to close the corresponding polygon. When XI. is drawn from *x*, if it only fails to go through *s* by a little deviation, that is the allowable error of drawing. In this way we have a *check* upon the construction. If it utterly fails to go through *s*, we must look back to the construction of the polygons for some point at which we did not keep to strict cyclic order. If with every care to draw exactly it still quite fails to close, that would mean that the truss is deficient in members—that it is not *rigid*. Of course this could not happen with the truss on fig. 279, which is necessarily rigid, being a set of triangles.

Anchor on Lee-side.—If we suppose the anchor shifted to *M*, that will make no difference in the magnitudes of the vertical reactions of the walls. For one way of calculating the magnitude of 8 is to take moments about *M*. Now, the horizontal reaction of the anchor passes through *M*, whether the anchor be at *M* or *T*, and so has no moment, and cannot affect the result. There is no need to draw the link-polygon anew, but only to shift *x* into the new position *z*, and the force-polygon is now 1_n, 1_v, 2_n, 2_v, 3_n, 3_v, 4_n, 4_v, 5, 6, 7, VIII_h, VIII_v, 9. Nor is there any need to redraw the stress diagram, polygon by polygon, for we may suppose the original stress diagram as having been constructed in two steps, as follows:—In the first step the six legs are drawn of indefinite length, one from each joint of the force-polygon. In the second step we might suppose a billiard-ball to start from *x*, describing the route 11, 13, 15, 16, rebounding from the legs in order at *m*, *n*, and *o*, and arriving on the horizontal at *p*, then completing its route XVI., XV., XIII.; rebounding from the legs at *q*, *r*, and *s*, and returning along XI. to the pocket at *x*. The dotted route is mapped out at once in this way, only beginning and ending at *z*.

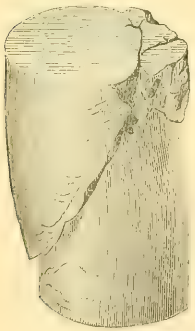
Scaling off the Results.—The full lines are scaled off and the results are written on a small *elevation* of the roof on fig. 279, having the anchor on the left, or storm side. Another *reverse elevation* is shown with the anchor shifted to the right, or lee side, and on it are written the loads as scaled off the dotted diagram. Now this is called a *reverse* elevation, because after all it is *not the anchor* which has really shifted, but the wind has, and we have gone round about to look at the back of the roof-frame. Hence this reverse elevation requires to be read through on the back of the paper when comparing the stresses with those marked on the first direct elevation. That is, the quantities scaled off for the bars bearing Arabic numbers on the first elevation, are to be compared with those scaled off for the bars bearing Roman numbers on the reverse elevation.

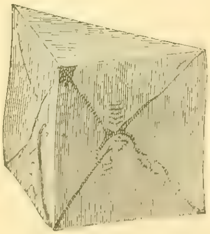
Both ends fixed.—If both ends of a truss be fixed, it is quite indeterminate in what proportions the two anchors will take up the horizontal load due to the wind. Suppose for a moment that both ends of the roof, fig. 279, are fixed, and that we assume that the two anchors share equally the horizontal load. It would be solved on this assumption, by simply placing the closing point of the force-polygon at a point midway between x and z, and of course the results would just be an average of those for the full and dotted stress diagrams shown there. These are marked on a half-elevation at bottom of the figure.

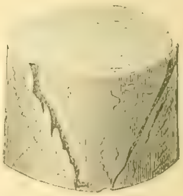
In every case, then, it is best to *assume* an anchor at the storm side for the reactions; then shift x horizontally in accordance with any *assumed* proportion between the holding power of the anchors.

With the anchor at the storm side, the stresses are *maxima*, so that it is a *safe* assumption to make in all cases. The dotted stress diagram, fig. 279, gives minima stresses; but in some examples these minima are very important. For a member which was a tie for the maxima, may have, in the minima diagram, decreases through zero, and become a *strut*.

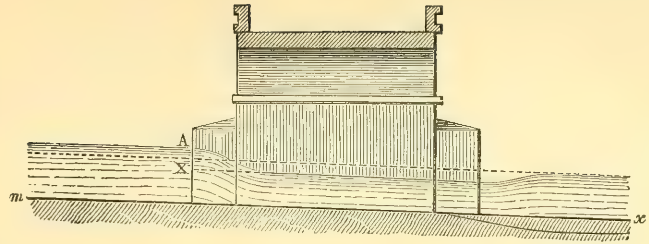
The following illustrations show Lowmoor cast-iron specimens, with the manner of splintering, when crushed, at angles from 48° to 58°.



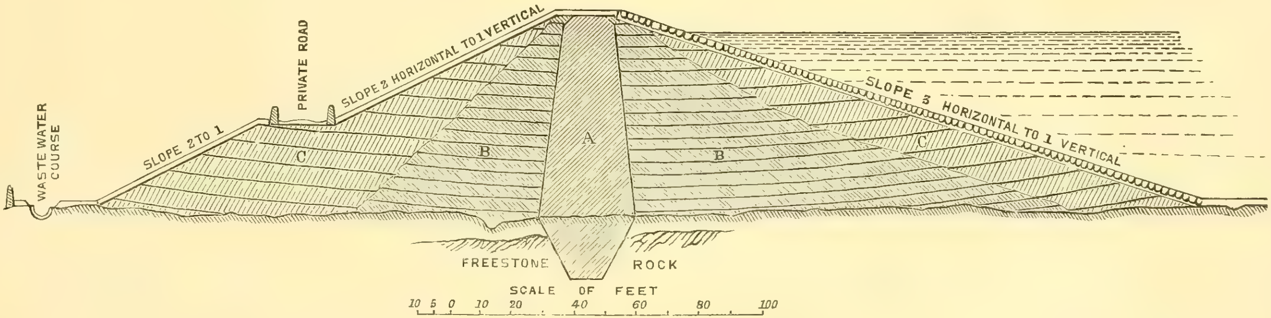




APPENDIX II.



Cross-section of a Bridge.—Showing the fall in water-level because of the increased velocity due to the contraction of channel between the piers.



Cross-section of a Built Earth Embankment.—*A* is the puddle wall, standing on the puddle trench cut in the water-tight stratum; *B* is selected stuff, and *C* is earth. The contours of the layers are approximately lines of principal stress. The water-slope is pitched with stone, and the land-slope finished with sods.

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